Games and Economic Behavior **30,** 22–43 (2000)
doi:10.1006/game.1998.0709, available online at http://www.idealibrary.com on IDEAL®

Difference-Form Contests and the Robustness of All-Pay Auctions

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Received June 25, 1998

In much of the existing literature on rent-seeking games, the outcome of the contest is either infinitely sensitive or relatively insensitive to contestants' efforts. The current paper presents a family of contest games that permit characterization of equilibrium for all levels of sensitivity of the outcome to contestants' efforts. Specifically, the outcome of the contest depends on the difference between efforts, which encompass the lottery and the all-pay auction as polar cases. The equilibrium converges to that of the all-pay auction as the probability of winning the prize grows infinitely sensitive to one's effort, and the main qualitative features of equilibrium persist over a large parameter region. *Journal of Economic Literature* Classification Numbers: D44, D72. © 2000 Academic Press

1. INTRODUCTION

Contests are an important fact of economic life. When firms bid for a defense contract, when scientists expend research effort to secure a patent, when lobbyists make contributions to gain political influence, and when politicians campaign for votes, the situation is one in which contestants make sunk investments in pursuit of a prize. An important element of each case is the rule by which the prize is allocated. The literature has largely taken a reduced-form approach by assuming the form of the success function that determines the probability that a contestant wins the prize.[1]

[1] One can also interpret the probability of winning as the share of the prize that accrues to the bidder.

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Two families of success functions have been studied frequently.[2] One comes from the so-called Tullock rent-seeking game in which a contestant's winning probability is proportional to a function of his "effort." Specifically, contestant i's probability of winning the prize is

$$\frac{e_i^R}{\sum_{j=1}^N e_j^R}, \qquad i = 1, \ldots, N$$

when contestants $j = 1, \ldots, N$ expend effort $e_j \geq 0$. Thus, a contestant's probability of winning depends on the ratio of effort levels. This success function has been studied by Tullock (1980), Baye *et al*. (1993, 1994, 1996), and Che and Gale (1997, 1998), among others. In the polar case of $R = \infty$, the success function of the Tullock game coincides with that of the all-pay auction: the contestant who expends the greatest effort wins the prize with probability one.

Despite its apparent flexibility, this success function has drawbacks. First, while some cases ($0 \leq R \leq 2$, and $R = \infty$) are well understood, the intermediate cases are not.[3] Thus, the model offers no prediction of how rent-seeking activity will change as the sensitivity of the success function changes. In particular, it is not known whether the qualitative features of the equilibrium when $R = 2$ and $R = \infty$ are preserved when R is perturbed slightly. Second, even for the cases that have been analyzed, the qualitative properties of the equilibrium change as one shifts from $0 \leq R \leq 2$ to $R = \infty$. In the former cases, an increased asymmetry in bidders' valuations does not lower the amount of rent dissipation, given moderate levels of asymmetry, whereas the all-pay auction always displays such a "preemption effect."

In another family of success functions, called "difference-form" success functions, contestant i's probability of winning is given by $f(\{g(e_i) - g(e_j)\}_{j \neq i})$, where $f(\cdot)$ and $g(\cdot)$ are increasing functions. The difference-form success function is appropriate in many contexts. For example, in many contests the prize is allocated according to an *absolute criterion* based on each contestant's performance measure. A worker may be promoted when his overall performance measure exceeds his peers' (even by a slim margin), and a politician wins an election when receiving just a single vote more. If each contestant's performance measure reflects his effort level exactly, then the all-pay auction is an appropriate model. In many cases, however, the performance measure may be an imperfect signal of the

[2] See Skaperdas (1996) for additional references and discussion.

[3] See Baye *et al*. (1994). They studied the intermediate cases, characterizing symmetric equilibria when two bidders with identical valuations select from a discrete set of bids, and they calculated an upper bound on total expected rent dissipation.

underlying variable of interest. For instance, a less able worker might achieve a better performance measure because of random events unrelated to effort.

Support that there are two contestants. Contestant i's performance measure, s_i, is given by his effort, e_i, garbled by an additive shock, ϵ_i. That is, for $i = 1, 2$,

$$s_i = e_i + \epsilon_i,$$

where ϵ_i is a random variable. Contestant i wins the prize if

$$s_i > s_j \Leftrightarrow e_i - e_j > \epsilon_j - \epsilon_i,$$

which occurs with probability $f(e_i - e_j)$, where $f(\cdot)$ is the cumulative distribution function of $\epsilon_j - \epsilon_i$. That is, the success function is of the difference form. As the noise disappears and the performance measures become perfectly precise, the difference-form success function approaches that of the all-pay auction [i.e., $f(\cdot)$ jumps from 0 to 1 at $e_i - e_j = 0$]. This shows that the all-pay auction is a difference-form contest, too.

Hirshleifer (1989) studied difference-form contests with two contestants, obtaining results that are dramatically different from those of the Tullock game.[4] For instance, in any pure-strategy equilibrium, one or both contestants expend zero effort. Baik (1997) studied a more general difference-form success function, showing that this property holds in all pure-strategy equilibria. By contrast, zero effort does not arise in the Tullock game when a pure-strategy equilibrium exists (i.e., when $0 \leq R \leq 2$).

The previous work on general difference-form contests was limited to settings in which a pure-strategy equilibrium exists, which means that at least one contestant is sufficiently insensitive to the outcome (has a sufficiently low valuation) that he expends zero effort. So, when one contestant expends zero effort, it is unclear whether this passivity is a result of the restriction to pure-strategy equilibria. The restriction to pure-strategy equilibria also meant that these papers did not treat cases close to the important case of the all-pay auction, so they could not address the robustness issue.

The current paper analyzes a class of difference-form contests without the restriction to pure-strategy equilibria. While our primary focus is on the cases in which both contestants are viable, we characterize equilibria for all parameter values. For much of this paper, we restrict attention to a piecewise-linear success function,

$$f(e_i - e_j) = \max\left\{\min\left\{1/2 + s(e_i - e_j), 1\right\}, 0\right\},$$

[4] In particular, a logistic success function was studied. Skaperdas (1996) provided an axiomatic basis for the logistic success function. Neither the all-pay auction nor the contests we consider in Section 3 satisfy the axioms completely.

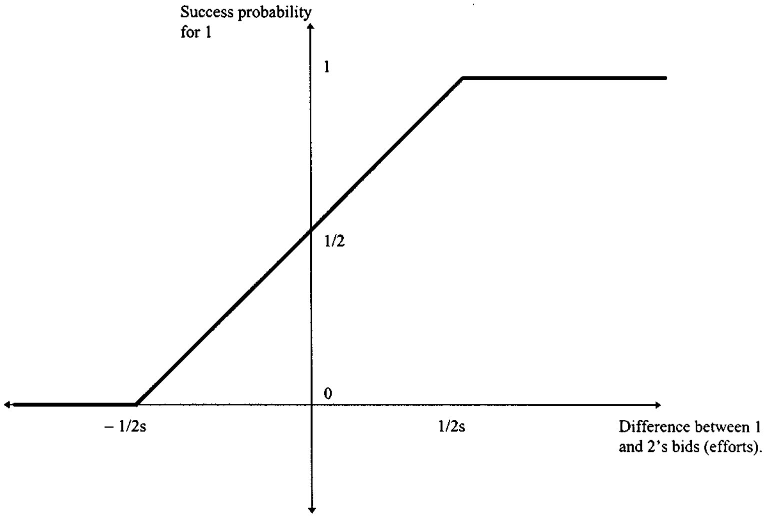

FIG. 1. Piecewise linear success function.

where $s > 0$. Thus, a contestant's probability of winning the prize increases at the rate s, until the probability reaches one. (See Fig. 1). This success function turns out to be very flexible. At one extreme, when $s = 0$, the outcome is completely insensitive to effort. At the other extreme, when $s = \infty$, the contest coincides with the all-pay auction, which is infinitely sensitive at the margin. For $0 < s < \infty$, a bidder can only guarantee success by exerting at least $1/(2s)$ more effort than his opponent.

Our success function proves to be more satisfying than the Tullock success function in the two dimensions noted above. First, it permits characterization of equilibrium for all possible valuations, and for all values of s. In the next section, we determine when pure-strategy equilibria fail to exist. Section 3 identifies two types of mixed-strategy equilibrium in that latter region. In these equilibria, the contestants place probability mass on a collection of isolated points. We are then able to show that the equilibrium converges to that of the all-pay auction as s goes to infinity (Section 5). This result shows that the all-pay auction is robust to randomness in how the prize is allocated, and to noise in the performance measure. Second, certain qualitative features of the equilibrium are preserved for all values of the sensitivity parameter. Of greatest interest is the preemption effect—the property that an increased asymmetry in bidders' valuations leads to reduced rent dissipation. Section 4 shows that this effect persists for a broad set of parameter values, confirming that preemption is a general feature of difference-form contests (not just all-pay auctions).

2. PRELIMINARY RESULTS

We first establish some basic results about contests with a difference-form success function. In keeping with much of the existing literature, we refer to the contestants as "bidders" and to their effort levels as "bids." The first result establishes that at least one bidder bids zero in any pure-strategy equilibrium, for a very general family of difference-form success functions. While Hirshleifer (1989) and Baik (1997) showed this for certain twice-differentiable success functions, we need a more general version of their result.

Let $f(x - y)$ be a general success function, which represents the probability that bidder 1 wins the prize when he expends $x \geq 0$ and bidder 2 expends $y \geq 0$ (both measured in monetary units). Bidders 1 and 2 value the prize at v_1 and v_2 (again measured in monetary units), respectively. The bidders are risk neutral, and they bid simultaneously.

Assumption 1. If $v_1 = v_2 \equiv v$, then $f(\cdot)$ does not have a linear portion with slope $1/v$.

This assumption is very mild as it does not require continuity or differentiability of $f(\cdot)$. It simply rules out linear segments with a specific slope when bidders have identical valuations.

PROPOSITION 1. *Given Assumption* 1, *one bidder bids zero in any pure-strategy equilibrium.*

Proof. Suppose, to the contrary, that there exists a pure-strategy equilibrium with $(x^*, y^*) \gg (0, 0)$. Then, bidder 1's expected payoff,

$$v_1 f(x - y^*) - x,$$

must attain a local maximum at $x = x^*$. This requires that $f(\cdot)$ be concave in an open neighborhood of $x^* - y^*$. Similarly, local concavity of bidder 2's expected payoff,

$$v_2 [1 - f(x^* - y)] - y,$$

requires that $f(\cdot)$ be convex in an open neighborhood of $x^* - y^*$. For both conditions to be met, $f(\cdot)$ must be linear in an open neighborhood of $x^* - y^*$. Hence, in this putative equilibrium, both bidders must face locally linear expected payoff functions. If $v_1 \neq v_2$, then at most one of the expected payoff functions can be locally linear with a slope of zero, so at least one bidder would strictly benefit from changing his bid. If $v_1 = v_2$, then Assumption 1 implies that neither bidder has an expected payoff function that is locally linear with a slope of zero. Thus, in either case, at least one bidder has an incentive to deviate. Q.E.D.

Equilibria in contest games with difference-form success functions are often in mixed strategies. Since it is a daunting task to characterize mixed-strategy equilibria for asymmetric bidders and a completely general success function, we focus on the aforementioned function, $f(x - y) = \max\{\min\{1/2 + s(x - y), 1\}, 0\}$. Given our success function, the above proposition tells us when pure-strategy equilibria exist. From here on, we assume $v_1 \geq v_2$, without loss of generality.

PROPOSITION 2. *With the piecewise-linear success function, a pure-strategy equilibrium fails to exist if* $v_2 s > 1$. *If* $v_2 s \leq 1$, *there is a pure-strategy equilibrium in which bidder* 2 *bids zero*.

Proof. Start with the case of $v_2 s > 1$, and suppose that a *pure-strategy* equilibrium exists. Proposition 1 indicates that some bidder, say bidder i, bids zero. Bidder $j \neq i$ then bids his best response, $1/2s$. Since bidder j wins the prize with probability one, bidder i will receive a payoff of zero in equilibrium. Consider a deviation by bidder i to a bid of $1/s$. With the latter bid, bidder i wins the prize with probability one (against the bid of $1/2s$ by bidder j). The payoff from deviating is therefore $v_i - 1/s > 0$, since $v_i s > 1$ for $i = 1, 2$. Since we have found a contradiction, a pure-strategy equilibrium does not exist when $v_2 s > 1$.

Now consider $v_2 s \leq 1$. A bid of zero is optimal for bidder 2, whatever bidder 1 does. Formally, bidder 2's expected payoff,

$$v_2[1 - f(x^* - y)] - y = v_2 f(y - x^*) - y,$$

is nonincreasing in y.[5] Thus, zero is a weakly dominant bid if $v_2 s \leq 1$. Bidder 1 optimizes against a bid of zero by bidding zero himself if $v_1 s \leq 1$, or by bidding $1/2s$ if $v_1 s > 1$. These pure-strategy equilibria are unique except in the nongeneric case that at least one of the valuations satisfies $v_i s = 1$. Q.E.D.

A pure-strategy equilibrium fails to exist unless at least one bidder is sufficiently weak (i.e., has a sufficiently low valuation). Thus, the more interesting cases in which both bidders are viable are ruled out when one focuses on pure-strategy equilibria. From here on, we assume that $v_1 \geq v_2 > 1/s$, so the equilibria will be in mixed strategies.

3. ANALYSIS OF MIXED-STRATEGY EQUILIBRIA

In this section we characterize the mixed-strategy equilibria. We find two types of mixed-strategy equilibrium. In the ‘‘overlapping’’ type, the

[5] The slope of the payoff function with respect to y takes on values of $v_2 s - 1 \leq 0$ or -1.

bidders put mass on zero, and on additional points whose locations depend on their respective valuations. It arises when the valuations are sufficiently close to each other and/or s is sufficiently small. Otherwise, a "staggered" equilibrium exists. In it, the bidders put mass on evenly staggered points, with bidder 2 alone putting mass on zero. We also show that these equilibria are mutually exclusive (generically) and exhaustive in the region with mixed-strategy equilibria.

A. Overlapping Equilibrium

We first examine the overlapping equilibrium. Consider the following strategies. Bidder 1 assigns mass of $m_1, \ldots, m_k$ to bids $x_1, \ldots, x_k$, for some integer $k \geq 2$, and bidder 2 assigns mass of $n_1, \ldots, n_k$ to $y_1, \ldots, y_k$. Suppose that

$$x_1 = y_1 = 0, \qquad x_2 \in \left[\frac{1}{2s}, \frac{1}{s}\right], \quad y_2 \in \left[\frac{1}{2s}, \frac{1}{s}\right], \qquad \text{and}$$

$$x_i - x_{i-1} = y_i - y_{i-1} = \frac{1}{s}, \qquad \text{for } i = 3, \ldots, k. \quad (1)$$

Then, once we pin down x_2 and y_2, we will have pinned down the locations of all mass points.

For any $x \geq 0$, let $i(x) \in \operatorname{argmin}_j |x - y_j|$ be the index of y_j closest to x. Similarly, for any $y \geq 0$, let $j(y) \in \operatorname{argmin}_i |y - x_i|$ be the index of x_i closest to y. The payoff to bidder 1 from bidding $x \geq 0$, when bidder 2 employs the suggested strategy, is as follows:

$$U_1(x) \equiv \begin{cases} v_1 n_1 \left[\frac{1}{2} + sx\right] - x & \text{if } x \leq y_2 - \frac{1}{2s}, \\ v_1 \left\{ n_1 \left[\frac{1}{2} + sx\right] + n_2 \left[\frac{1}{2} + s(x - y_2)\right] \right\} - x & \\ \quad \text{if } x \in \left[y_2 - \frac{1}{2s}, \frac{1}{2s}\right], & \\ v_1 \left\{ \sum_{j=1}^{i(x)-1} n_j + n_{i(x)} \left[\frac{1}{2} + s(x - y_{i(x)})\right] \right\} - x & \\ \quad \text{if } x \in \left[\frac{1}{2s}, y_k + \frac{1}{2s}\right], & \\ v_1 - x & \text{if } x \geq y_k + \frac{1}{2s}. \end{cases}$$

Adjacent mass points are $1/s$ units apart, except for the first two, which are closer. Since $f(-1/2s) = 0$, $f(1/2s) = 1$, and $f(d) \in (0,1)$ if $d \in (-1/2s, 1/2s)$, at most one mass point is "in play" for a typical $x \geq 1/2s$. That is, only one of bidder 2's mass points would affect bidder 1's payoff if he were to change his bid slightly.

When bidder 1 follows the suggested strategy, bidder 2's expected payoff from bidding $y \geq 0$ is

$$U_2(y) \equiv \begin{cases} v_2 m_1\left[\dfrac{1}{2} + sy\right] - y & \text{if } y \leq x_2 - \dfrac{1}{2s}, \\ v_2\left\{m_1\left[\dfrac{1}{2} + sy\right] + m_2\left[\dfrac{1}{2} + s(y - x_2)\right]\right\} - y \\ \quad \text{if } y \in \left[x_2 - \dfrac{1}{2s}, \dfrac{1}{2s}\right], \\ v_2\left\{\displaystyle\sum_{i=1}^{j(y)-1} m_i + m_{j(y)}\left[\dfrac{1}{2} + s(y - x_{j(y)})\right]\right\} - y \\ \quad \text{if } y \in \left[\dfrac{1}{2s}, x_k + \dfrac{1}{2s}\right], \\ v_2 - y & \text{if } y \geq x_k + \dfrac{1}{2s}. \end{cases}$$

Several conditions are necessary for these to be equilibrium strategies. We now use some of these conditions to pin down when this type of equilibrium exists. First, bidder 1 must attain local maxima at $x_2, \ldots, x_k$. Since (1) implies that $i(x_j) = j$, a sufficient condition is

$$U_1'(x_i) = v_1 n_i s - 1 = 0,$$

which implies that

$$n_i = \frac{1}{v_1 s}, \tag{2}$$

for $i = 2, \ldots, k$.

The assumption that $x_1 = 0$ means that bidder 1's expected payoff must not rise as he raises his bid from zero, which implies

$$U_1'(0) = v_1 n_1 s - 1 \leq 0,$$

or

$$n_1 \le \frac{1}{v_1 s}. \tag{3}$$

Since bidder 2's mass must sum to one, we have

$$n_1 = 1 - \frac{k-1}{v_1 s}. \tag{4}$$

Combining (3) with (4), and using the fact that $n_1 \ge 0$, yields

$$v_1 s \le k \le v_1 s + 1. \tag{5}$$

Repeating the exercise for bidder 2 pins down $m_1, \ldots, m_k$, and it yields the additional necessary condition:

$$v_2 s \le k \le v_2 s + 1. \tag{6}$$

This type of equilibrium only exists if both (5) and (6) hold:

$$v_1 s \le k \le v_2 s + 1. \tag{7}$$

Imposing indifference between x_1 and x_2 for bidder 1, and between y_1 and y_2 for bidder 2, pins down x_2 and y_2, which pins down the locations of all other mass points, via (1).

Generically, at most one integer, $k \ge 2$, satisfies (7). (Recall that we assume $v_2 s > 1$ here.) Note that (7) is only possible if $v_2 s + 1 \ge v_1 s$, which implies that v_1 is sufficiently close to v_2 and/or s is sufficiently small. In particular, fixing $v_1 > v_2$, the region disappears as s approaches ∞. We now establish the result.[6]

PROPOSITION 3. *If $v_1 s \le k \le v_2 s + 1$, for some integer $k \ge 2$, then there exists an equilibrium in which bidder 1 places mass $m_1, \ldots, m_k$ on $x_1, \ldots, x_k$, and bidder 2 places mass $n_1, \ldots, n_k$ on $y_1, \ldots, y_k$, where*

$$m_1 = 1 - \frac{k-1}{v_2 s}, \quad m_i = \frac{1}{v_2 s}, \qquad \forall i = 2, \ldots, k,$$

$$x_1 = 0, \quad x_i = \frac{v_2}{2} - \frac{k+2-2i}{2s}, \qquad \forall i = 2, \ldots, k,$$

$$n_1 = 1 - \frac{k-1}{v_1 s}, \quad n_i = \frac{1}{v_1 s}, \qquad \forall i = 2, \ldots, k,$$

and

$$y_1 = 0, \quad y_i = \frac{v_1}{2} - \frac{k+2-2i}{2s}, \qquad \forall i = 2, \ldots, k.$$

[6] Generically, there exists a continuum of equilibria that are payoff equivalent to the overlapping equilibrium. See Che and Gale (1998) for details.

Proof. By construction, $U_1(x)$ decreases (weakly) in x for $x \in [0, y_2 - 1/2s]$. Since $n_i = 1/v_1 s$, for $i = 2, \ldots, k$, by (2), we have $n_1 + n_2 \geq 1/v_1 s$, which implies that $U_1(x)$ is increasing in x for $x \in [y_2 - 1/2s, 1/2s]$. Next, (2) implies that $U_1(x)$ is constant for $x \in [1/2s, y_k + 1/2s]$, and is strictly decreasing in x for $x \geq y_k + 1/2s$. Finally, we have $U_1(0) = U_1(x)$ for all $x \in [1/2s, y_k + 1/2s]$. Arguing symmetrically, $U_2(y)$ decreases for $y \in [0, x_2 - 1/2s]$, rises for $y \in [x_2 - 1/2s, 1/2s]$, is constant for $y \in [1/2s, x_k + 1/2s]$, and decreases thereafter. In particular, $U_2(0) = U_2(y)$ for all $y \in [1/2s, x_k + 1/2s]$. It then suffices to note that $x_k \leq y_k + 1/2s$ and $y_k \leq x_k + 1/2s$, which holds since

$$|x_k - y_k| = |x_2 - y_2| = \frac{v_1 - v_2}{2} \leq \frac{1}{2s},$$

where the inequality follows from (7). The expected payoff to each bidder is graphed in Fig. 2, which shows that neither bidder can benefit strictly by deviating from the putative equilibrium strategies. Q.E.D.

In this overlapping equilibrium, bidder 2 places mass on higher bids (i.e., $x_i \leq y_i$, $i = 2, \ldots, k$) but places less mass on each non-zero mass point (i.e., $m_i \geq n_i$, $i = 2, \ldots, k$). We can show that these two effects completely offset each other.

COROLLARY 1. *In the overlapping equilibrium presented in Proposition* 3, *the expected payment from each bidder is equal to* $(k - 1)/(2s)$, *which yields total expected rent dissipation of* $(k - 1)/s$. *For* $s < \infty$, *the dissipated rent is* (*strictly*) *less than* v_2 (*generically*).

Proof. The first statement is obtained by straightforward computation. The second statement holds since, by (7), the total expected rent dissipation satisfies

$$\frac{k - 1}{s} \leq \frac{(v_2 s + 1) - 1}{s} = v_2.$$

The inequality is strict, generically. Q.E.D.

It is instructive to examine the expected surplus accruing to each bidder. Proposition 3 shows that each bidder places strictly positive mass on zero when s is finite. This implies that each bidder can receive a strictly positive (expected) surplus by bidding zero, which implies the following:

COROLLARY 2. *In the overlapping equilibrium presented in Proposition* 3, *each bidder receives strictly positive expected surplus if* $s < \infty$.

An important special case of the overlapping equilibrium involves symmetric values. Suppose that $v_1 = v_2 \equiv v$. Then, (7) reduces to $vs \leq k \leq vs + 1$, and the equilibrium is symmetric. We summarize the result.

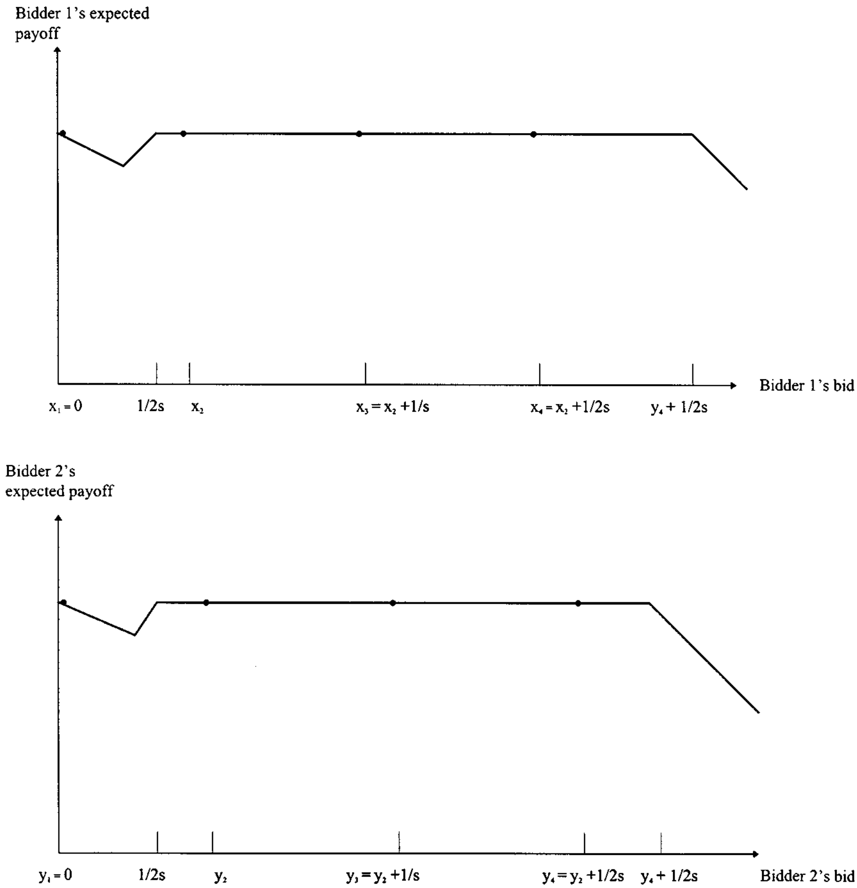

FIG. 2. Bidder 1 and 2's expected payoffs in equilibrium ($k = 4$).

COROLLARY 3. *Suppose that $v_1 = v_2 = v$ and $vs > 1$. Then, there exists an equilibrium in which each bidder places mass of $n_1, \ldots, n_k$ on $x_1, \ldots, x_k$, where*

$$n_1 = 1 - \frac{k-1}{vs}, \quad n_i = \frac{1}{vs}, \qquad \forall i = 2, \ldots, k,$$

and

$$x_1 = 0, \quad x_i = \frac{v}{2} - \frac{k+2-2i}{2s}, \quad \forall i = 2, \ldots, k.$$

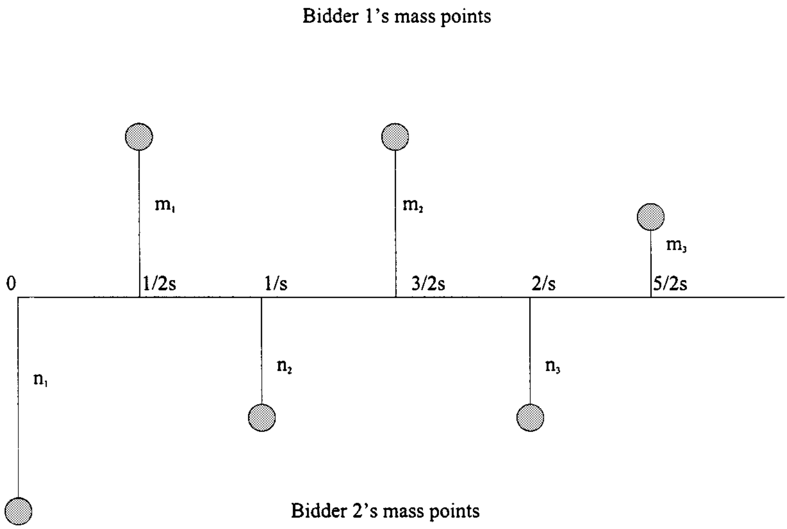

FIG. 3. Schematic representation of the staggered equilibrium (with $k = 3$).

To illustrate this equilibrium, suppose that $1 < vs < 2$. Each bidder places probability mass of $(vs - 1)/vs$ and $1/vs$ on 0 and $v/2$, respectively.

B. Staggered Equilibrium

The previous section concerned the case in which v_1 and v_2 were close to each other. We now treat the complementary case. As before, suppose that bidder 1 puts probability mass $m_1, \ldots, m_k$ on $x_1, \ldots, x_k$, and bidder 2 puts mass $n_1, \ldots, n_k$ on $y_1, \ldots, y_k$, for some integer, $k \geq 2$. Now, the mass points are staggered in the following way:

$$y_1 = 0, \quad x_1 = \frac{1}{2s}, \quad \text{and } x_i - x_{i-1} = y_i - y_{i-1} = \frac{1}{s}, \quad \text{for } i = 2, \ldots, k. \tag{8}$$

For a given bidder, the mass points are spaced $1/s$ units apart. This mixed-strategy profile is sketched in Fig. 3 for $k = 3$.

As before, let $i(x) \in \text{argmin}_j |x - y_j|$ for any $x \geq 0$, and let $j(y) \in \text{argmin}_i |y - x_i|$, for any $y \geq 0$. Then, the expected payoff to bidder 1 from bidding $x \geq 0$, when bidder 2 follows the suggested strategy, is

$$u_1(x) \equiv \begin{cases} v_1 \left\{ \sum_{j=1}^{i(x)-1} n_j + n_{i(x)} \left[\frac{1}{2} + s(x - y_{i(x)}) \right] \right\} - x \\ \quad \text{if } x \in \left[0, y_k + \frac{1}{2s}\right], \\ v_1 - x \qquad \text{if } x \geq y_k + \frac{1}{2s}. \end{cases}$$

Likewise, when bidder 1 follows the suggested strategy, bidder 2's expected payoff from bidding $y \geq 0$ is

$$u_2(y) \equiv \begin{cases} v_2 \left\{ \sum_{i=1}^{j(y)-1} m_i + m_{j(y)} \left[\frac{1}{2} + s(y - x_{j(y)}) \right] \right\} - y \\ \quad \text{if } y \in \left[0, x_k + \frac{1}{2s}\right], \\ v_2 - y \qquad \text{if } y \geq x_k + \frac{1}{2s}. \end{cases}$$

The first necessary condition for such an equilibrium to exist is that bidder 1 must attain local maxima at $x_1, \ldots, x_k$. For $x < x_1$, this requires

$$u_1'(x) = v_1 n_1 s - 1 \geq 0,$$

or

$$n_1 \geq \frac{1}{v_1 s}. \tag{9}$$

For any $x \in (x_1, x_k)$, a sufficient condition is

$$u_1'(x) = v_1 n_{i(x)} s - 1 = 0.$$

(Recall that $y_{i(x)}$ is bidder 2's closest mass point to x, and $x_k = y_k + 1/2s$.) The latter condition implies that

$$n_i = \frac{1}{v_1 s}, \tag{10}$$

for all $i = 2, \ldots, k$. Since bidder 2's mass must sum to 1, we have

$$n_1 = 1 - \frac{k-1}{v_1 s}. \tag{11}$$

Combining (9) and (11) yields

$$k \leq v_1 s \tag{12}$$

Bidder 2 must attain local maxima at $y_1, \ldots, y_k$. For $y < x_k - 1/2s = y_k$,

$$u_2'(y) = v_2 m_{j(y)} s - 1.$$

Clearly, $y_1, \ldots, y_k$ are local maxima if

$$m_i = \frac{1}{v_2 s}, \tag{13}$$

for $i = 1, \ldots, k-1$.

Bidder 2's expected payoff must not rise for $y \geq y_k = x_k - 1/2s$. Since $u_2'(y) = v_2 m_k s - 1$, for $y \in (y_k, y_k + 1/s)$, this implies that

$$m_k \leq \frac{1}{v_2 s}. \tag{14}$$

At the same time, bidder 1's mass must sum to one, so

$$m_k = 1 - \frac{k-1}{v_2 s}. \tag{15}$$

Combining (15) and (14), and using the fact that mass must be nonnegative, yields the necessary condition

$$v_2 s \leq k \leq v_2 s + 1. \tag{16}$$

Combining (16) and (12) gives

$$v_2 s \leq k \leq \min\{v_2 s + 1, v_1 s\}. \tag{17}$$

The equilibrium is characterized as follows.[7]

PROPOSITION 4. *If* $v_2 s \leq k \leq \min\{v_2 s + 1, v_1 s\}$, *for some integer* $k \geq 2$, *then there exists an equilibrium in which bidder* 1 *places mass* $m_1, \ldots, m_k$ *on* $x_1, \ldots, x_k$, *and bidder* 2 *places mass* $n_1, \ldots, n_k$ *on* $y_1, \ldots, y_k$, *where* m_i *satisfies* (13) *and* (15), n_i *satisfies* (11) *and* (10), *and* x_i *and* y_i *satisfy* (8).

Proof. Given $v_2 s \leq k \leq \min\{v_2 s + 1, v_1 s\}$, it follows by construction that $u_1(x)$ increases for $x \in [0, 1/2s]$, is constant for $x \in [1/2s, y_k + 1/2s]$ and declines for $x > y_k + 1/2s = x_k$. Likewise, $u_2(y)$ is constant for

[7] Unlike with the overlapping equilibrium, there exist no other *payoff-equivalent* equilibria here.

$y \in [0, x_k - 1/2s] = [0, y_k]$ and declines for $y > x_k - 1/2s$. It follows that the prescribed strategies are mutual best responses. Q.E.D.

At most one integer, $k \geq 2$, satisfies (17), generically. No solution exists if v_1 and v_2 are sufficiently close to each other. But, fixing $v_2 < v_1$, as s goes to infinity there must be an equilibrium of the staggered type. This holds since (17) amounts to $v_2 s \leq k \leq v_2 s + 1$ when s is sufficiently large.

To illustrate the equilibrium, suppose that $k = 2$ satisfies (17). In this case, bidder 1 puts mass of $1/v_2 s$ and $(v_2 - 1)/(v_2 s)$ on $1/(2s)$ and $3/(2s)$, respectively, and bidder 2 puts mass of $1/(v_1 s)$ and $(v_1 s - 1)/(v_1 s)$ on zero and $1/s$, respectively.

It is again instructive to study the expected surplus of each bidder.

COROLLARY 4. *In the staggered equilibrium presented in Proposition* 4, *bidder* 1 *receives strictly positive expected surplus and bidder* 2 *receives zero expected surplus*.

Proof. Bidder 2 puts mass on zero. Such a bid loses for sure, since the infimum of bidder 1's bids is $x_1 = 1/(2s)$. This implies that the equilibrium expected surplus for bidder 2 is zero. Since bidder 2 places mass on zero, bidder 1 must receive a strictly positive expected surplus. Q.E.D.

This result is commonly observed in contest models such as the all-pay auction and the modified war-of-attrition game (see Riley, 1998) in which the bidder with the higher valuation can profitably preempt this opponent. It is noteworthy that the same result holds here, even though preemption is more difficult.[8]

C. Existence of an Equilibrium

We now establish existence of equilibrium for the general case in which $v_2 > 1/s$.

PROPOSITION 5. *If* $v_2 s > 1$, *there exists an equilibrium of the overlapping or the staggered type, but* (*generically*) *not both*.

Proof. An overlapping equilibrium exists when $v_1 s \leq k \leq v_2 s + 1$ for some integer k, while a staggered equilibrium exists when $v_2 s \leq k \leq \min\{v_2 s + 1, v_1 s\}$ for some integer k. To see that one of the two types exists, note that

$$[v_1 s, v_2 s + 1] \cup [v_2 s, \min\{v_2 s + 1, v_1 s\}] = [v_2 s, v_2 s + 1],$$

[8] In the all-pay auction, the bidder with the higher valuation need only bid more than the lower valuation to guarantee a win. Here, by contrast, ensuring a win requires outbiding by at least $1/2s$. Note that this preemption result may hold here even when the bidders are almost symmetric, provided that $[v_2 s, v_1 s]$ contains an integer.

which contains an integer. We now show that there cannot be both types of equilibrium, generically. Note first that $[v_1 s, v_2 s + 1]$ is empty if $v_1 s > v_2 s + 1$, so suppose that $v_1 s \leq v_2 s + 1$. In that case, the intersection of the two regions is

$$[v_1 s, v_2 s + 1] \cap [v_2 s, v_1 s] = v_1 s.$$

Thus, both types of equilibrium can exist only if $k = v_1 s$ for some integer k, which is a nongeneric case. Q.E.D.

4. PROPERTIES OF THE EQUILIBRIUM

We can now study how the equilibrium varies as parameter values change. This will enable us to show that increases in the higher valuation cause total expected rent dissipation to fall, meaning that preemption is a general feature. The all-pay auction provides a useful benchmark. Total expected rent dissipation from the all-pay auction is $(1 + v_2/v_1)v_2/2$, which falls as the valuations become more asymmetric. As the high-valuation bidder's valuation rises, the low-valuation bidder becomes more pessimistic about his prospect of winning and becomes less aggressive, which enables the high-valuation bidder to win with a relatively lower bid (on average).

Assume that a staggered equilibrium exists, with $k \geq 2$ satisfying (17). Straightforward calculations yield the total expected rent dissipation:

$$R(k, v_1, v_2) \equiv \frac{k-1}{s} + \frac{1}{2s} - \frac{k(k-1)}{2s^2}\left(\frac{1}{v_2} - \frac{1}{v_1}\right).$$

Suppose that v_1 rises. Then, the original value of k still satisfies (17). The effect on bidder 2's strategy is that $n_2, \dots, n_k$ fall, and n_1 rises. This means that bidder 2's bids fall in the sense of first-order stochastic dominance. Similarly, if v_2 falls and the original k still satisfies (17), then $m_1, \dots, m_{k-1}$ rise, and m_k falls. This means that bidder 1's bids fall in the sense of first-order stochastic dominance. Finally, if v_2 falls sufficiently that the original k no longer satisfies (17), then rent dissipation falls discretely. To see this last point, consider the transition value, $v_2 = (k-1)s$, at which we go from a staggered equilibrium with k mass points to one with $k - 1$. The change in rent dissipation is

$$R\left(k-1, v_1, \frac{k-1}{s}\right) - R\left(k, v_1, \frac{k-1}{s}\right) = -\frac{k-1}{s^2 v_1} < 0.$$

These observations imply that total expected rent dissipation falls as v_1 rises and/or v_2 falls. Thus, preemption is a robust feature of the difference-form contest, not just a special feature of the all-pay auction. It is noteworthy that preemption occurs when $s < \infty$ even though the bidder with the higher valuation cannot guarantee a win as easily as when $s = \infty$.

The results are different for the overlapping equilibrium. Suppose that an integer $k \geq 2$ satisfies (7). As v_1 rises or v_2 falls, (7) becomes more onerous: either the same k continues to satisfy (7) or no integer satisfies it, in which case there is a staggered equilibrium.

As v_1 rises from $v_1 = v_2$, the equilibrium remains in the overlapping regime initially. Consider changes in the parameter values that are sufficiently small that the original k continues to satisfy (7). We know from the previous section that the expected payment from each bidder does not vary with v_1 or v_2, for fixed k. Hence, total expected rent dissipation is constant as valuations become more asymmetric, within the overlapping regime. As v_1 rises, $v_1 s$ will eventually reach k. At that point, the equilibrium shifts to the staggered regime.[9] Thereafter, the total expected payments go down as v_1 rises, as shown above. The results are now summarized.

PROPOSITION 6. *Total expected rent dissipation increases with v_2 and decreases with v_1 when the equilibrium is of the staggered type, but it is invariant to changes in valuations when the equilibrium is of the overlapping type.*

Using this result, we can bound the total expected rent dissipation.

PROPOSITION 7. *Total expected rent dissipation is (weakly) less than* $\max\{v_2, 1/2s\}$. *It is strictly less (generically) when* $v_2 > 1/2s$.

Proof. First, assume that $v_2 s < 1$. In any equilibrium, bidder 2 bids zero and bidder 1 bids a best response against zero. There are two cases. If $v_1 s < 1$, then bidder 1 also bids zero, so the result is trivially true. If $v_1 s \geq 1$, then bidder 1 bids at most $1/2s$, so the result again holds.

Now suppose that $v_2 s \geq 1$. We know that there exists an overlapping equilibrium or a staggered equilibrium. Since the stated result holds for the overlapping equilibrium (Corollary 1), it suffices to show the result for the staggered equilibrium. Fix a staggered equilibrium, and let $k \geq 2$ be an integer satisfying (17). The remainder of the proof consists of showing that

[9] Total expected rent dissipation may jump up when the regime shifts. To see this, note that we are considering $k = v_1 s \leq v_2 s + 1$. If we have $v_1 s = v_2 s + 1$, then expected rent dissipation is $(k - 1)/s$ in both regimes. An increase in $v_2 s$, all else equal, raises rent dissipation in the staggered regime while leaving it unchanged in the overlapping regime. Thus, if $v_1 s = k < v_2 s + 1$, rent dissipation jumps up when moving to the staggered regime.

the expected rent dissipation, $R(k, v_1, v_2)$ is strictly less than v_2, generically. Observe that

$$\frac{\partial R(k, v_1, v_2)}{\partial v_2} = \frac{k(k-1)}{2v_2^2 s^2} \leq \frac{(v_2 s + 1)(v_2 s)}{2v_2^2 s^2} = \frac{1}{2} + \frac{1}{2v_2 s} < 1.$$

The first inequality follows from (17). The second follows from $v_2 s \geq 1$, and it is strict for $v_2 s > 1$. Now let

$$\phi(k, v_1, v_2) \equiv R(k, v_1, v_2) - v_2.$$

By the above calculation, $\phi(k, v_1, v_2)$ is decreasing in v_2 in this region. From Proposition 6, $\phi(k, v_1, v_2)$ is strictly decreasing in v_1 in the same region. It follows that, for parameter values satisfying (17), $\phi(k, v_1, v_2)$ is maximized when $v_1 s = v_2 s + 1 = k$. We therefore have

$$\phi(k, v_1, v_2) \leq \phi(k, k/s, (k-1)/s) = 0,$$

in this region. The inequality is strict, generically. Q.E.D.

Since $v_2 \geq 1/s > 1/2s$ in the mixed-strategy equilibrium region, total expected rent dissipation is less than v_2 in that region.

5. ROBUSTNESS OF THE ALL-PAY AUCTION

We now study the limiting behavior as the sensitivity of the success function approaches $s = \infty$, which is the case of the all-pay auction. In particular, we show that the cumulative distribution function of equilibrium bids converges uniformly to that under the all-pay auction, which implies that the all-pay auction is robust to randomness in how the prize is allocated. In doing so, we must distinguish the cases of symmetric and asymmetric valuations: the overlapping equilibrium holds in the former case and the staggered equilibrium holds in the latter, for sufficiently large s. We begin with the symmetric case.

A. Symmetric Valuations

Let $v_1 = v_2 \equiv v$. There exists an integer, $k \geq 2$, that satisfies (7), provided that $vs \geq 1$. Then, an overlapping equilibrium of the form described in Proposition 3 exists. Fix any $s > 1/v$, and let $k(s)$ denote the corresponding value of k that satisfies (7). The cumulative distribution function of equilibrium bids, $F_s(x) \equiv \text{Prob}\{i\text{'s bid is no greater than } x\}$, is given by

$$F_s(x) = \sum_{j=1}^{i} m_j \text{ if } x \in [x_i, x_{i+1}),$$

where $x_{k(s)+1} \equiv \infty$.

In the all-pay auction, the cumulative distribution function of equilibrium bids is

$$F(x) = \min\{x/v, 1\}$$

for $x \geq 0$ (Hillman and Riley, 1989; Baye *et al.*, 1993).

The following proposition establishes the convergence result.

PROPOSITION 8. *$F_s(\cdot)$ converges uniformly to $F(\cdot)$ as s goes to ∞.*

Proof. Fix $s < \infty$. Let

$$\| F_s(\cdot) - F(\cdot) \| \equiv \sup_x | F_s(x) - F(x) |.$$

Since $F_s(\cdot)$ is a step function, $|F_s(\cdot) - F(\cdot)|$ attains its supremum at a point where $F_s(\cdot)$ jumps. Thus,

$$\| F_s(\cdot) - F(\cdot) \| = \max_{i=1,\ldots,k(s)} \left\{ \max\left\{ | F_s(x_i) - F(x_i) |, | F_s(x_i^-) - F(x_i) | \right\} \right\},$$

where $F_s(x_i^-)$ denotes the left-hand limit of $F_s(\cdot)$ at x_i. We show that the right-hand side of the above equation approaches zero as s tends to ∞.

Observe that

$$F_s(x) - F(x) = \begin{cases} 1 - \dfrac{k(s) - 1}{vs} & \text{if } x = x_1, \\ \dfrac{1}{2} - \dfrac{k(s) - 2}{2vs} & \text{if } x = x_i, \quad \forall i = 2, \ldots, k(s), \\ \dfrac{1}{2} - \dfrac{k(s) - 2}{2vs} - \dfrac{1}{vs} & \text{if } x = x_i^-, \quad \forall i = 2, \ldots, k(s). \end{cases}$$

Condition (7) and the assumption that $v_1 = v_2 = v$ yield

$$v \leq \frac{k(s)}{s} \leq v + \frac{1}{s},$$

so $\lim_{s \to \infty} k(s)/s = v$. This, in turn, implies that the three expressions for $F_s(x) - F(x)$ all go to zero as s approaches ∞, so

$$\lim_{s \to \infty} \| F_s(\cdot) - F(\cdot) \| = 0.$$

Thus, $F_s(\cdot)$ converges uniformly to $F(\cdot)$. Q.E.D.

Uniform convergence also implies *convergence in the mean* (see de Barra, 1981, p. 126), which gives the following result.

COROLLARY 5. *Given symmetric valuations, total expected rent dissipation converges to v, the level under the all-pay auction, as s approaches ∞.*

B. Asymmetric Valuations

Assume that $v_1 > v_2$. For sufficiently large s, there exists an integer, $k \geq 2$, that satisfies (17). Then, a staggered equilibrium of the form described in Proposition 4 exists. Fixing s, let $k(s)$ denote the solution to (17). The cumulative distribution function of the equilibrium bids for bidder 1 is

$$G_s(x) \equiv \sum_{j=1}^{i} m_j \text{ if } x \in [x_i, x_{i+1}),$$

where $x_{k(s)+1} \equiv \infty$, and $G_s(x) = 0$ for $x \in [0, x_1)$. For bidder 2, it is

$$H_s(y) \equiv \sum_{j=1}^{i} n_j \text{ if } y \in [y_i, y_{i+1}),$$

where $y_{k(s)+1} \equiv \infty$.

In an all-pay auction, the cumulative distribution functions are: $G(x) = \min\{x/v_2, 1\}$ and $H(y) = \min\{1 - (v_2/v_1) + (y/v_1), 1\}$. We now establish uniform convergence of $G_s(\cdot)$ and $H_s(\cdot)$ to $G(\cdot)$ and $H(\cdot)$, respectively.

PROPOSITION 9. *$G_s(\cdot)$ and $H_s(\cdot)$ converge uniformly to $G(\cdot)$ and $H(\cdot)$, respectively, as s goes to ∞.*

Proof. It suffices to show that $|G_s(x) - G(x)|$ and $|H_s(y) - H(y)|$ converge to zero at all points where $G_s(\cdot)$ and $H_s(\cdot)$ jump, respectively. (Let $x_1^- \equiv 0$). For bidder 1, we do not check $|G_s(x) - G(x)|$ for $x = x_{k(s)}$ since the last jump is smaller than the rest, making the corresponding gaps smaller in absolute value.[10] At other jumps, the gap is

$$G_s(x) - G(x) = \begin{cases} \dfrac{1}{2v_2 s} & \text{if } x = x_i, \forall i = 1, \dots, k(s) - 1, \\[2ex] \dfrac{-1}{2v_2 s} & \text{if } x = x_i^-, \forall i = 1, \dots, k(s) - 1, \end{cases}$$

[10] Moreover, the last mass point for bidder 1 may exceed v_2, which is the supremum in the all-pay auction, again making the jump smaller.

and

$$H_s(y) - H(y)$$
$$= \begin{cases} -\dfrac{k(s)-1}{v_1 s} + \dfrac{v_2}{v_1} & \text{if } y = y_i, \forall i = 1, \ldots, k(s), \\ -\dfrac{k(s)-1}{v_1 s} + \dfrac{v_2}{v_1} - \dfrac{1}{v_1 s} & \text{if } y = y_i^-, \forall i = 1, \ldots, k(s). \end{cases}$$

The two expressions for $G_s(x) - G(x)$ clearly approach zero as s goes to ∞. From (17), we have

$$v_2 \leq \frac{k(s)}{s} \leq \min\left\{ v_2 + \frac{1}{s}, v_1 \right\}.$$

Hence, $\lim_{s \to \infty} k(s)/s = v_2$, so the two expressions for $H_s(y) - H(y)$ approach zero. This proves the proposition. Q.E.D.

Again, uniform convergence implies convergence in the mean.

COROLLARY 6. *Given asymmetric valuations, the expected payment by each bidder approaches that under the all-pay auction, as s approaches* ∞.

6. CONCLUSION

We have analyzed contests in which a contestant's probability of winning the prize depends on the difference between his effort and his rival's. Our model has advantages over the previous work in this area. First, it allows different levels of sensitivity of the success function to contestants' efforts, with the lottery and all-pay auction as special cases, and it permits characterization of equilibrium for all possible values of the sensitivity parameter. Previous work on difference-form contests and all-pay auctions largely treated extreme cases in which the outcome of the contest was infinitely sensitive or relatively insensitive to effort levels, or at least one agent had a low valuation of the prize. This limited scope had left unclear whether the earlier findings were a direct result of parameter restrictions. Our complete characterization fills a gap in the understanding of contests.

Second, our model allows us to check whether properties of contest games such as the all-pay auction are robust. We demonstrated the robustness of the limiting cases by showing that the equilibrium of our difference-form contest converges to that of the all-pay auction, as s approaches infinity.[11]

[11] It is trivial that the equilibrium approaches that of the lottery as s approaches zero, since the bidders both bid zero in that case.

Finally, we showed that the main qualitative features of the equilibrium are very general. In particular, the amount of rent dissipation falls as the gap between the bidders' valuations increases, over a large parameter region. This result buttresses the assessment that asymmetries among bidders are key to explaining why real-world contests appear to display significant underdissipation (see Riley, 1998). While previous work had shown that preemption occurs in the all-pay auction (Hillman and Riley, 1989; Baye *et al.*, 1993), our results indicate that such "preemption-induced underdissipation" is a general feature of difference-form contests, not just all-pay auctions.

Our model is a step toward building a general theory of contests, but more work is needed in two directions. First, we have focused on a particular class of difference-form success functions. It is important to know whether the main results extend to more general classes of difference-form success functions. Second, it is important to know whether there are other equilibria, and, if so, whether the main results of this paper are preserved.

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