Supplementary Note

for "Assigning Resources to Budget-Constrained Agents"

1 Proof of Proposition 2: Omitted Parts

Throughout the proof, we assume that τ satisfies Property (M); i.e. there is a positive mass of agents who choose from $(\underline{x}, x_\varepsilon)$ under τ. Note that Property (M) implies $\tau(x_\varepsilon) > 0$ since otherwise almost every agent chooses either $\underline{x}$ or some $x \geq \underline{x}$.

Let f and g denote the density for F and G, respectively. Since f and g are bounded by assumption, we let $M_f := \sup_{v \in [0,1]} f(v)$ and $M_g := \sup_{w \in [0,1]} g(w)$. We will repeatedly use the following fact: for any v and $v' > v$, $F(v') - F(v) \leq M_f(v' - v)$; and likewise for G.

To first analyze the properties of τ^s in Lemma S.1-S.3 (below), let us introduce a few notations. For each $s \in [v_\varepsilon, 1]$, let $\lambda^s(x) := \tau(\underline{x}) - \varepsilon + s(x - \underline{x})$. Then, $\tau^s(x) := \min\{\tau(x), \lambda^s(x)\}$ for $x \in [\underline{x}, x_\varepsilon)$ and $\tau^s(x) := \tau(x)$ for $x \in [x_\varepsilon, 1] \cap X$. Recall that $x(w, v)$ and $x^s(w, v)$ denote the incentive compatible choices associated with τ and τ^s for each type (w, v), respectively. We choose x^s such that $x^s(w, v) = x(w, v)$ if the contract $x(w, v)$ continues to be optimal for (w, v) under τ^s. Define $X^s_\lambda := \{x \in (\underline{x}, x_\varepsilon) : \tau^s(x) = \lambda^s(x)\}$ and $X^s_\tau := \{x \in (\underline{x}, x_\varepsilon) \cap X : \tau^s(x) = \tau(x) < \lambda^s(x)\}$.

First of all, the agent types with either $v \leq v_\epsilon$ or $v > v_\varepsilon$ and $w \geq \frac{\tau(x_\varepsilon)}{x_\varepsilon}$ do not change their consumption as s changes. (Given any τ^s, the former types always choose $\underline{x}$ while the latter types choose $x(w, v) \geq x_\varepsilon$.) Next, we study how the agent types with $v > v_\varepsilon$ and $w < \frac{\tau(x_\varepsilon)}{x_\varepsilon}$ change their consumption behavior. Let $x^s_w \in X^s_\lambda$ for each $w \in [0, \frac{\tau(x_\varepsilon)}{x_\varepsilon}]$ be such that $\lambda^s(x^s_w) = wx^s_w$.[1] Then, we obtain:

LEMMA S.1. *Fix any $s \in [v_\varepsilon, 1]$ and fix any $w \in [0, \frac{\tau(x_\varepsilon)}{x_\varepsilon}]$. Then,*

1. for all $v \in (v_\varepsilon, s)$, $x^s(w, v)$ is equal to either $\underline{x}$ or $x(w, v)$ so $x^s(w, v) \leq x(w, v)$;

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[1] Note that x^s_w is well defined since we assume that $\tau(x_\varepsilon) > 0$.

2. *for all $v > s$, if $x^s(w, v) \neq x(w, v)$, then $x^s(w, v) = x_w^s > x(w, v)$;*

3. *for all $v > v_\varepsilon$, both $x^s(w, v)$ and $\tau^s(x^s(w, v))$ increase weakly as s decreases.*

Proof. Part 1 follows immediately from observing that all agent types with $v \in (v_\varepsilon, s)$ prefers $\underline{x}$ to any $x \in X_\lambda^s$, so they either choose $\underline{x}$ or keep the same level as under τ.

To prove part 2, note that if any agent type with $v > s$ chooses $x^s(w, v) \neq x(w, v)$ under τ^s, then we must have $x^s(w, v) \in X_\lambda^s$. Also such a type must exhaust her budget by choosing $x^s(w, v) = x_w^s$. Next, to show that $x^s(w, v) > x(w, v)$, suppose this is not true. Then, since $x^s(w, v) \neq x(w, v)$, we must have $x(w, v) > x_w^s$. Hence, we obtain

$$vx^s(w, v) - \tau^s(x^s(w, v)) = x_s^w(v - \frac{\tau^s(x_w^s)}{x_w^s}) < x(w, v)(v - w) \leq x(w, v)v - \tau(w, v),$$

where the first equality holds since $x^s(w, v) = x_w^s$, and the strict inequality holds since $v > s \geq v_\epsilon = \frac{\tau(x_\varepsilon) - \tau(\underline{x}) + \varepsilon}{x_\varepsilon - \underline{x}} > \frac{\tau(x_\varepsilon)}{x_\varepsilon} \geq w$.[2] This is a contradiction since $(x(w, v), \tau(w, v))$ is available under τ^s.

To prove part 3, consider $s_1, s_2 \in [v_\varepsilon, 1]$ with $s_1 > s_2$ and denote $x^i(w, v) = x^{s_i}(w, v)$ and $\tau^i(x) = \tau^{s_i}(x)$ for simplicity. Also, denote $\lambda^i(x)$, X_λ^i, X_τ^i, and x_w^i similarly. Suppose $x^1(w, v) > \underline{x}$ since otherwise the result is immediate. If $x^1(w, v) \in X_\lambda^1$, then it is straightforward to see that $x^1(w, v) = x_w^1 < x_w^2 = x^2(w, v)$,[3] which implies that $\tau^2(x^2(w, v))) = wx_w^2 > wx_w^1 = \tau^1(x^1(w, v))$, as desired. If $x^1(w, v) \in X_\tau^1$, we have either $x^2(w, v) = x^1(w, v) \in X_\tau^2$ or $x^2(w, v) = x_w^2 \neq x^1(w, v)$. In the latter case, the same proof that was used to prove part 2 above can be used to show $x_w^2 > x^1(w, v)$. Also, in that case, we have $\tau^2(x^2(w, v)) = wx_w^2 > wx^1(w, v) \geq \tau^1(x^1(w, v))$, as desired. ∎

LEMMA S.2. *There exists $\hat{\varepsilon} > 0$ such that, for any $\varepsilon \in [0, \hat{\varepsilon})$, τ^s does not run a budget deficit for any $s \in [v_\varepsilon, 1]$.*

Proof. Note that the total revenue under τ^s rises when s falls, given part 3 of Lemma S.1 and the fact that the agent types with either $v \leq v_\epsilon$ or $v > v_\varepsilon$ and $w \geq \frac{\tau(x_\varepsilon)}{x_\varepsilon}$ do not change their consumption, and thus their spending, as s changes. So it suffices to show that τ^s does not run a deficit when $s = 1$. Assume $s = 1$ from now on.

First, it is clear that no agent chooses $x \in X_\lambda^s$ since $v \leq 1 = s$ with probability 1. Also, for any agent type (w, v) with $x^s(w, v) \neq x(w, v)$, we must have $x^s(w, v) = \underline{x} < x(w, v)$, and

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[2]The strict inequality in turn follows since $\tau(x_\varepsilon) > 0 > \tau(\underline{x}) - \varepsilon$.

[3]That $x^2(w, v) = x_w^2$ follows directly from a few simple observations: x_w^2 is strictly better for type (w, v) than x_w^1 and all other x in X_λ^2; $x^1(w, v) = x_w^1$ is (weakly) preferred to any $x \in X_\tau^1 \cup \{\underline{x}\}$; and $X_\tau^2 \subset X_\tau^1$. Also $x_w^1 < x_w^2$ since x_w^s is decreasing in s, as can be easily checked.

this is the only source of decrease in the total revenue. We show that this decrease can be made arbitrarily small by making ε sufficiently small. To do so, we fix any $x \in (\underline{x}, 1) \cap X$ and obtain the measure of valuation types who may decrease their consumption from x to $\underline{x}$. The valuation v of any agent dosing so must satisfy two conditions: (1) she prefers $(x, \tau(x))$ to $(\underline{x}, \tau(\underline{x}))$; (2) she prefers $(\underline{x}, \tau(\underline{x}) - \varepsilon)$ to $(x, \tau(x))$. Then, (1) and (2) require $v \in [\frac{\tau(x) - \tau(\underline{x})}{x - \underline{x}}, \frac{\tau(x) - \tau(\underline{x}) + \varepsilon}{x - \underline{x}}]$. The mass of valuation types in this interval, denoted m_x, is bounded by $F(\frac{\tau(x) - \tau(\underline{x}) + \varepsilon}{x - \underline{x}}) - F(\frac{\tau(x) - \tau(\underline{x})}{x - \underline{x}}) \leq M_f(\frac{\tau(x) - \tau(\underline{x}) + \varepsilon}{x - \underline{x}} - \frac{\tau(x) - \tau(\underline{x})}{x - \underline{x}}) = \frac{M_f \varepsilon}{x - \underline{x}}$. Using this, we can bound the decrease in the total revenue as follow:

$$
\begin{aligned}
\mathbb{E}[(\tau(x(\tilde{w}, \tilde{v})) - \tau^s(x^s(\tilde{w}, \tilde{v}))1_{\{x^s(\tilde{w}, \tilde{v}) = \underline{x} < x(\tilde{w}, \tilde{v})\}}] &= \mathbb{E}[(\tau(x(\tilde{w}, \tilde{v})) - \tau(\underline{x}) + \varepsilon)1_{\{x^s(\tilde{w}, \tilde{v}) = \underline{x} < x(\tilde{w}, \tilde{v})\}}] \\
&\leq \int_{\underline{x}}^1 (\tau(x) - \tau(\underline{x}))m_x dx + \varepsilon \\
&\leq \int_{\underline{x}}^1 \left(\frac{\tau(x) - \tau(\underline{x})}{x - \underline{x}} \right) M_f \varepsilon dx + \varepsilon \\
&\leq M_f \varepsilon + \varepsilon = (M_f + 1)\varepsilon,
\end{aligned}
$$

where the third inequality follows from the fact that $\frac{\tau(x) - \tau(\underline{x})}{x - \underline{x}} \leq 1$ for any $x \in (\underline{x}, x_\varepsilon) \cap X$ and $1 - \underline{x} \leq 1$. Hence, by making ε sufficiently small, τ^s will not run a budget deficit, given that τ generates a budget surplus. ∎

LEMMA S.3. *For $s \in (v_\varepsilon, 1]$, $\mathbb{E}[x^s(w, v)]$ is increasing continuously as s decreases.*

Proof. Consider any $s_1, s_2 \in (v_\varepsilon, 1)$ with $s_2 < s_1$. We continue to use all the notations from the proof of Lemma S.1. Note that all agent types consume weakly more under τ^2 than under τ^1. Then, the agent types who strictly increase their consumption can be divided into two sets: the set of agent type (w, v), denoted T_a, who chooses $\underline{x}$ under τ^1 and some $x^2(w, v) \in (\underline{x}, x_\varepsilon)$ under τ^2; the set of agent type (w, v), denoted T_b, who chooses some $x^1(w, v) > \underline{x}$ under τ^1 and $x^2(w, v) \in (x^1(w, v), x_\varepsilon)$ under τ^2. In the subsequent proof, we fix a wealth type at any $w \in [0, \frac{\tau(x_\varepsilon)}{x_\varepsilon}]$ and show that for each $i = a, b$, the increase in the total consumption by the agent types in T_i with wealth w can be made arbitrarily small by making $s_1 - s_2$ sufficiently small.

To this end, we let $\Delta x_w := x_w^2 - x_w^1$. It is straightforward to see that Δx_w can be made arbitrarily small (uniformly) for all $w \in [0, \frac{\tau(x_\varepsilon)}{x_\varepsilon}]$ by making $s_1 - s_2$ sufficiently small. Throughout the proof, we will repeatedly use the following observation (without explicitly mentioning), which results from combining parts 1 and 2 of Lemma S.1: any agent type (w, v) with $v > v_\varepsilon$ chooses among $\underline{x}$, $x(w, v)$, and x_w^s.

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Consider first any agent type in T_a. We argue that such an agent must have $v \in [s_2, s_1]$. Clearly, any agent (w, v) with $v > s_1$ would not choose $\underline{x}$ under τ^1 since x_w^1 is better. Suppose for a contradiction that some agent type (w, v) with $v < s_2$ chooses $\underline{x}$ under τ^1 and some $x > \underline{x}$ under τ^2. First, $x^2(w, v)$ can equal neither x_w^2 (since an agent with $v < s_2$ strictly prefers $\underline{x}$ to x_w^2) nor $\underline{x}$ (since this would contradict (w, v) being in T_a). It follows that $x^2(w, v) = x(w, v)$. This in turn implies that the agent type under consideration likes $x(w, v)$ at least as much as $\underline{x}$, which means that both $\underline{x}$ and $x(w, v)$ are optimal under τ^1. Then, $\underline{x}$ cannot be chosen under τ^1 due to our assumption that $x^1(w, v) = x(w, v)$ if $x(w, v)$ continues to be optimal for (w, v) under τ^1. Consequently, the mass of agents in T_a is bounded by $F(s_1) - F(s_2) \le M_f(s_1 - s_2)$. Then, the increase in their total consumption is bounded by $M_f(s_1 - s_2)(x_\varepsilon - \underline{x})$, which can be made arbitrarily small by making $(s_1 - s_2)$ sufficiently small.

Turning to the agent types in T_b, note that any agent type $(w, v) \in T_b$ must have $x^2(w, v) \le x_w^2$. If $x^2(w, v) > x_w^2$, then we must have $x^2(w, v) = x(w, v)$, which implies that $x^1(w, v) = x(w, v) = x^2(w, v)$, yielding a contradiction. We now divide T_b into two subsets: $T_b^+ := \{(w, v) \in T_b : x^1(w, v) \ge 2x_w^1 - x_w^2\}$ and $T_b^- := T_b \backslash T_b^+$. Given that $x^2(w, v) \le x_w^2$ for all $(w, v) \in T_b$, the increase in the consumption by the agent types in T_b^+ is bounded by $x_w^2 - (2x_w^1 - x_w^2) = 2\Delta x_w$ multiplied by their mass, which can be made arbitrarily small by making $s_1 - s_2$, and thus Δx_w, sufficiently small.

Turning to T_b^-, we fix w and observe that $x^1(w, v) \in (\underline{x}, x_w^1)$ for any $(w, v) \in T_b^-$, which implies $x^1(w, v) = x(w, v)$. Since $x(w, v) = x^1(w, v) < x^2(w, v)$ (by assumption), we must have $x^2(w, v) = x_w^2$. Thus, letting $x = x(w, v)$, any agent type $(w, v) \in T_b^-$ must prefer $(x, \tau(x))$ to $(x_w^1, \lambda^1(x_w^1))$ and prefer $(x_w^2, \lambda^2(x_w^2))$ to $(x, \tau(x))$. This requires that for any $(w, v) \in T_b^-$, we have $v \in [\frac{\lambda^2(x_w^2) - \tau(x)}{x_w^2 - x}, \frac{\lambda^1(x_w^1) - \tau(x)}{x_w^1 - x}]$. The mass of valuation types in this interval, denoted $\bar{m}_x$, is bounded by

$$
\begin{aligned}
F(\frac{\lambda^1(x_w^1) - \tau(x)}{x_w^1 - x}) - F(\frac{\lambda^2(x_w^2) - \tau(x)}{x_w^2 - x}) &\le M_f \left(\frac{\lambda^1(x_w^1) - \tau(x)}{x_w^1 - x} - \frac{\lambda^2(x_w^2) - \tau(x)}{x_w^2 - x} \right) \\
&\le M_f \left(\frac{\lambda^2(x_w^2) - \tau(x)}{x_w^1 - x} - \frac{\lambda^2(x_w^2) - \tau(x)}{x_w^2 - x} \right) \\
&= M_f \left(\frac{\lambda^2(x_w^2) - \tau(x)}{x_w^2 - x} \right) \left(\frac{x_w^2 - x_w^1}{x_w^1 - x} \right) \le \frac{M_f \Delta x_w}{x_w^1 - x},
\end{aligned}
$$

where the last inequality follows from the fact that $\frac{\lambda^2(x_w^2) - \tau(x)}{x_w^2 - x} \le 1$.[4] Using this and the fact that $\frac{x_w^2 - x}{x_w^1 - x} \le 2$ if $x \le 2x_w^1 - x_w^2 (< x_w^1)$, we can bound the increase of total consumption by

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[4]Otherwise, no agent with any $v \in [v_\varepsilon, 1]$ would choose x_w^2 over x.

the agent types in T_b^- as in

$$\mathbb{E}\big[(x^2(\tilde{w}, \tilde{v}) - x^1(\tilde{w}, \tilde{v}))1_{\{(\tilde{w}, \tilde{v}) \in T_b^-\}}\big|\tilde{w} = w\big] \leq \int_{\underline{x}}^{2x_w^1 - x_w^2} \bar{m}_x(x_w^2 - x)dx$$

$$\leq \int_{\underline{x}}^{2x_w^1 - x_w^2} M_f \Delta x_w \left(\frac{x_w^2 - x}{x_w^1 - x}\right) dx \leq 2M_f \Delta x_w,$$

which can be made arbitrarily small by making $s_1 - s_2$, and thus Δx_w, sufficiently small. ∎

Lemmas S,4-S.6 below characterize the properties of τ_δ used in Step 2 in the proof of Proposition 2. We choose x_δ to satisfy the following property, which we call Property (C): if $x_\delta(w, v) > x(w, v)$, then the type (w, v) prefers $x_\delta(w, v)$ strictly to any feasible $x < x_\delta(w, v)$.

LEMMA S.4. *If ε is sufficiently small, then τ_δ does not run a budget deficit for any $\delta \in (0, \varepsilon]$.*

Proof. Let $d := \mathbb{E}[\tau(x(w, v))] > 0$ denote the total budget collected under τ. Recall the tariff function $\bar{\tau}^\sigma$ for $\sigma \in (0, \varepsilon]$: $\bar{\tau}^\sigma(x) = \tau(x)$ for all $x \in X \setminus \{\underline{x}\}$; $\bar{\tau}^\sigma(\underline{x}) = \tau(\underline{x}) - \sigma$. Let $\bar{x}^\sigma$ denote the associated incentive compatible choice. Compared to τ, this mechanism only induces some agent types to switch to $\underline{x}$ and therefore generates a lower total revenue. However, one can use the same argument as in the proof of Lemma S.2 to show that the decrease in the total revenue can be made arbitrarily small for any $\sigma \in (0, \varepsilon]$ if ε is sufficiently small. Now set ε small enough to satisfy that for any $\sigma \in (0, \varepsilon]$,

$$\mathbb{E}[\tau(x(w, v)) - \bar{\tau}^\sigma(\bar{x}^\sigma(w, v))] < d/2. \tag{1}$$

Consider another mechanism $\bar{\tau}_\sigma$ defined as follows: $\bar{\tau}_\sigma(x) = \tau(\underline{x}) - \sigma + v_\sigma(x - \underline{x})$ for $x \in [\underline{x}, x_\sigma)$; and $\bar{\tau}_\sigma(x) = \tau(x)$ for $x \in [x_\sigma, 1] \cap X$. Let $\bar{x}_\sigma$ denote the associated incentive compatible choice. This mechanism generates no lower total revenue than $\bar{\tau}^\sigma$ does for the following reasons: no agent types consume less under $\bar{\tau}_\sigma$; all agent types who choose from the interval $[\underline{x}, x_\sigma)$ under $\bar{\tau}^\sigma$ (i.e. those with $v > v_\sigma$ and $w < \frac{\tau(x_\sigma)}{x_\sigma}$) increase their consumption (at least weakly) under $\bar{\tau}_\sigma$ to the point of exhausting their budgets; all other agent types whose consumption remains the same between $\bar{\tau}^\sigma$ and $\bar{\tau}_\sigma$ pay the same amount. Thus, using (1), we have for any $\sigma \leq \varepsilon$

$$\mathbb{E}[\bar{\tau}_\sigma(\bar{x}_\sigma(w, v))] \geq \mathbb{E}[\bar{\tau}^\sigma(\bar{x}^\sigma(w, v))] > \mathbb{E}[\tau(x(w, v))] - d/2. \tag{2}$$

Recall

$$\tau_\delta(x) = \begin{cases} \tau(\underline{x}) - \varepsilon + v_{\varepsilon-\delta}(x - \underline{x}) & \text{for } x \in [\underline{x}, x_{\varepsilon-\delta}) \\ \tau(x) - \delta & \text{for } x \in [x_{\varepsilon-\delta}, 1] \cap X. \end{cases}$$

Observe that $\tau_\delta(x) = \overline{\tau}_{\varepsilon-\delta}(x) - \delta$ for all $x \in [\underline{x}, x_{\varepsilon-\delta}) \cup X$. Since all agent types consume weakly more under τ_δ than under $\overline{\tau}_{\varepsilon-\delta}$, we must have $\tau_\delta(x_\delta(w,v)) = \overline{\tau}_{\varepsilon-\delta}(x_\delta(w,v)) - \delta \geq \overline{\tau}_{\varepsilon-\delta}(\overline{x}_{\varepsilon-\delta}(w,v)) - \varepsilon$ for all $\delta \in [0, \varepsilon]$ and $(w,v) \in [0,1]^2$. Then, the desired conclusion follows from choosing ε to be the smaller of $d/2$ and the value satisfying (1) since

$$\mathbb{E}[\tau_\delta(x_\delta(w,v))] \geq \mathbb{E}[\overline{\tau}_{\varepsilon-\delta}(\overline{x}_{\varepsilon-\delta}(w,v))] - \varepsilon > \mathbb{E}[\tau(x(w,v))] - 2/d - \varepsilon \geq \mathbb{E}[\tau(x(w,v))] - d = 0,$$

where the second inequality follows from (2). ∎

To establish the continuity of $\mathbb{E}[x_\delta(w,v)]$ with respect to δ, we need the following preliminary result:

LEMMA S.5. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any mechanism $\tilde{\tau}$ and associated incentive compatible choices $\tilde{x}$, there exist incentive compatible choices $\tilde{x}^\delta$ associated with $\tilde{\tau}^\delta(x) := \tau(x) - \delta$ such that $\mathbb{E}[\tilde{x}^\delta(w,v) - \tilde{x}(w,v)] < \varepsilon$.*

Proof. Fix any $\delta > 0$. Then for $\tilde{\tau}^\delta$, there exists an incentive compatible choice $\tilde{x}^\delta$ such that: $\tilde{x}^\delta(w,v) \geq \tilde{x}(w,v)$ for all (w,v); if $\tilde{x}^\delta(w,v) > \tilde{x}(w,v)$, then the type (w,v) prefers $\tilde{x}^\delta(w,v)$ strictly to any feasible $x < \tilde{x}^\delta(w,v)$; and $\tilde{x}^\delta(\cdot, v)$ is nondecreasing for all v. Fix any such $\tilde{x}^\delta$. Fix a $v \in [0,1]$. Define

$$\tilde{x}_1^\delta(v) := \inf\{\tilde{x}^\delta(w,v) | \tilde{x}^\delta(w,v) - \tilde{x}(w,v) \geq \sqrt{\delta}\}.$$

Let

$$W_1(v) := \{w | w < \frac{\tilde{\tau}(\tilde{x}_1^\delta(v))}{\tilde{x}_1^\delta(v)} \text{ and } \tilde{x}^\delta(w,v) - \tilde{x}(w,v) \geq \sqrt{\delta}\}.$$

We show that $W_1(v) \subset \tilde{W}_1(v) := \{w | \frac{\tilde{\tau}(\tilde{x}_1^\delta(v)) - \delta}{\tilde{x}_1^\delta(v)} \leq w < \frac{\tilde{\tau}(\tilde{x}_1^\delta(v))}{\tilde{x}_1^\delta(v)}\}$. Suppose to the contrary that $w \in W_1(v)$ but $w < \frac{\tilde{\tau}(\tilde{x}_1^\delta(v)) - \delta}{\tilde{x}_1^\delta(v)}$. Then, $\frac{\tilde{\tau}(\tilde{x}^\delta(w,v)) - \delta}{\tilde{x}^\delta(w,v)} \leq w < \frac{\tilde{\tau}(\tilde{x}_1^\delta(v)) - \delta}{\tilde{x}_1^\delta(v)}$. Since $w \in W_1(v)$, $\tilde{x}^\delta(w,v) \geq \tilde{x}_1^\delta(v)$, by definition of $\tilde{x}_1^\delta(v)$. But since $\tilde{x}^\delta(w,v) \neq \tilde{x}_1^\delta(v)$, we must have $\tilde{x}^\delta(w,v) > \tilde{x}_1^\delta(v)$. It follows that any agent who can choose $\tilde{x}_1^\delta(v)$ will strictly prefer $\tilde{x}^\delta(w,v)$ under $\tilde{\tau}^\delta$, but this contradicts the definition of $\tilde{x}_1^\delta(v)$ as the limit point of choices under $\tilde{\tau}^\delta$.[5]

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[5]Since X is a closed, $\tilde{x}_1^\delta(v) \in X$. Recall that $\tilde{\tau}(x)$ (and therefore $\tilde{\tau}^\delta(x)$) is lower semi-continuous in x for $x \in X$. Therefore, if $\tilde{x}_1^\delta(v)$ is strictly worse and requires strictly higher budget than $\tilde{x}^\delta(w,v)$, then all quantities sufficiently close to $\tilde{x}_1^\delta(v)$ are equally suboptimal and will never be chosen as well.

For $i \geq 2$, define recursively

$$\tilde{x}_i^\delta(v) := \inf\{\tilde{x}^\delta(w,v) | w \geq \frac{\tilde{\tau}(\tilde{x}_{i-1}^\delta(v))}{\tilde{x}_{i-1}^\delta(v)} \text{ and } \tilde{x}^\delta(w,v) - \tilde{x}(w,v) \geq \sqrt{\delta}\}.$$

Let

$$W_i(v) := \{w | \frac{\tilde{\tau}(\tilde{x}_{i-1}^\delta(v))}{\tilde{x}_{i-1}^\delta(v)} \leq w < \frac{\tilde{\tau}(\tilde{x}_i^\delta(v))}{\tilde{x}_i^\delta(v)} \text{ and } \tilde{x}^\delta(w,v) - \tilde{x}(w,v) \geq \sqrt{\delta}\}.$$

The same argument as above establishes

$$W_i(v) \subset \tilde{W}_i := \{w | \frac{\tilde{\tau}(\tilde{x}_i^\delta(v)) - \delta}{\tilde{x}_i^\delta(v)} \leq w < \frac{\tilde{\tau}(\tilde{x}_i^\delta(v))}{\tilde{x}_i^\delta(v)}\}. \tag{3}$$

Fix any $w \in W_i(v)$ and v such that $\tilde{x}^\delta(w,v) - \tilde{x}(w,v) \geq \sqrt{\delta}$. Then, we must have $\tilde{x}(w,v) \geq \tilde{x}_{i-1}^\delta(v)$. To see this, suppose to the contrary that $\tilde{x}(w,v) < \tilde{x}_{i-1}^\delta(v)$. Recall that there exists a budget type $w' < \frac{\tilde{\tau}(\tilde{x}_{i-1}^\delta(v))}{\tilde{x}_{i-1}^\delta(v)}$ (and valuation v) who picks some $x' \geq \tilde{x}_{i-1}^\delta(v)$ with $x' > \tilde{x}(w',v)$. Then, either $w' < \frac{\tilde{\tau}(\tilde{x}(w,v)) - \delta}{\tilde{x}(w,v)}$ or $w' \geq \frac{\tilde{\tau}(\tilde{x}(w,v)) - \delta}{\tilde{x}(w,v)}$ but the agent with w' prefers strictly x' to $\tilde{x}(w,v)$ under $\tilde{\tau}^\delta$.[6] In the former case, $\tilde{x}(w,v)$ is strictly dominated by x' since $w' \geq \frac{\tilde{\tau}(x') - \delta}{x'}$ and $x' > \tilde{x}(w,v)$. Thus, the agent with valuation v prefers x' strictly to $\tilde{x}(w,v)$. Observe now that either there is some w' with $\tilde{x}^\delta(w',v) = \tilde{x}_{i-1}^\delta(v)$ or there is some w' such that x' and $\tilde{\tau}(x')$ are arbitrarily close to $\tilde{x}_{i-1}^\delta(v)$ and $\tilde{\tau}(\tilde{x}_{i-1}^\delta(v))$, respectively.[7] Thus, we must have that the agent with valuation v must strictly prefer $\tilde{x}_{i-1}^\delta(v)$ to $\tilde{x}(w,v)$ under $\tilde{\tau}$. This is a contradiction since $w \geq \frac{\tilde{\tau}(\tilde{x}_{i-1}^\delta(v))}{\tilde{x}_{i-1}^\delta(v)}$. We thus conclude that $\tilde{x}(w,v) \geq \tilde{x}_{i-1}^\delta(v)$. Since $\tilde{x}^\delta(w,v) - \tilde{x}(w,v) \geq \sqrt{\delta}$ for all such pair (w,v), the same must be true at the limit as $\tilde{x}_i^\delta(v)$ is obtained, so $\tilde{x}_i^\delta(v) - \tilde{x}_{i-1}^\delta(v) \geq \sqrt{\delta}$. The largest n such that $\tilde{x}_n(v)^\delta \leq 1$ is given by $n := 1/\sqrt{\delta}$.

Further, for any such (w,v), $\tilde{x}^\delta(w,v) \leq \tilde{x}^\delta(w',v)$ for all $w' \in W_{i+1}$ since $w < w'$ and $\tilde{x}^\delta(\cdot,v)$ is nondecreasing. It follows that $\tilde{x}^\delta(w,v) \leq \tilde{x}_{i+1}^\delta(v)$. It thus follows that for any $i = 1, ..., n$,

$$\sup\{\tilde{x}^\delta(w,v) - \tilde{x}(w,v) | w \in W_i\} \leq \tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v), \tag{4}$$

where we let $\tilde{x}_0^\delta(v) := 0$ and $\tilde{x}_{n+1}^\delta(v) := 1$.

We can now bound the increase in aggregate consumption of the good when the tariff

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[6] This follows from our hypothesis that if $\tilde{x}^\delta(w,v) > \tilde{x}(w,v)$, then the type (w,v) prefers $\tilde{x}^\delta(w,v)$ strictly to any feasible $x < \tilde{x}^\delta(w,v)$.

[7] Note that $\tilde{\tau}$ cannot jump up discontinuously at $\tilde{x}_{i-1}^\delta(v)$ since then no one would choose $\tilde{x}_{i-1}^\delta(v)$ or any x close to $\tilde{x}_{i-1}^\delta(v)$, meaning that $\tilde{x}_{i-1}^\delta(v)$ cannot be a limit point.

changes from $\tilde{\tau}$ to $\tilde{\tau}^\delta$:

$$\mathbb{E}[\tilde{x}^\delta(w,v) - \tilde{x}(w,v)]$$

$$= \int_0^1 \int_0^1 (\tilde{x}^\delta(w,v) - \tilde{x}(w,v))g(w)dwf(v)dv$$

$$= \int_0^1 \int_0^1 (\tilde{x}^\delta(w,v) - \tilde{x}(w,v))1_{\{\tilde{x}^\delta(w,v)-\tilde{x}(w,v)<\sqrt{\delta}\}}g(w)dwf(v)dv$$

$$+ \int_0^1 \int_0^1 (\tilde{x}^\delta(w,v) - \tilde{x}(w,v))1_{\{\tilde{x}^\delta(w,v)-\tilde{x}(w,v)\geq\sqrt{\delta}\}}g(w)dwf(v)dv$$

$$\leq \sqrt{\delta} + \int_0^1 \sum_{i=1}^n \left(\int_{w\in W_i} (\tilde{x}^\delta(w,v) - \tilde{x}(w,v))g(w)dw \right) f(v)dv$$

$$\leq \sqrt{\delta} + \int_0^1 \sum_{i=1}^n \left(\int_{w\in W_i} [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)]g(w)dw \right) f(v)dv$$

$$\leq \sqrt{\delta} + \int_0^1 \sum_{i=1}^n \left(\int_{w\in \tilde{W}_i} [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)]g(w)dw \right) f(v)dv$$

$$\leq \sqrt{\delta} + \int_0^1 \sum_{i=1}^n [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)]M_g \left(\int_{\frac{\tilde{\tau}(\tilde{x}_i^\delta(v))-\delta}{\tilde{x}_i^\delta(v)}}^{\frac{\tilde{\tau}(\tilde{x}_i^\delta(v))}{\tilde{x}_i^\delta(v)}} dw \right) f(v)dv$$

$$= \sqrt{\delta} + \int_0^1 \sum_{i=1}^n [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)]M_g \left(\frac{\delta}{\tilde{x}_i^\delta(v)} \right) f(v)dv$$

$$\leq \sqrt{\delta} + \int_0^1 \sum_{i=1}^n [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)]M_g\sqrt{\delta}f(v)dv$$

$$= \sqrt{\delta} + M_g\sqrt{\delta} \int_0^1 \sum_{i=1}^n [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)]f(v)dv$$

$$\leq \sqrt{\delta} + M_g\sqrt{\delta} \int_0^1 (1 + \tilde{x}_n^\delta(v))f(v)dv \leq \sqrt{\delta} + 2M_g\sqrt{\delta} = (2M_g + 1)\sqrt{\delta},$$

where the second inequality follows from (4), the third from (3), the fourth from the fact that $g(w) \leq M$ for all w, the fifth from the fact that $\tilde{x}_i^\delta(v) \geq \sqrt{\delta}$, the sixth from the fact that $\sum_{i=1}^n [\tilde{x}_{i+1}^\delta(v) - \tilde{x}_{i-1}^\delta(v)] = \tilde{x}_{n+1}^\delta(v) + \tilde{x}_n^\delta(v) - \tilde{x}_1^\delta(v) \leq 1 + \tilde{x}_n^\delta(v)$, and the seventh (last) from the fact that $\tilde{x}_n^\delta(v) \leq 1$.

Now, the lemma can be proven by setting $\delta = \left(\frac{\varepsilon}{2M_g+1} \right)^2$. ∎

LEMMA S.6. $\mathbb{E}[x_\delta(w,v)]$ *is continuous and nondecreasing in* δ.

Proof. First, it is straightforward to verify that for any δ and $\delta' > \delta$, $\tau_{\delta'}(x) \leq \tau_\delta(x)$ for any

x and $x_{\delta'}(w, v) \geq x_\delta(w, v)$ for all (w, v).[8] We fix δ and show that for any $\eta > 0$, there exists some $\bar{\delta} > \delta$ such that for any $\delta' \in (\delta, \bar{\delta})$, $\mathbb{E}[x_{\delta'}(w, v) - x_\delta(w, v)] < \eta$.

First of all, by the continuity of v_z with respect to z, we can find some δ_1 such that for any $\delta' \in (\delta, \delta_1)$, $F(v_{\varepsilon-\delta}) - F(v_{\varepsilon-\delta'}) < \eta/2$. Second, we invoke Lemma S.5 with $\tilde{\tau}(\cdot) = \tau_\delta(\cdot)$ to find some $\delta_2 > \delta$ such that

$$\mathbb{E}[\tilde{x}^{\delta_2-\delta}(w, v) - x_\delta(w, v)] < \eta/2, \tag{5}$$

where $\tilde{x}^{\delta_2-\delta}$ denotes the choice associated with $\tilde{\tau}^{\delta_2-\delta}(\cdot) = \tilde{\tau}(\cdot) - (\delta_2 - \delta) = \tau_\delta(\cdot) - (\delta_2 - \delta)$. Third, we can find $\delta_3 > \delta$ sufficiently close to δ so that for any $\delta' \in (\delta, \delta_3)$, we have $\tilde{\tau}^{\delta_2-\delta}(x) < \tau_{\delta'}(x)$ for any x. Figure 1 below illustrates the tariff functions, $\tilde{\tau}$, $\tau_{\delta'}$, and $\tilde{\tau}^{\delta_2-\delta}$.
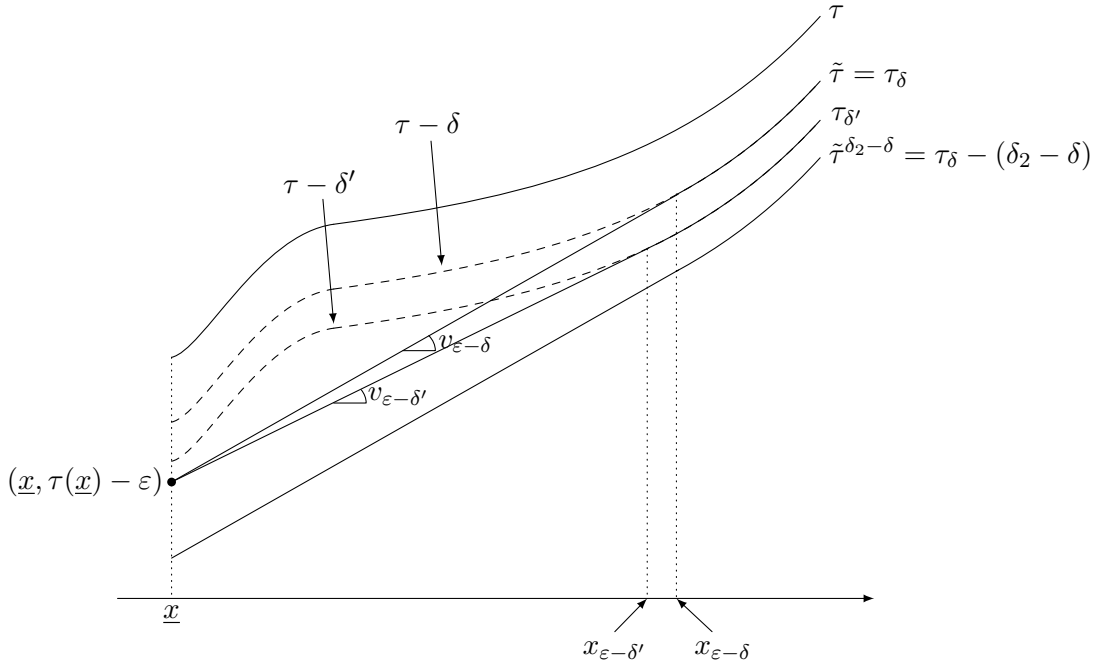

Figure 1: Illustration of the tariff functions, $\tilde{\tau}$, $\tau_{\delta'}$, and $\tilde{\tau}^{\delta_2-\delta}$ with $\delta' \in (\delta, \delta_3)$

We then prove the following claim:

CLAIM 1. *For any $\delta' \in (\delta, \delta_3)$, we have $x_{\delta'}(w, v) \leq \tilde{x}^{\delta_2-\delta}(w, v)$ for all $v \in [0, 1] \setminus [v_{\varepsilon-\delta'}, v_{\varepsilon-\delta}]$.*

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[8]As a reminder, we write

$$\tau_\delta(x) = \begin{cases} \tau(\underline{x}) - \varepsilon + v_{\varepsilon-\delta}(x - \underline{x}) & \text{for } x \in [\underline{x}, x_{\varepsilon-\delta}) \\ \tau(x) - \delta & \text{for } x \in [x_{\varepsilon-\delta}, 1] \cap X, \end{cases}$$

and note that $v_{\varepsilon-\delta}$ is strictly decreasing in δ while $x_{\varepsilon-\delta}$ is weakly decreasing in δ.

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Proof. First, for any agent type with $v < v_{\varepsilon-\delta'}$, $x_{\delta'}(w,v) = x_\delta(w,v) = \tilde{x}^{\delta_2-\delta}(w,v) = \underline{x}$, as desired. Consider now the agent types with $v > v_{\varepsilon-\delta}$. We consider two cases depending on whether $x_{\delta'}(w,v) \leq x_{\varepsilon-\delta}$ or not. Suppose first $x_{\delta'}(w,v) \leq x_{\varepsilon-\delta}$. Then, since $\tilde{\tau}^{\delta_2-\delta}(x) < \tau_{\delta'}(x)$ for any x and since $\tilde{\tau}^{\delta_2-\delta}(x)$ is linear in x with slope equal to $v_{\varepsilon-\delta}$ for any $x \leq x_{\varepsilon-\delta}$, any agent type with $v > v_{\varepsilon-\delta}$ will choose some $x \geq x_{\delta'}(w,v)$ under $\tilde{\tau}^{\delta_2-\delta}$, as desired. Next consider the case $x_{\delta'}(w,v) > x_{\varepsilon-\delta}$, and suppose for a contradiction that there is some agent type (w,v) such that $x_{\delta'}(w,v) > \tilde{x}^{\delta_2-\delta}(w,v)$. Clearly, we must have $\tilde{x}^{\delta_2-\delta}(w,v) \geq x_{\varepsilon-\delta}$. Since both tariffs $\tau_{\delta'}$ and $\tilde{\tau}^{\delta_2-\delta} = \tau_\delta - (\delta_2 - \delta)$ are obtained by shifting τ downward for x greater than or equal to $x_{\varepsilon-\delta}(\geq x_{\varepsilon-\delta'})$, it means that the agent type (w,v) must be indifferent between $x_{\delta'}(w,v)$ and $\tilde{x}^{\delta_2-\delta}(w,v)$ under both $\tau_{\delta'}$ and $\tilde{\tau}^{\delta_2-\delta'}$. This contradicts with Property (C) that $x_{\delta'}$ must satisfy. ∎

Now set $\overline{\delta} = \min\{\delta_1, \delta_3\}$. Then, letting $I_v := [v_{\varepsilon-\delta'}, v_{\varepsilon-\delta}]$, we obtain for any $\delta' \in (\delta, \overline{\delta})$

$$\begin{aligned}
\mathbb{E}[x_{\delta'}(w,v) - x_\delta(w,v)] &= \mathbb{E}\big[\big(x_{\delta'}(w,v) - x_\delta(w,v)\big)1_{\{v\notin I_v\}}\big] + \mathbb{E}\big[\big(x_{\delta'}(w,v) - x_\delta(w,v)\big)1_{\{v\in I_v\}}\big] \\
&< \mathbb{E}\big[\big(\tilde{x}^{\delta_2-\delta}(w,v) - x_\delta(w,v)\big)1_{\{v\notin I_v\}}\big] + F(v_{\varepsilon-\delta}) - F(v_{\varepsilon-\delta'}) \\
&< \mathbb{E}[\tilde{x}^{\delta_2-\delta}(w,v) - x_\delta(w,v)] + \eta/2 < \eta,
\end{aligned}$$

where the first inequality follows from Claim 1 and the fact that $x_{\delta'}(w,v) - x_\delta(w,v) \leq 1$ for all (w,v), the second inequality from the fact that $\tilde{x}^{\delta_2-\delta}(w,v) \geq x_\delta(w,v)$ for all (w,v),[9] and the third inequality from (5). ∎

2 Proof of Lemma 8

Suppose, to the contrary, that $x_{LH}^s < \max\{x_{HL}^s, x_{LL}^s\}$ for some $s \in \{\ell, h\}$, i.e. $x_{LH}^s < x_{HL}^s$. Then, the same argument as in the proof of Claim 2 (in the proof of Lemma 4) can be used to show that $x_{LH}^s = x_{LL}^s$. So $x_{LL}^s = x_{LH}^s < x_{HL}^s \leq x_{HH}^s$ and also $x_{HH}^s = 1$ by Lemma 7. Then, in order to be revenue-maximal, $\Gamma \in \mathcal{M}_m$ must satisfy the following: (i) $t_{LH}^s = t_{LL}^s = w_L x_{LH}^s$; (ii) the type (w_H, v_L) is indifferent between (x_{HL}^s, t_{HL}^s) and (x_{LH}^s, t_{LH}^s); (iii) the type (w_H, v_H) is indifferent between (x_{HH}^s, t_{HH}^s) and (x_{HL}^s, t_{HL}^s). Thus, we have

$$t_{HL}^s = v_L x_{HL}^s - (v_L x_{LH}^s - t_{LH}^s) \text{ and } t_{HH}^s = v_H - (v_H x_{HL}^s - t_{HL}^s). \tag{6}$$

Now construct an alternative mechanism, $\tilde{\Gamma}$, as follows: let $\tilde{\Gamma}^{s'} = \Gamma^{s'}$ for $s' \neq s$, and let

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[9]This is true since $\tilde{\tau}^{\delta_2-\delta}$ is a downward shift of τ_δ.

$\tilde{x}_{HL}^s = x_{HL}^s - \varepsilon$ and $\tilde{x}_{LH}^s = \tilde{x}_{LL}^s = x_{LH}^s + \varepsilon'$ while $\tilde{x}_{HH}^s = x_{HH}^s = 1$, where $\varepsilon, \varepsilon' > 0$ are chosen to satisfy $\tilde{x}_{HL}^s > \tilde{x}_{LH}^s$ and

$$h_{HL}^s \varepsilon = (n_{LH}^s + n_{LL}^s)\varepsilon'. \tag{7}$$

Clearly, $\tilde{\Gamma}^s$ generates a higher surplus than Γ^s while $\sum_{i,j} n_{ij}^s x_{ij}^s = \sum_{i,j} n_{ij}^s \tilde{x}_{ij}^s$. To determine payments, first set $\tilde{t}_{LH}^s = \tilde{t}_{LL}^s = w_L \tilde{x}_{LH}^s = w_L(x_{LH}^s + \varepsilon')$. Then, the payments $\tilde{t}_{HH}^s$ and $\tilde{t}_{HL}^s$ are determined to preserve the indifference; (ii) and (iii) given the new assignments. After substitution of (6), this yields

$$\tilde{t}_{HL}^s = v_L \tilde{x}_{HL}^s - (v_L \tilde{x}_{LH}^s - \tilde{t}_{LH}^s) = t_{HL}^s - \varepsilon v_L - \varepsilon'(v_L - w_L) \tag{8}$$

$$\tilde{t}_{HH}^s = v_H - (v_H \tilde{x}_{HL}^s - \tilde{t}_{HL}^s) = t_{HH}^s + \varepsilon(v_H - v_L) - \varepsilon'(v_L - w_L). \tag{9}$$

Letting B^s and $\tilde{B}^s$ denote the revenue generated from Γ^s and $\tilde{\Gamma}^s$, respectively, using (8) and (9) we obtain

$$\Delta B^s := \tilde{B}^s - B^s = \sum_{i,j \in \{L,H\}} n_{ij}^s(\tilde{t}_{ij}^s - t_{ij}^s) = (n_{LH}^s + n_{LL}^s)\varepsilon' w_L - n_{HL}^s\big[\varepsilon v_L + \varepsilon'(v_L - w_L)\big]$$

$$+ n_{HH}^s\big[\varepsilon(v_H - v_L) - \varepsilon'(v_L - w_L)\big].$$

Using $n_{HH}^s = n_{HL}^s$ and (7), rewrite this as

$$\Delta B^s = n_{HH}^s \varepsilon(v_H - 2v_L + w_L) - 2n_{HH}^s \varepsilon'(v_L - w_L)$$

$$= n_{HH}^s \varepsilon\left[v_H - 2v_L + w_L - \frac{2n_{HL}^s}{n_{LH}^s + n_{LL}^s}(v_L - w_L)\right].$$

So, $\Delta B^s \geq 0$ if $v_H \geq 2v_L - w_L + \frac{2n_{HL}^s}{n_{LH}^s+n_{LL}^s}(v_L - w_L)$. The right-hand side of this inequality is larger with $s = h$ than with $s = \ell$ since $v_L > w_L$ and $\frac{2n_{HL}^h}{n_{LH}^h+n_{LL}^h} = \frac{\rho}{1-\rho} > \frac{1-\rho}{\rho} = \frac{2n_{HL}^\ell}{n_{LH}^\ell+n_{LL}^\ell}$. This implies that if $v_H \geq 2v_L - w_L + \frac{2n_{HL}^h}{n_{LH}^h+n_{LL}^h}(v_L - w_L) = \frac{(2-\rho)v_L}{(1-\rho)} - \frac{w_L}{1-\rho}$, then both ΔB^h and ΔB^ℓ are nonnegative. To sum up, whether $s = \ell$ or h, the alternative mechanism $\tilde{\Gamma}^s$ generates at least as much revenue as Γ^s and yields a higher surplus, which gives a contradiction.