Weak Cartels and Collusion-Proof Auctions*

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November 25, 2014

Abstract

We study collusion in auctions by cartels whose members cannot exchange side-payments (i.e., weak cartels). We provide a complete characterization of outcomes that are implementable in the presence of weak cartels, identifying the set of circumstances under which standard auctions are susceptible to them. We then solve for optimal collusion-proof auctions and show that they can be made robust to the specific details of how cartels are formed and operated.

Keywords: Weak cartels, weakly collusion-proof auctions, optimal auctions, robustly collusion-proof auctions.
JEL-Code: D44, D82.

1 Introduction

Collusion is a pervasive problem in auctions, especially in public procurement. A canonical example is the famous "Great Electrical Conspiracy" in the 1950s, in which more than 40 manufacturers of electrical equipment colluded in sealed bid procurement auctions, using a bid rotation scheme also known as "phase of the moon" agreement (see Smith (1961)).

*We thank Philippe Jehiel, Navin Kartik, Michihiro Kandori, Debasis Mishra, and Jidong Zhou for their comments, and Cheonghum Park for his excellent research assistantship. We thank seminar participants at University of Bonn Summer Conference, Columbia University, University College London, HKUST, University of Iowa, Korea Development Institute, Northwestern University, and the 10th World Congress in Shanghai. We acknowledge the financial support from the National Research Foundation through its Global Research Network Grant (NRF-2013S1A2A2035408).

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More recently, in 2012, the largest six construction companies of Korea—so-called "Big 6" according to the competition authority—were involved in bid rigging in the Four River Restoration Project.[1] As a result, each of the Big 6 won 2 sections of the rivers while two other companies, also part of the collusive agreement, won 1 section each.

Many bid-rigging cases uncovered by competition authorities fall into the category of what McAfee and McMillan (1992) labeled weak cartels, namely cartels that do not involve exchange of side payments among cartel members.[2] Weak cartels usually operate by designating a winning bidder and suppressing competition from other cartel members. The winning bidder is designated through "market sharing" agreements (e.g., the Korean construction case), through "bid rotation" whereby firms took turns in winning contracts (e.g., the U.S. case of electrical equipment conspiracy), or through more complicated schemes. The designated bidders place bids somewhere around the reserve price, and bids from other cartel members are either altogether suppressed (the practice of "bid suppression") or submitted at non-competitive levels (the practice of "cover bidding").

Cartels may avoid side payments for fear that they will leave a trail of evidence for antitrust authorities.[3] Compensating losing bidders in money may also lure "pretenders" who join a cartel solely to collect "the loser compensation" without ever intending to win. In fact, we show in Appendix A that the ability to use side payments and reallocate the winning object (e.g., via a "knock-out" auction) adds little value to a cartel if entry by such pretenders cannot be controlled.[4] While transfers and knockout auctions are sometimes

[1] This construction project (with objectives such as securing water resources; implementing comprehensive flood control measures; improving water quality and restoring river ecosystems) is considered the biggest national infrastructural project in Korean history and has received a great deal of attention. We emphasize that many large national procurement auctions are "one-off" kind. These auctions are often so important for bidders that, even though they know they may face each other in future auctions, they naturally perceive the interaction as a static one.

[2] For example, among 16 bidding rigging cases in Korea that have been filed by the Korea Fair Trade Commission during the first half of year 2014, some evidence of side transfers was found only in 2 cases while there was no such evidence in 8 cases. It is also unclear whether transfers have been used in other cases. Another recent example of weak cartel is producers of high voltage power cables that have been fined about 0.3 billion euros by the European Commission (see the press release in http://europa.eu/rapid/press-release_IP-14-358_en.htm). According to the press release, "the European and Asian producers would stay out of each other's home territories and most of the rest of the world would be divided amongst them. In implementing these agreements, the cartel participants allocated projects between themselves according to the geographic region or customer."

[3] In practice, cartels may hide side payments under different guises. For instance, Marshall et al. (1994) suggests that members bring bogus lawsuits against one another and exchange settlements. Such settlements must pass the scrutiny of a legal system, and must involve lawyers, so they entail transaction costs.

[4] If transfers cannot be used, the ability to reallocate the object (e.g., via a knockout auction) makes no

important features of bidding rings (see Marshall and Marx (2012)), it is therefore also important to study weak cartels.

A key question is how weak cartels can profitability suppress competition in a way that is beneficial to all its members. Given that losing bidders cannot be paid off by a winning bidder to stay out of competition, the only scope for its profitable operation is to manipulate the allocation, sometimes allocating the good to bidders that do not have the highest value for it. But since the latter entails efficiency loss, it is not clear when and how such a distortion may benefit a cartel. That weak cartels can profit from such a manipulation was first demonstrated by McAfee and McMillan (1992, henceforth MM). They showed that in a first-price auction, symmetric bidders would benefit *ex-ante* from agreeing to randomly select a single bidder to bid the reserve price (as opposed to playing the symmetric equilibrium of the auction) whenever their value distribution exhibits increasing hazard rate. Further, they suggest that the optimal response by the seller is to allocate the good randomly at a fixed price.

To the extent that the increasing hazard rate is a mild condition, MM's theory suggests that a first-price auction is "virtually always" susceptible to a weak cartel, and that in its presence, the seller can never hope to realize the efficiency gain from bidding competition. However, as we will show, this largely negative view rests on the analysis of ex-ante benefit from collusion. Importantly, their model does not consider bidders' (interim) incentives to participate in a cartel. Even though a cartel promises to yield strictly positive surplus to its members on average, the surplus may not accrue to all bidder types so that bidders may actually be worse off from participating in the cartel, depending on the realization of their types. In practice, the lack of interest alignment is often what causes a cartel to break up.

In the current paper, we explicitly consider the bidders' interim incentive to participate in a cartel. By doing so, we offer a theory of weak cartels that differs from existing theories not only in terms of what auctions are susceptible to collusion and under what conditions, how a weak cartel would behave when it is active, but also in terms of how the auctioneer should respond to the threat of collusion. The key observation is that when a cartel seeks to reduce competition from a certain bidder, for instance by requesting him to place the same bid for a certain subset of his possible valuations, the resulting efficiency loss is not borne uniformly across the different valuation types. Instead, the type with the highest valuation in that subset suffers most acutely from the manipulation, and is the most likely to

difference. Further, since we assume risk neutrality for bidders, a fractional/probabilistic assignment entails no loss of generality per se. Hence, arrangements such as counter-purchase agreements which may be used to fine-tune market shares add no additional value to our weak cartel. In other words, our notion of a weak cartel already subsumes such an arrangement via random assignment.

object. However, when the "average" type with whom he shares the same winning probability has a relatively high valuation, his expected payment is reduced enough as to make the manipulation profitable for him as well. This observation leads us to identify so-called "susceptible types"—namely those that would benefit from colluding—to be an interval of the bidder's valuation types above the reserve price for which the distribution is "convex in a certain sense."[5]

Restricting attention to a large class of what we call "winner-payable" auctions, we show that any such auction is susceptible to a weak cartel if it seeks to implement non-constant allocation across susceptible types—i.e., if, absent collusion, the winning probability of a bidder were to strictly increase over some susceptible types (Theorem 1). We also prove that, given additional mild conditions, the converse also holds—namely, any winner-payable auction that implements a constant winning probability for any susceptible types is unsusceptible to a weak cartel (Theorem 2). An implication of this characterization is that efficient auctions as well as the revenue-maximizing auction (à la Myerson) are unsusceptible to weak cartels if the value distributions of bidders are strictly concave.

Our characterization of collusion-proof allocation also leads to a positive theory of how cartel behaves. We show that under our interim participation constraint, a weak cartel may implement a random allocation among all bidder types above the reserve price, as predicted by MM, but only when *all* types are susceptible. When some bidder types are not susceptible, a cartel must coax them into participation by allowing them to separate, i.e., for a high type to win with higher probability than a low type. Consequently, a weak cartel would induce some types to be separated and others to be pooled—which a weak cartel can accomplish by employing cheap talk communication among its members. Further, any cartel which seeks to collude in an interim Pareto optimal fashion (i.e., collusion producing an interim payoff which is not Pareto dominated by that from any other form of collusion) would choose an allocation that is itself collusion-proof.

Finally, the complete characterization of collusion-proof auctions enables us to study a normative question: *How should one design an auction in the presence of a weak cartel?* Restricting attention to winner-payable auctions, we identify the optimal collusion-proof auction for the seller up to the choice of the individual reserve prices (Theorem 3). The optimal mechanism allocates the good to maximize the virtual value functions that are suitably ironed out for the susceptible types. An interesting feature of the optimal mechanism

[5]More precisely, this property can be defined via a notion of concave closure. A concave closure of a function F is the smallest concave function G satisfying $G \geq F$. The susceptible types of a bidder are then defined to be an interval for which the concave closure of the bidder's value distribution, truncated to the valuations above the reserve price, is linear. See Theorem 1 for the detail.

is asymmetric treatment of bidders who are ex-ante identical. For instance, when the bidders' value distributions are convex, the optimal mechanism takes the form of a sequential negotiation: the seller engages in a take-it-or-leave-it negotiation with each of the bidders sequentially in a predetermined order. The reason for this asymmetric treatment is the collusion-proofness constraint that prevents the seller from discriminating across different *types* of bidders. Facing this constraint, the seller finds it optimal to discriminate across *bidders* instead.

Modeling a bidder's decision to participate in a cartel involves a methodological issue. A bidder's willingness to join a cartel depends on the payoff he expects to receive if he refuses to join the cartel. This payoff in turn depends on how the remaining bidders update their beliefs about the refusing bidder, whether they will still form a cartel among themselves, and, if so, to what extent they can credibly punish the refusing bidder. In dealing with these issues, we initially follow the weak collusion-proofness notion of Laffont and Martimort (1997, 2000) by assuming that when a bidder refuses to participate in a cartel, the cartel collapses and the remaining bidders do not update their beliefs.

In Section 6, we consider a much broader set of circumstances in terms of how a weak cartel is formed and operated. For instance, any informed bidder(s) as well as an uninformed mediator may propose a cartel manipulation; there can be partial or multiple cartels in operation; and participants in a cartel may punish those who have refused to participate. We show that outcomes that are weakly collusion-proof can be also implemented by the auctioneer in these environments, as long as no cartel employs a strategy profile weakly dominated by another profile for all cartel members. (Theorem 4).

The current paper is related to a number of papers on collusion in auction. Seminal contributions include Robinson (1985), Graham and Marshall (1987), von Ungern-Stenberg (1988), Mailath and Zemsky (1991), and MM, who studied whether a collusive agreement can be beneficial to its members.[6] Unlike the current paper, these papers largely focus on strong cartels, where side-payments play a crucial role for achieving efficient collusion. As mentioned above, MM does consider weak cartels and show that they involve random allocation of the object for sale, much consistent with often observed practice of bid rotation.[7] As highlighted above, our approach is differentiated by its explicit consideration of the bidders'

[6]These authors, like us, abstract from the enforcement issue—how members of a cartel may sustain collusion without a legally binding contract.

[7]See also Condorelli (2012). This paper analyzes the optimal allocation of a single object to a number of agents when payments made to the designer are socially wasteful and cannot be redistributed. The problem addressed is analogous to that of a cartel-mediator designing an ex-ante optimal weak cartel agreement at a standard auction with no reserve price.

interim incentive for participation in the cartel. In practice, our model is suited to analyze environments where bidders are likely to have some private information at the moment of cartel formation, as opposed to forming a cartel under complete information. To the extent that the two approaches treat distinct sets of circumstances, they complement each other.

Aside from the timing of participation decision, our model is more general than MM in several respects. First, we consider a more general class of auctions called "winner-payable auctions." These are the auctions in which bidders can coordinate, if they so choose, so that only one bidder can pay to win the object. Winner-payable auctions include all standard auctions such as first-price sealed-bid, second-price sealed-bid, Dutch and English auctions, or any hybrid forms, and sequential negotiation. Considering such a general class of auctions helps to isolate the features of auctions that make them vulnerable to cartels. Second, we relax the monotone hazard rate and symmetry assumptions. One may view bidder symmetry as favoring the emergence of a cartel especially when the use of side payments is limited. In practice, however, bidders are unlikely to be symmetric, so it is useful to know to what extent bidder asymmetry affects the sustainability of weak cartels.

Several authors study enforceability of collusion through repeated interaction (see Aoyagi (2003), Athey et al. (2004), Blume and Heidhues (2004), and Skrzypacz and Hopenhayn (2004)) or via implicit collusive strategies (see Engelbrecht-Wiggans and Kahn (2005), Brusco and Lopomo (2002), Marshall and Marx (2007, 2009), Garratt et al. (2009)). If types are distributed independently over time, repeated interaction enables members of a weak cartel to use their future market shares in a way similar to monetary transfers. If the types are persistent over time, as we envision to be more realistic, however, tampering with future market shares involves severe efficiency loss (see Athey and Bagwell (2008)). In this sense, our approach—including the focus on interim participation—remains valid in a repeated interaction setting where market shares cannot be adjusted in a frictionless manner due to incentive constraints arising from persistent valuations.

The current paper is also related to the literature that studies collusion-proof mechanism design. This literature, pioneered by Laffont and Martimort (1997, 2000) (henceforth LM) and further generalized by Che and Kim (2006, 2009) (henceforth CK), models cartel as designing an optimal mechanism for its members (given the underlying auction mechanism they face), assuming that the members have necessary wherewithal to enforce whatever agreement they make.[8] Similar to Laffont and Martimort (1997, 2000) and Che and Kim (2006), we explicitly consider the bidders' incentives for participating in the cartel. Unlike

[8]The likely scenario of enforcement involves the threat of retaliation through future interaction, multi-market contact, or organized crime.

the current paper, though, their models allow a cartel to be formed only after bidders enter into the grand auction noncooperatively. This modeling assumption, while realistic in some internal organization setting, is not applicable to auction environments where the collusion often centers around the participation into auction.

Che and Kim (2009) and Pavlov (2008) do consider collusion on participation. And they show that the second-best outcome (i.e., the Myerson (1981) benchmark) can be achieved even in the presence of a strong cartel as long as the second-best outcome involves a sufficient amount of exclusion of bidders. The mechanism that accomplishes this has features not observed in the standard auctions, however. For instance, it requires losing bidders not only to pay the winning bidders but also to incur strict loss in some states, i.e., it fails ex-post individual rationality of the bidders. Such auctions, while theoretically interesting, are never observed in practice. By contrast, the current paper restricts attention to a more realistic, still broad, class of auctions rules, particularly those that ensure ex-post individual rationality. Further, the results we obtain here are more in line with the casual empiricism, namely that even weak cartels can pose a serious problem for auctions. These two approaches ultimately complement each other in the sense that they clarify the features of auctions that make them vulnerable to bidder collusion.

The rest of the paper is organized as follows. In the next section, we illustrate our main results via two simple examples. Then, section 3 introduces the class of "winner payable" auction rules that we study and the model of collusion. Section 4 characterizes the susceptibility of auctions to weak cartels. Section 5 characterizes optimal collusion-proof auctions. Section 6 presents a more robust concept of collusion-proofness. Appendix A deals with the equivalence between strong cartel and weak cartel in the presence of the entry exclusion constraint. Appendixes B-E contain all the proofs not presented in the main body of the paper.

2 Illustrative Examples

We first illustrate via simple examples how bidders' interim incentives to participate in weak cartels dramatically affect the formation of cartels and their behavior. As will be seen, we obtain new predictions on when weak cartels will form and how they behave, relative to the MM's analysis. We present two examples here, and others will be interspersed throughout the analysis.

For the first example, suppose there are two bidders vying for a single object in a second-price auction (with zero reserve price). Each bidder has a valuation drawn from the interval

$[0,1]$ according to a distribution function $F(v) = 1 - (1-v)^2$. Its hazard rate $\frac{f}{1-F}$ is increasing, and, according to MM, this implies that bidders would benefit ex ante from a weak cartel. Specifically, if bidders were to bid non-cooperatively, both bidding their values, each bidder would earn an ex-ante payoff of $\frac{2}{15}$, but if they form a cartel and select one bidder at random to win the object at zero price, each would enjoy a strictly higher ex-ante payoff of $\frac{1}{6}$.

However, if bidders have private information at the cartel formation stage, then the fact that a cartel is beneficial ex-ante need not guarantee it will form. To see this, suppose initially that both bidders participate in the cartel regardless of their valuations. And suppose the cartel has each bidder win with probability one half at zero price. Then, a bidder would enjoy the "interim" payoff of $\frac{v}{2}$ if his valuation is v.

Suppose the same bidder refuses to join the cartel. Then, the cartel collapses, and in the ensuing noncooperative play, each bidder employs a dominant strategy of bidding his valuation. The bidder would earn the "interim" payoff of

$$U^0(v) := \int_0^v (v-s)dF(s) = v^2 - \frac{v^3}{3}.$$

As depicted in Figure 1, $U^0(v) > v/2$ if $v > \frac{1}{2}\left(3-\sqrt{3}\right) =: \bar{v}_0$. That is, any bidder with valuation greater than $\bar{v}_0$ will be better off from refusing to join the cartel.

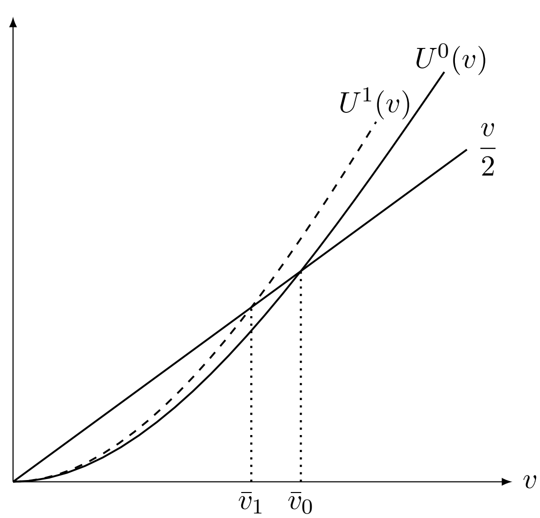


Figure 1: Unraveling of Cartel

Given this, bidders *may* attempt to form a cartel that only operates when their valuations are both less than $\bar{v}_0$. Will such a "partial cartel" form? The answer is no. To see this, suppose to the contrary that a cartel forms if and only if the bidders' values are both less than $\bar{v}_0$.

And suppose the cartel operates as before, randomly selecting a winner and having the loser bid zero. Given the agreement, the bidder will enjoy the interim payoff of $\frac{v}{2}$ as before, conditional on a cartel having been formed. But given the same event (i.e., his opponent having $v < \bar{v}_0$), he would have earned

$$U^1(v) = \frac{\int_0^v (v-s)dF(s)}{F(\bar{v}_0)}$$

if he refused to join the cartel and bid his valuation in the noncooperative play. It turns out that $U^1(v) > v/2$ if and only if $v > \frac{1}{2}\left(3 - \sqrt{9 - 12\bar{v}_0 + 6\bar{v}_0^2}\right) =: \bar{v}_1$, which is strictly less than $\bar{v}_0$, as described in Figure 1. In other words, no bidder with valuation $v \in (\bar{v}_1, \bar{v}_0]$ will participate in the cartel.

Arguing recursively in this manner, one can see that no types of bidders are willing to participate in the cartel. Simply put, a cartel unravels here! We shall later show that the unraveling is due to the density decreasing in v. Intuitively, declining density means that a higher valuation type forgoes relatively more from a non-cooperative play, in terms of the chance of winning the good. This creates the iterative process of high valuation types successively dropping out of collusion, leading to a full collapse, despite the fact that it is beneficial ex-ante.

The next example deals with a situation in which a cartel is sustainable, but the way a cartel operates is crucially affected by the interim participation constraints. Suppose again two bidders participate in a second-price auction to obtain an object. Each bidder draws his valuation from a triangular distribution F with density $f(v) = 8v$ if $v \in [0, 1/4]$ and $f(v) = \frac{8}{3}(1-v)$ if $v \in [1/4, 1]$. The hazard rate is increasing everywhere, so bidders ex-ante payoff would be maximized by a random allocation. However, since the density is decreasing in $[1/4, 1]$, a random allocation is not implementable by the cartel.[9]

[9] To see this, suppose that the bidders form a cartel and randomly allocate the object between them. A bidder will then earn the payoff of $v/2$ if his valuation is v. Suppose the same bidder refuses to form a cartel. From the ensuing non-cooperative bidding, the bidder will earn the payoff of

$$U(v) = \int_0^v (v-s)dF(s) = \int_0^v F(s)ds, \tag{1}$$

where

$$F(v) = \begin{cases} 4v^2 & \text{if } v \in [0, 1/4] \\ -\frac{1}{3} + \frac{8v}{3} - \frac{4v^2}{3} & \text{if } v \in [1/4, 1]. \end{cases} \tag{2}$$

A simple calculation reveals that $U(v) > v/2$ for v sufficiently close to 1, meaning that a high valuation bidder will refuse to join such a cartel.

Unlike the previous example, the density is not decreasing everywhere, and this feature will ensure profitability of a cartel, as our results in Section 4 will show. Such a cartel will, however, require a different arrangement then complete pooling. Suppose the cartel has each participating member send a cheap talk message, either H or L, depending on whether their values are above or below $\tilde{v} = 1/2$, respectively. Their bids are then coordinated as follow. If a bidder sends H, he bids his value. If bidder i with value v_i sends L, he bids $\frac{6}{7}v_i$ if bidder $j \neq i$ sends H. If bidder j also sends L, then one bidder is chosen at random to bid zero while the other bids his value.[10]

Given this coordination, a type-v bidder obtains the object with probability

$$\tilde{Q}(v) := \begin{cases} \frac{F(\tilde{v})}{2} = \frac{1}{3} & \text{if } v \leq \tilde{v} \\ F(v) & \text{if } v > \tilde{v}, \end{cases}$$

and enjoys the payoff of

$$\tilde{U}(v) := \begin{cases} \frac{1}{3}v & \text{if } v \leq \tilde{v} \\ \frac{1}{3}\tilde{v} + \int_{\tilde{v}}^{v} F(s)ds & \text{if } v > \tilde{v}. \end{cases}$$

It can be checked that bidders have the incentives to follow the coordination, once they join the cartel.[11] Second, as seen in Figure 2, the bidders will have the incentive to participate in the cartel since the collusive payoff $\tilde{U}$ dominates the non-collusive payoff U (specified in (1)) for all v. Later we shall show (Corollary 5) that the above cartel behavior is Pareto optimal among all sustainable cartel behaviors. This example demonstrates a way in which a cartel may operate, which differ from the simple randomization scheme that MM suggests as a possible collusive behavior.

As will be seen, the collusion participation constraint also affects the way in which the seller should design her auction mechanism. An example will be later presented to illustrate the optimal response by the seller.

[10] As noted in the Introduction, we assume that the cartel has the ability to enforce the agreed coordination. In a second price auction, the needed enforcement power is minimal; for instance, a bidder who is supposed to bid zero may deviate and bid his valuation, but this can be prevented by having the other bidder bid $\tilde{v} = \frac{1}{2}$ (instead of his value).

[11] More precisely, the type $\tilde{v}$ is indifferent between sending L and H, which means that each bidder with $v < (>)\tilde{v}$ prefers sending $L(H)$.

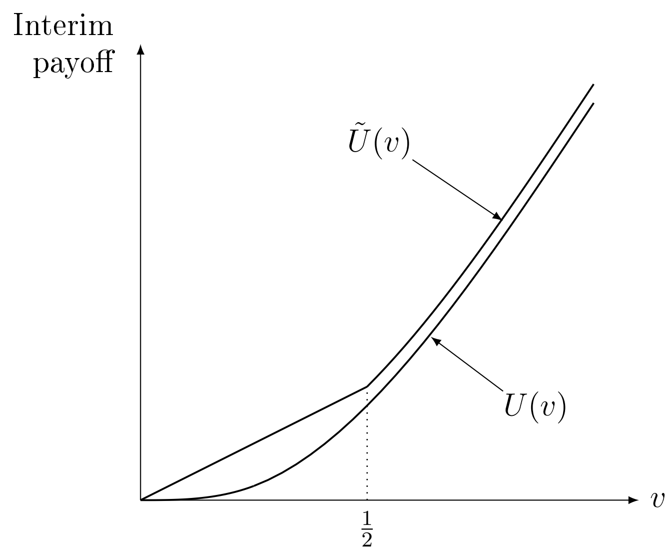


Figure 2: Profitability of Cartel Manipulation

3 Model

3.1 Environment

A risk neutral seller has a single object for sale. The seller's valuation of the object is normalized at zero. There are $n \geq 2$ risk neutral bidders, and $N := \{1, ..., n\}$ denotes the set of bidders. We assume that bidder i is privately informed of his valuation of the object, v_i, drawn from an interval $\mathcal{V}_i := [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$ according to a strictly increasing and continuous cumulative distribution function F_i (with density f_i). We let $\mathcal{V} := \times_{i \in N} \mathcal{V}_i$ and assume that bidders' valuations are independently distributed. Each bidder's payoff from not obtaining the object and paying (or receiving) no money is normalized to zero.

The object is sold via an auction. An **auction** is defined by a triplet, $A := (\mathcal{B}, \xi, \tau)$, where $\mathcal{B} := \times_{i \in N} \mathcal{B}_i$ is a profile of message spaces (with $\mathcal{B}_i$ being i's message space), $\xi : \mathcal{B} \to \mathcal{Q}$ is a rule mapping a vector of messages ("bids") to a (possibly random) allocation of the object in $\mathcal{Q} := \{(x_1, \ldots, x_n) \in [0,1]^n | \sum_{i \in N} x_i \leq 1\}$, and $\tau : \mathcal{B} \to \mathbb{R}^n_+$ is a rule determining expected payments as a function of the messages. Let ξ_i and τ_i be i-th element of ξ and τ that corresponds to the allocation and payment rule for bidder i, respectively. We assume that the seller cannot force bidders to participate in the auction. Therefore, for each bidder, we require the message space $\mathcal{B}_i$ to include a non-participation option, b_i^0, the exercise of which results in no winning and no payment for bidder i, $\xi_i(b_i^0, \cdot) = \tau_i(b_i^0, \cdot) = 0$. It is useful to define the set $\mathcal{B}^i = \{b \in \mathcal{B} | \xi_i(b) > 0\}$ of bid profiles that lead bidder i to win with positive probability. Bidder i's **reserve price** under A is then defined as

$$r_i := \inf \left\{ \frac{\tau_i(b)}{\xi_i(b)} \geq 0 \middle| b \in \mathcal{B}^i \right\}, \tag{3}$$

the minimum per-unit price bidder i must pay to win with positive probability, which will turn out to play an important role in our analysis. Likewise, the maximum per-unit price bidder i could pay under auction A is given by

$$R_i := \sup \left\{ \frac{\tau_i(b)}{\xi_i(b)} \leq \overline{v}_i \middle| b \in \mathcal{B}^i \right\}.$$

If $\mathcal{B}^i = \emptyset$, then set $r_i = R_i = \overline{v}_i$.

Whether and how a cartel operates in an auction depends crucially on the details of its allocation and payment rule. CK (2009) show that if the seller faces no constraints in designing an auction, any outcome that involves no sale with sufficient probability can be implemented even in the presence of a cartel that can use side payment and even reallocate the object ex-post. The idea is that the seller hands over the object to the cartel at a fixed price, whereby most of the scope for the cartel to manipulate the outcome is removed. Implementing this idea, however, requires losing bidders to make payments as well—a feature seldom observed in practice.

Standard auctions do not often collect payments from losing bidders, so they are potentially susceptible to bidder collusion in a way not recognized by CK (2009). In the current paper, we focus on these more realistic auction formats. Specifically, we restrict attention to a set $\mathcal{A}^*$ of auction rules that are winner-payable in the following sense.

DEFINITION 1. *An auction A is **winner-payable** if, for all $i \in N$, there exist bid profiles $\underline{b}^i, \overline{b}^i \in \mathcal{B}^i$ such that $\xi_i(\underline{b}^i) = \xi_i(\overline{b}^i) = 1$, $\tau_i(\underline{b}^i) = r_i$, $\tau_i(\overline{b}^i) = R_i$, and $\tau_j(\underline{b}^i) = \tau_j(\overline{b}^i) = 0$, for $j \neq i$.*

In words, an auction is winner-payable if it is possible for bidders to coordinate their bids (possibly including non-participation) so that any given bidder can win the object for sure at the minimum per-unit price r_i (i.e., his reserve price) or at the maximum per-unit price R_i allowed by the bidding rule, and the other (losing) bidders pay nothing. One can see that most of commonly observed auctions are winner-payable.[12]

[12]Lotteries represent a notable exception. For instance, consider a mechanism where there is a fixed number $n \geq 2$ of lottery tickets, each bidder can buy a single ticket at a fixed price $p \in \mathbb{R}_+$, the auctioneer retains the unsold tickets, and the object is assigned to the holder of a randomly selected ticket. In this mechanism $\mathcal{B}_i := \{0, 1\}$, $\xi_i(0, b_{-i}) = \tau_i(0, b_{-i}) = 0$, $\xi_i(1, b_{-i}) = 1/n$, $\tau_i(1, b_{-i}) = p$. Winner-payability fails as there is no message profile that can guarantee the object to any of the players. On the other hand, fixed-prize raffles (see Morgan (2000)) are winner-payable.

- **First-Price (or Dutch) Auctions with Reserve Price**: Winner-payability holds because each bidder can obtain the object for sure at any positive price above the reserve price, if he places a bid at that price and all the other bidders place lower bids or do not participate in the auction.

- **Second-Price (or English) Auctions with Reserve Price**: Winner-payability holds because each bidder i can be guaranteed to win the good at any price above the reserve price, if another bidder bids exactly that price, i bids anything above that price, and all other bidders bid strictly lower or do not participate.

- **Sequential Take-It-or-Leave-It Offers**: Suppose the seller approaches the buyers in a given exogenous order and makes to each of them a single take-it-or-leave-it offer. This format is winner-payable because each bidder can win the object for sure if all other prior bidders reject their offers.[13]

We note that our main results (Theorem 1 and 2) apply beyond winner-payable rules as long as only the winner of the auction pays for the object and its equilibrium allocation is deterministic (i.e. for each profile of bids, the object is assigned with probability one to only one of the bidders, whenever it is assigned), or randomization is limited to tie-breaking and occurs with zero probability.

3.2 Characterization of Collusion-Free Outcomes

In the absence of collusion, an auction rule A in $\mathcal{A}^*$ induces a game of incomplete information where all bidders simultaneously submit messages (i.e. bids) to the seller. A pure strategy for player i is denoted as $\beta_i : \mathcal{V}_i \to \mathcal{B}_i$, and $\beta = (\beta_1, \cdots, \beta_n)$ denotes its profile.

Given a profile of equilibrium bidding strategies β^* of an auction, its **outcome** corresponds to a **direct mechanism** $M_A \equiv (q, t) : \mathcal{V} \to \mathcal{Q} \times \mathbb{R}^n$, where for all $v \in \mathcal{V}$, $q(v) = \xi(\beta^*(v))$ is the allocation rule for the object and $t(v) = \tau(\beta^*(v))$ is the payment rule. Given M_A, we define the interim winning probability $Q_i(v_i) = \mathbb{E}_{v_{-i}}[q_i(v_i, v_{-i})]$ and interim

[13] More precisely, suppose the seller approaches the buyers in the order of the bidder index, say, and makes a take-it-or-leave-it offer of p_i for bidder i in his turn (i.e., when all bidders before i have rejected the seller's offers). A bid profile $b = (b_1, ..., b_n)$ in this rule may represent the highest offers bidders are willing to accept. Given this interpretation, $\xi_i(b)$ represents the probability of the event that bidder i is approached by the seller and accepts her offer of p_i, so $\xi_i(b) > 0$ means that $b_i > p_i$. Further, conditional on that event, bidder i pays p_i, so $\tau_i(b)/\xi_i(b) = p_i$ whenever $\xi_i(b) > 0$. In this case, $\underline{b}^i = \overline{b}^i$ can be set so that for $j \neq i$, $\underline{b}^i_j = \overline{b}^i_j = 0$ (so all other bidders than i reject the seller's offers), and $\underline{b}^i_i = \overline{b}^i_i = b_i$ (i.e, the same as the original bid for bidder i). Then, $\xi_i(\underline{b}^i) = \xi(\overline{b}^i) = 1$, and $\tau(\underline{b}^i) = \tau(\overline{b}^i) = p_i = \tau_i(b)/\xi_i(b)$.

payment $T_i(v_i) = \mathbb{E}_{v_{-i}}[t_i(v_i, v_{-i})]$ for bidder $i \in N$ with type $v_i \in \mathcal{V}_i$. We will refer to the mapping $Q = (Q_i)_{i \in N}$ and $T = (T_i)_{i \in N}$ as interim allocation and transfer rules, respectively. The equilibrium payoff of player i with value v_i is then expressed as

$$U_i^{M_A}(v_i) := Q_i(v_i)v_i - T_i(v_i).$$

Any collusion-free equilibrium outcome M_A must be **incentive compatible** (by definition of equilibrium) and **individually rational** (because bidders are offered the non-participation option). That is, for all $i \in N$ and $v_i \in \mathcal{V}_i$:

$$(IC) \qquad U_i^{M_A}(v_i) \geq v_i Q_i(\tilde{v}_i) - T_i(\tilde{v}_i), \text{ for all } \tilde{v}_i \in \mathcal{V}_i,$$

$$(IR) \qquad U_i^{M_A}(v_i) \geq 0.$$

As is well known, (IC) and (IR) are equivalent to the following conditions:

$$(M) \qquad Q_i \text{ is nondecreasing, } \forall i \in N;$$

$$(Env) \qquad T_i(v_i) = v_i Q_i(v_i) - \int_{\underline{v}_i}^{v_i} Q_i(s)ds + T(\underline{v}_i) - \underline{v}_i Q_i(\underline{v}_i), \forall v_i \in \mathcal{V}_i, \forall i \in N;$$

$$(IR') \qquad U_i^{M_A}(\underline{v}_i) = \underline{v}_i Q_i(\underline{v}_i) - T(\underline{v}_i) \geq 0, \forall i \in N.$$

In the later analysis, we often restrict attention to direct mechanisms (that may not be winner-payable.) This restriction is without loss, however, as is shown next:

LEMMA 1. *Given any direct mechanism $M = (q, t)$ that satisfies (IC) and (IR), there is a winner-payable auction rule $A \in \mathcal{A}^*$ whose equilibrium outcome yields the same interim outcome as M.*

Proof. See Appendix A. (page 34). ▌

3.3 Models of Collusion

Members of a weak cartel can only coordinate the "bids" submitted to the seller. Since non-participation is regarded as a possible bid in our model, this means that the bidders can also coordinate on their participation decisions. As mentioned in the introduction, we abstract from the question of how a cartel can enforce an agreement among its members, but rather focus on whether there will be an incentive compatible agreement that is beneficial for all bidders.[14]

Formally, a **cartel agreement** is a mapping $\alpha : \mathcal{V} \to \Delta(\mathcal{B})$ that specifies a lottery over possible bid profiles in auction A for each profile of valuations for the bidders. We envision bidders in the cartel to commit to submitting their private information to the cartel (e.g., an uninformed mediator) and bidding according to its subsequent recommendation. A cartel agreement leads bidders to play a game of incomplete information where each player's strategy is to report his type to the cartel and then outcomes are determined by the lottery α over bids and auction rule A. By the revelation principle, it is without loss to restrict attention to cartel agreements that make bidders report their true valuation to the mediator. Hence, for any cartel agreement α, one can equivalently consider a direct mechanism it induces as follows.

Definition 2. *A direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ is a* ***cartel manipulation*** *of A if there exists a cartel agreement α such that*

$$\tilde{q}_i(v) = \mathbb{E}_{\alpha(v)}[\xi_i(b)] \quad \text{and} \quad \tilde{t}_i(v) = \mathbb{E}_{\alpha(v)}[\tau_i(b)], \forall v \in \mathcal{V}, i \in N, \tag{4}$$

where $\mathbb{E}_{\alpha(v)}[\cdot]$ denotes the expectation taken using the probability distribution $\alpha(v) \in \Delta(\mathcal{B})$.

Since $\tilde{M}_A$ results from bidders' equilibrium play in the incomplete information game described above, it is incentive compatible, i.e. satisfies (IC). Our goal is to investigate whether any auction $A \in \mathcal{A}^*$ is susceptible to some cartel manipulation $\tilde{M}_A$. To do this, we must first identify the set of cartel manipulations that would be agreed upon by the bidders, which, in turn, requires postulating what happens if some bidder refuses to participate in a proposed manipulation. The latter depends on the beliefs formed by other bidders about the bidder who refuses a proposed manipulation and on their abilities to punish such a bidder.

To address these issues, we initially follow the notion of collusion-proofness originally developed by LM. According to this notion, an auction is *not* collusion-proof if it is an equilibrium for all bidders to accept a collusive agreement proposed by a third-party under the assumption that an out-of-equilibrium rejection of the proposal will not lead to revision of the prior belief about the values of rejecting bidders. Given that playing the auction under prior beliefs yields $U_i^{M_A}(v_i)$ while playing a certain manipulation provides $U_i^{\tilde{M}_A}(v_i)$, the notion of weak collusion-proofness is formalized as follows.

[14]This is consistent with MM and LM and most of the literature analyzing static models of collusion in auctions.

Definition 3. *Given an auction A, its collusion-free equilibrium outcome M_A is **weakly collusion-proof (or WCP)** if there exists* no *cartel manipulation $\tilde{M}_A$ of A satisfying* (IC) *and*

$$(C-IR) \qquad U_i^{\tilde{M}_A}(v_i) \geq U_i^{M_A}(v_i), \forall v_i, i, \text{ with strict inequality for some } v_i, i.$$

According to this definition, an auction is susceptible to bidder collusion if and only if there exists a cartel manipulation that interim Pareto dominates its collusion-free outcome.[15] This condition provides a reasonable test for the collusion-proofness of an auction rule. The presence of an interim Pareto dominating manipulation would make it a common knowledge that everyone will gain from collusion (as argued by Holmstrom and Myerson (1983)), making cartel-forming a clear cause for concern. By contrast, its absence would mean that no consensus exists among bidders to form a cartel.[16]

Our analysis focuses on weak-cartels. Weak cartels are characterized by two important restrictions on their behavior that differentiate them form strong-cartels. First, they are unable to use side payments to compensate losers. Second, they cannot reallocate the object once it leaves the seller's hand. While realistic in many settings, these limitations are non-trivial. Therefore, one might expect that strong cartels will always serve collusive bidders better then weak cartels. For instance, transfers could be used to prevent the sort of cartel unraveling described in Section 2, by providing compensation for high-value bidders and allowing them to join the cartel. This is not necessarily the case. As we formally show in Appendix A, *a winner-payable auction that is resistant to weak-cartels will be also resistant to strong-cartels, if bidders with values below the reserve price do not expect a positive payoff from joining the strong cartel.*

The latter condition is a natural one. MM were the first to recognize that a strong cartel may need to avoid making positive compensation to non serious bidders in case their entry into the cartel is difficult to restrict. The idea is that if a large pool of low-value bidders exists and positive compensation is offered to bidders that would never make a profit in the auction, then it would be difficult to prevent entry of such bidders into the cartel, and this would dissipate collusive rents for serious bidders. In light of this, following MM, we call

[15]Definition 3 implies that at least one type of one bidder must have a strict incentive to accept the cartel manipulation. If that was not the case, then the manipulation would yield exactly the same outcome as the original auction, including the same revenue for the seller. In this case, collusion would not be a concern.

[16]One could argue this test to be rather weak; namely, a weak cartel may still form despite an auction being WCP. For instance, a WCP requirement does not preclude a scheme that may benefit only a subset of bidders perhaps for some types, possibly at the expense of the other bidders. Also, collusion could be sustained by letting the cartel maintain arbitrary beliefs on the value of bidders who refuse to collaborate. To address this concern, a more robust notion of collusion-proofness is introduced later in Section 6.

entry exclusion constraint (or EEC) the requirement that in a strong cartels the expected utility of bidders with values below their reserve prices is zero.

The equivalence between weak cartels and strong cartels that satisfy the EEC means that the necessary and sufficient condition for an auction to be collusion-proof, which we will present in section 4, remains valid in the presence of strong cartels that satisfy the EEC. Furthermore, this implies that the optimal auction in the presence of collusion by weak-cartels characterized in section 5 is still optimal in the presence of strong cartels that satisfy the EEC.

4 When Are Auctions Susceptible to Weak Cartels?

In this section, we study the conditions that make a winner-payable auction weakly collusion-proof. First, we provide a necessary condition for an auction to be weakly collusion-proof (Theorem 1). Next we prove that this condition is also sufficient, when two further natural requirements are satisfied (Theorem 2). We begin by introducing one key definition.

DEFINITION 4. *For each $i \in N$, $r \in [\underline{v}_i, \overline{v}_i]$ and $v \in [r_i, \overline{v}_i]$, we define $G_i(v; r_i)$ as:*

$$G_i(v; r_i) := \max\{sF_i(v') + (1-s)F_i(v'') | s \in [0,1], v', v'' \in [r_i, \overline{v}_i], \text{ and } sv' + (1-s)v'' = v\}.$$

For fixed r_i, the function $G_i(\cdot; r_i)$ is the concave closure of F_i over $[r_i, \overline{v}_i]$, that is, the lowest concave function such that $G_i(v; r_i) \geq F_i(v)$ for all $v \in [r_i, \overline{v}_i]$. To simplify notation, we will henceforth write $G_i(\cdot; r_i)$ as G_i.[17] Figure 3 depicts the concave closure G_i for a value distribution F_i that has a single-peaked density. Concave closure G_i is always linear on regions where F_i is linear or convex, but it may also be linear in areas where F_i is concave.[18] Note that each concave function G_i admits density, denoted $g_i(v)$, for almost every $v \in [r_i, \overline{v}_i]$, whose derivative is well defined and satisfies $g'_i(v) \leq 0$ for almost every $v \in [r_i, \overline{v}_i]$. For each bidder i, we call $\mathcal{V}_i^0 := \{v \geq r_i | g'_i(v) = 0\}$ **susceptible types**—namely a subset of types above r_i where the concave closure is linear. In Figure 3, the susceptible types are an interval $[r_i, v^*(r_i)]$, while in general the set $\mathcal{V}_i^0$ is a collection of disjoint intervals.

The intuition provided in the introduction suggests that susceptible types are prone to a cartel manipulation. This intuition is formalized in the next theorem which provides a necessary condition for a winner-payable auction to be WCP .

[17] We stress that G_i depends not only on type distribution F_i but also indirectly on the on the specific auction rule, which determines r_i.

[18] In the case of single-peaked density, the upper bound of the linear segment, $v^*(r_i)$, is always above the peak of the density $\hat{v}$ and falls as r rises, an observation we shall come back to in Section 5.

Theorem 1. *If an equilibrium outcome M_A of an auction $A \in \mathcal{A}^*$ is weakly collusion-proof, then its interim allocation rule Q satisfies the following property:*

$$(CP) \qquad Q_i(v) = Q_i(v') \text{ if } [v, v'] \subset \mathcal{V}_i^0 \quad \forall v < v', \forall i \in N,$$

Proof. See Appendix B (page 35). ▮

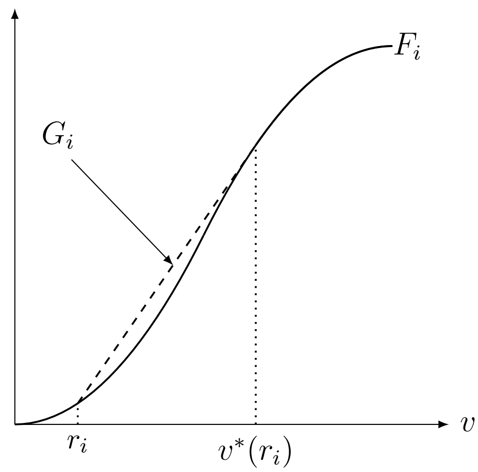


Figure 3: Type Distribution F and Its Concave Closure G

This result implies that if there is a positive measure of susceptible types for any bidder, then there is a scope for a profitable cartel manipulation unless that bidder's winning probability is constant within each interval of susceptible types. (The result is anticipated by the second example of Section 2, where the collusive manipulation involves pooling the types in the interval $[0, 1/2]$ in which G_i is linear.)

To see the logic behind this result, suppose that in (collusion-free) equilibrium, bidder i's winning probability $Q_i(v_i)$ is strictly increasing within some interval $[v, v']$ of susceptible types (i.e., $[v, v'] \subset \mathcal{V}_i^0$). Consider a maximal interval $[a, b] \subset \mathcal{V}_i^0$ that contains $[v, v']$. (In Figure 3, $[a, b] = [r_i, v^*(r_i)]$.) Then, one can construct a cartel manipulation, labeled $\tilde{M}_A$, that: (i) leaves unchanged the interim winning probability and expected payments of all bidders other than bidder i and also of bidder i when his value is outside $[a, b]$ and (ii) gives the good to bidder i with a constant probability $\bar{p}$ if his value is inside $[a, b]$, where

$$\bar{p} = \frac{\int_a^b Q_i(s) f_i(s) ds}{F_i(b) - F_i(a)}, \tag{5}$$

that is, bidder i's average winning probability over the interval $[a, b]$ in M_A.

Our proof in Appendix A shows that (i) this cartel manipulation can be implemented by the bidders in auction A; and (ii) it is acceptable to all bidders in the sense of satisfying $(C-IR)$, thus making the auction not weakly collusion-proof.

Let us first show (ii), recalling that bidders other than i are unaffected by the manipulation. Absent collusion, we know that bidder i will earn the interim payoff of

$$U_i^{M_A}(v_i) = U_i^{M_A}(a) + \int_a^{v_i} Q_i(s)ds, \tag{6}$$

when his valuation is $v_i \in [a,b]$.

Observe next that

$$\frac{\int_a^b Q_i(s)f_i(s)ds}{F_i(b)-F_i(a)} \geq \frac{\int_a^b Q_i(s)g_i(s)ds}{G_i(b)-G_i(a)} = \frac{\int_a^b Q_i(s)ds}{b-a}, \tag{7}$$

where the first inequality holds since $G_i(b) = F_i(b), G_i(a) = F_i(a)$, $G_i(v) \geq F_i(v)$, so $F_i(v)$ first-order stochastically dominates $G_i(v)$ conditional on $v \in [a,b]$, and the equality holds since G_i is linear.[19]

Under the manipulation $\tilde{M}_A$, bidder i with valuation $v_i \in [a,b]$ will earn

$$U_i^{\tilde{M}_A}(v_i) = U_i^{M_A}(a) + \int_a^{v_i} \bar{p}ds = U_i^{M_A}(a) + \frac{v_i - a}{F_i(b)-F_i(a)}\int_a^b Q_i(s)dF_i(s), \tag{8}$$

where the last equality follows from (5).

Next, note that the payoff in both (6) and (8) rises at a speed equal to the winning probability. Since Q_i rises strictly whereas $\bar{p}$ is constant, the payoff without manipulation is strictly convex whereas the payoff under manipulation is linear, as depicted by Figure 4. Essentially, the manipulation speeds up the rate of payoff increase for lower values and slows down the rate for higher values. As was seen in (7),

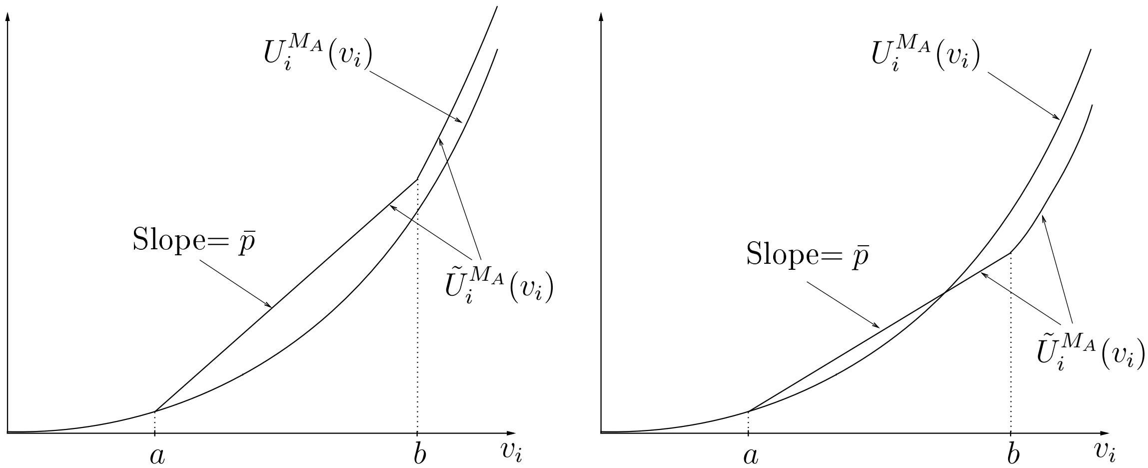
$$\int_a^b Q_i(s)dF_i(s) \geq \frac{F_i(b)-F_i(a)}{b-a}\int_a^b Q_i(s)ds. \tag{9}$$

Substituting (9) into (8) for $v_i = b$, we get

$$U_i^{\tilde{M}_A}(b) \geq U_i^{M_A}(a) + \int_a^b Q_i(s)ds = U_i^{M_A}(b). \tag{10}$$

In other words, bidder i with valuation $v_i = b$ will be at least weakly better off from the manipulation. Given the curvatures of these two payoff functions, the bidder will be strictly better off from the manipulation for any intermediate value $v_i \in (a,b)$ (see Figure 4a).

[19] Note that the two equalities, $G_i(a) = F_i(a)$ and $G_i(b) = F_i(b)$, follow from the fact that $[a,b]$ is a maximal interval in $\mathcal{V}_i^0$.



(a) G_i is linear on $[a, b]$ (b) G_i is strictly concave on $[a, b]$

Figure 4: Profitability of Manipulation

The same argument explains why this manipulation may not work if G_i is strictly concave (or equivalently, F_i is strictly concave). In this case, the inequality of (9) is reversed. Hence, as shown in Figure 4b, bidder i will be strictly worse off from the manipulation when his valuation is $v_i \approx b$. This will be formally shown in the next theorem.

To complete our argument, we need to verify (i), i.e. that the cartel can implement the desired manipulation. In fact, pooling the types of bidder i in $[a, b]$ requires shifting the winning probability away from high value types toward low value types of bidder i, and it is not clear whether and how such a shifting of the winning probabilities can be engineered, especially without altering the payoffs of the other bidders.

As a first step, we observe that the interim allocation from $\tilde{M}_A$, $\tilde{Q}$, is feasible in the sense that there is an ex-post allocation rule $\tilde{q}$ that gives rise to $\tilde{Q}$ as the associated interim allocation rule. The tricky part is how to replicate the interim transfer $\tilde{T}$, which makes $\tilde{M}_A$ incentive compatible, along with the above allocation $\tilde{q}$ via a weak cartel manipulation (that does not use any side payments among the cartel members). The winner-payability plays a role here: it allows the cartel to find, for each profile of reported values, a distribution of bids that produces $\tilde{q}$ and $\tilde{T}$ (in expectation) for the proposed manipulation.[20]

Theorem 1 suggests that a winner-payable auction which assigns the object with higher probability to bidders with higher values is vulnerable to weak cartels unless each bidder's value distribution is strictly concave wherever the object is allocated with a positive prob-

[20]Winner-payability is sufficient for the cartel to attain any incentive compatible allocation for values above reserve prices. Therefore, focusing on a set of auctions larger than $\mathcal{A}^*$ would not make collusion any easier for the cartel.

ability. The following three corollaries state (under certain technical qualifications) that (i) standard auctions, (ii) revenue maximizing auctions (i.e. those which implement Myerson's optimal auction), and (iii) efficient auctions are all susceptible to weak cartels unless all distributions of values are strictly concave.

Corollary 1. *Letting* $\overline{v} := \min_{i \in N} \overline{v}_i$ *and* $\underline{v} := \max_{i \in N} \underline{v}_i$*, assume that* $\overline{v} > \underline{v}$*. Then, the collusion-free equilibrium outcomes (in weakly undominated strategies) of first-price, second-price, English, or Dutch auctions, with a reserve price* $r < \overline{v}$*, are not WCP if* G_i *is linear in some interval* $(a, b) \subset \mathcal{V}_i$ *with* $b > r$ *and* $a \geq \underline{v}$ *for some bidder* i*.*

Proof. See Appendix A (page 39). ▮

Corollary 2. *Suppose that the virtual valuation,* $J_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$*, is strictly increasing in* v_i *for all* $i \in N$*. Suppose also that* G_i *is linear in some interval* $(a, b) \subset (r_i, \overline{v}_i]$*,* $J_i(b) > 0$*, and* $\max_{j \neq i} J_j(\underline{v}_j) < J_i(b) < \max_{j \neq i} J_j(\overline{v}_j)$*, for some bidder* i*. Then, all auction rules in* $\mathcal{A}^*$ *that maximize the seller's revenue are not WCP.*

Proof. The hypotheses guarantee that there exists an interval $[b - \epsilon, b]$ with $\epsilon > 0$, where $Q_i(v_i)$ is strictly increasing in the optimal auction. The result follows from Theorem 1. ▮

Corollary 3. *Suppose that* G_i *is linear in some interval* $(a, b) \in (r_i, \overline{v}_i]$ *and* $\max_{j \neq i} \underline{v}_j < b < \max_{j \neq i} \overline{v}_j$*, for some bidder* i*. Then, all auction rules in* $\mathcal{A}^*$ *whose equilibrium outcomes are efficient are not WCP.*

Proof. The hypotheses guarantee that there exists an interval $[b - \epsilon, b]$ with $\epsilon > 0$, where $Q_i(v_i)$ is strictly increasing in any efficient auction. The result follows from Theorem 1. ▮

The next result establishes a converse of Theorem 1: a sufficient condition for an auction rule to be weakly collusion-proof. The sufficiency requires two further conditions that, roughly speaking, represent minimal optimality requirements from the seller's perspective.

Theorem 2. *Suppose (a) that an auction rule* $A \in \mathcal{A}^*$ *satisfies* $r_i \geq \underline{v}_i$*, and (b) that* $\sum_{i \in N} q_i(v) = 1$ *if* $v_i \geq r_i$ *for at least one* $i \in N$*. Then, the* M_A *is weakly collusion-proof if it satisfies* (CP)*.*

Proof. See Appendix B (page 40). ▮

The two conditions rule out auctions that are clearly undesirable from the seller's perspective, because these either leave the object unsold even though selling would raise her revenue without sacrificing incentives, or sell the object to some bidder at a price below his lowest possible value. Given these additional optimality conditions, the intuition behind this result is essentially the flip-side of the intuition behind Theorem 1. In other words, starting from the suggested equilibrium, any manipulation including, but not limited to, those that involve pooling of some types, must leave some bidder types strictly worse off.

Theorem 2 has the following immediate corollary, which collects in a single statement the natural counterparts to the three previous corollaries to Theorem 1.

Corollary 4. *If F_i is strictly concave for all $i \in N$, then the following auctions are WCP: (i) the collusion-free equilibrium equilibria of first-price, second-price, English, or Dutch auctions, with reserve price $r \geq \max_{i \in N} \underline{v}_i$ (ii) any equilibrium of any auction with $r_i \geq \underline{v}_i, \forall i$ that results in an efficient allocation, and (iii) any equilibrium of any auction that maximizes the seller's revenue.*

Proof. The proof is immediate given Theorem 2 and the fact that there is no interval in $\mathcal{V}_i$ for any $i \in N$ where G_i is linear. ▮

Our characterization of collusion-proof auctions in Theorem 1 and 2 contrasts with that of MM, who assume the cartel can successfully form if bidders benefit ex-ante from collusion. They show that if the hazard rates of the value distributions are increasing, then a cartel will always form, implementing a random allocation for all bidders types. In contrast, our theory suggests, and the example of Section 2 illustrates, that this need not be the case once we take into consideration the bidders' interim incentives to participate in the cartel. That is, ignoring the latter could significantly overstate the incidence of cartel formation.

Even when an auction is susceptible to a weak cartel, our theory leads to cartel behavior that is different from what MM suggests. In particular, the necessity to reward high-value bidders may cause the cartel to refrain from implementing a random allocation for all bidder types. How a cartel would behave when an auction is not weakly collusion-proof can be understood using Theorems 1 and 2. To do so, let's define a cartel manipulation as **optimal** if the manipulation is *not* interim Pareto dominated by another manipulation (i.e., if there is no other cartel manipulation which yields weakly higher payoff for all types of all bidders, and strictly higher payoff for a positive measure of types for at least one bidder). Then we have the following corollary, stated without proof.

Corollary 5. *Assume the an auction $A \in \mathcal{A}^*$ satisfies the conditions (a) and (b) in the statement of Theorem 2. Then a cartel manipulation of A is optimal if and only if the associated interim allocation satisfies condition (CP).*

To the extent that the assumption that the cartel operates in a Pareto efficient way is a mild one, Corollary 5 gives rise to a "positive" theory of cartel behavior. Specifically, it suggests that the the property emphasized in Theorem 1 and Theorem 2 should be an equilibrium feature of any standard (i.e., winner payable) auction with collusive bidders, *whether the auctioneer intervenes to deter collusion or not*.

By Corollary 5, one must expect pooling to arise *at least* for intervals of susceptible types. Yet, an optimal cartel manipulation need not limit random allocation of the object to the intervals in the type space to which condition (CP) applies, and this fact has an important consequence. To see the point, recall the triangular distribution in the second example of Section 2 and suppose the seller holds a standard second-price auction. With $r_i = 0$, the susceptible types are $[0, 1/2]$ and Section 2 exhibits an optimal cartel manipulation that induces a random allocation in that interval. However, one can find optimal cartel manipulations that implement random allocations in strictly larger sets; in fact, for any $\tilde{v} \in [1/2, 0.76]$ there exists a manipulation that implements a random allocation in $[0, \tilde{v}]$.[21] It follows that if in this example the auctioneer simply holds a second price auction, then a random allocation may arise for all types in $[0, 0.76]$. On the other hand, if the auctioneer deliberately induces a random allocation only for those types in $[0, 1/2]$, then no further cartel manipulation would occur, since the auction of the seller is now WCP. Hence, because random allocations are inefficient, there is a sense in which an active intervention by the auctioneer is desirable. This brings us to the problem of designing an optimal auction in the presence of collusion, which we take up in the next section.

5 Optimal Collusion-Proof Auctions

Corollary 2 shows that in a wide range of circumstances, the auction that maximizes the seller's revenue in the absence of collusion will not be collusion-proof. If an auction is not WCP, then bidders can gain from coordinating their bidding strategies and this will typically leave the seller worse off. Therefore, in this section, we look for an auction that maximizes the seller's revenue among all winner-payable WCP auctions.[22] Thanks to Lemma 1, we can focus on a direct auction mechanism.

Let us write $J_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ for the virtual valuation of bidder i with value v_i, and henceforth assume, as standard, that it is strictly increasing. Then, we can write the seller's

[21]While switching from an optimal manipulation to another manipulation must make some bidder worse off, observe that the larger the pooling area, the larger is the ex-ante welfare of bidders. This follows from the analysis of MM by observing that the hazard rate for the chosen triangular distribution is increasing.

[22]Corollary 5 implies that there is no loss of generality in restricting attention to WCP auctions where bidders do not manipulate the outcome (i.e., the *collusion-proofness principle* of LM applies in this setting).

maximization problem as

$$[P] \qquad \max_{(q_i,t_i)_{i\in N}} \sum_{i\in N} \int_{r_i}^{\overline{v}_i} J_i(v_i)Q_i(v_i)dF_i(v_i)$$

subject to the following constraints: (M) and (Env) (for incentive compatibility); (CP) (for collusion-proofness); and $r_i = \inf\{\frac{t_i(v)}{q_i(v)} \,|\, q_i(v) > 0\}$ (for definition of r_i as in (3)).

The objective function represents the seller's expected revenue.[23] The constraint (CP) arises from the characterization given by Theorems 1 and 2. It can be easily verified that the solution of the program $[P]$, which is presented in Theorem 3 below, also satisfies the conditions (a) and (b) in Theorem 2, which means that its equilibrium outcome is WCP.

Our first result identifies the optimal weakly collusion-proof auction up to the choice of the reserve prices $(r_1, \dots, r_n)$. To state our result, we need to introduce some further notations. Recall first that $\mathcal{V}_i^0 \subset [r_i, \overline{v}_i]$ denotes the set of susceptible types. Note that $\mathcal{V}_i^0 \subset [r_i, \overline{v}_i]$ is a disjoint union of countably many intervals $I_i^k = [a_i^k, b_i^k], k \in K_i \subset \mathbb{N}$, in each of which G_i is linear. Then, for every bidder i, we define the "ironed" virtual valuation as follow:

$$\bar{J}_i(v_i) := \begin{cases} J_i(v_i) & \text{if } v_i \in \mathcal{V}_i \backslash \mathcal{V}_i^0 \\ \frac{\int_{a_i^k}^{b_i^k} J_i(s)dF_i(s)}{F_i(b_i^k) - F_i(a_i^k)} & \text{if } v_i \in I_i^k \text{ for some } k \in K_i. \end{cases} \tag{11}$$

The ironed virtual value function is constant within each interval I_i^k for which (CP) requires bidder i to receive the object with a constant probability. For any value in I_i^k, it coincides with the conditional expected value of the virtual valuation in that interval.[24]

The following result shows that the optimal allocation rule is the one that, under optimal reserve prices, always assigns the object to the bidder with the highest ironed virtual value. As standard, the payoff equivalence allows us to focus on the allocation rule only.

THEOREM 3. *For any $v \in \mathcal{V}$, let $W(v) := \{j \in N \,|\, \bar{J}_j(v_j) = \max_{k\in N} \bar{J}_k(v_k)\}$ and let $\#W(v)$ denote the cardinality of this set. Then, there is an auction rule $(q_i^*, t_i^*)_{i\in N}$ that solves $[P]$*

[23] It is well known that the expression is obtained by substituting for the payments T_i into the original objective function using the condition (Env) and noting that (IR') must be binding at the optimum for the lowest types, i.e., for all $i \in N$, $T_i(\underline{v}_i) = \underline{v}_i Q_i(\underline{v}_i)$. In the above expression we also used the facts that $Q_i(v_i) = 0$ for all $v_i < r_i$.

[24] The idea of ironing is in the spirit of Myerson (1981). In our case, ironing is needed to deal with the collusion-proofness constraint even though the virtual valuation is increasing; in Myerson (1981), ironing is required to satisfy the the monotonicity constraint that becomes binding in regions where the virtual value is decreasing.

such that $r_i \geq J_i^{-1}(0), \forall i \in N$, *and where*

$$q_i^*(v) = \begin{cases} 0 & \text{if } v_i < r_i \text{ or } \bar{J}_i(v_i) < \max_{j \in N} \bar{J}_j(v_j) \\ \frac{1}{\#W(v)} & \text{if } v_i \geq r_i \text{ and } i \in W(v). \end{cases} \tag{12}$$

Proof. See Appendix C (page 42). ▮

The theorem characterizes the optimal WCP auction up to the choice of reserve prices. Therefore, once the allocation is chosen according to Theorem 3 for each $(r_1, \ldots, r_n)$, the revenue maximizing WCP auction is obtained by choosing $(r_1, \ldots, r_n)$ to maximize the resulting objective function in $[P]$. Note that $r_i \geq J_i^{-1}(0)$ for all $i \in N$ follows immediately as inspection of $[P]$ reveals that it is never optimal for the seller to sell to bidders with negative virtual valuations.

We can obtain a more complete characterization of the optimal auction by focusing on some special cases. We discuss, in turn, the case of nondecreasing densities and the case of symmetric bidders with single-peaked density.[25]

5.1 Monotone Nondecreasing Densities

If all bidders have nondecreasing densities, then for any $r_i \in [\underline{v}_i, \overline{v}_i]$, the function G_i will be linear in $[r_i, \overline{v}_i]$. Hence, bidder i must expect a constant probability of obtaining the object for all his values above r_i. This implies that the seller's problem takes on a much simpler form.

Corollary 6. *Suppose that all* f_i*'s are nondecreasing. Then, the program* $[P]$ *simplifies to*

$$\max_{(r_i)_{i \in N}} \Big[\sum_{i \in N} \Big(\prod_{j : \pi(j) < \pi(i)} F_j(r_j) \Big) (1 - F_i(r_i))\, r_i \Big], \tag{13}$$

where $\pi : N \to N$ *can be any permutation function that satisfies* $\pi(j) < \pi(i)$ *if* $r_j > r_i$.

Proof. See Appendix C (page 44). ▮

Interestingly, Corollary 6 suggests that the optimal WCP auction can be implemented via a sequential negotiation process. Bidders are ordered from first to last and the seller approaches them in sequence and makes them take-it-or-leave-it offers. If bidder i refuses the offer, the seller proceeds to make an offer to the next bidder; and the process continues until either some bidder j accepts an offer and pays r_j to the seller, or the object remains

[25] We have already argued that if all bidders have monotone *decreasing* density, then the Myerson's revenue maximizing auction is WCP (see Corollary 4).

unsold. Not surprisingly, the seller's optimal offer falls with each rejected offer and the last bidder in the sequence must receive an offer at price $J_i^{-1}(0)$.[26]

This result suggests that the seller can compute optimal reserve prices, recursively, for all possible orders of bidders and then select the order that is optimal. The order becomes irrelevant when bidders are ex-ante symmetric, and the problem is further simplified in this case as illustrated by the next corollary, which is stated without proof.

COROLLARY 7. *Suppose that $f_i = f$ for all $i \in N$ and f is nondecreasing. Then, it is an optimal WCP auction to approach all bidders in sequence (i.e., in any arbitrary sequence) and offer to the k-th buyer a price r_k that maximizes*

$$r_k(1 - F(r_k)) + F(r_k)V_{n-k},$$

where V_{n-k} is the revenue the seller gets from an subproblem dealing only with $n - k$ bidders (and $V_0 = 0$).

One insight that emerges from this corollary is that, contrary to Myerson (1981), reserve prices in an optimal WCP auction may be different even when bidders are ex-ante symmetric. To see this point, consider the recursive nature of the problem. It is straightforward to see that $0 = V_0 < V_1 < \cdots < V_{n-1}$, which implies that $r_n = J_n^{-1}(0) < r_{n-1} < \cdots < r_1$. Therefore, the optimal reserve price charged to bidder i will be different from the one charged to bidder j, for any $i, j \in N$.

The optimality of treating ex ante identical bidders asymmetrically extends beyond this case. Because virtual valuations are strictly increasing, optimal price discrimination calls for assigning the object to bidders with the highest values. However, the collusion-proof constraint makes this allocation infeasible. The asymmetric allocation, implicit in the sequential negotiation, accomplishes partial price discrimination without violating the collusion-proof constraint.

5.2 Single-Peaked Density

Suppose now that bidders are ex-ante symmetric and that the (common) virtual valuation J is strictly increasing. In addition, assume that the (common) density f is (weakly) increasing in $[\underline{v}, \hat{v}]$ and strictly decreasing $[\hat{v}, \overline{v}]$ for some $\hat{v} \in [\underline{v}, \overline{v}]$. Let $\hat{v} > r^M := J^{-1}(0)$ to avoid the trivial result in which the Myerson's optimal auction is WCP. Observe that for any $r_i \geq \hat{v}$,

[26] This is the optimal take-it-or-leave-it offer for a single bidder, which also corresponds to i's reserve price in the Myerson's optimal auction without collusion.

one can find some $v^*(r_i) \leq \hat{v}$ such that $V_i^0 = [r_i, v^*(r_i)]$ (refer to Figure 3), and also that $v^*(r_i)$ is decreasing in r_i while $v^*(\hat{v}) = \hat{v}$.[27] Let v_i^* denote $v^*(r_i)$ to simplify notation.

Treating the single-peaked density case allows us to illustrate how the choice of the reserve prices interacts with the (endogenous) level of ironing at the optimal mechanism. Moreover, the case of single-peaked density covers a general class of many plausible and well-known distributions, including Uniform, Triangular, Cauchy, Exponential, Logistic, Normal, and Weibull.

General Auctions. We can use Theorem 3 to identify the optimal WCP auction under the assumptions stated above. To simplify the explanation, we focus on the case of two bidders. We discuss the extension to multiple bidders at the end of the section.

To begin, consider a reserve price $r_i \in [r^M, \hat{v}]$ for $i = 1, 2$.[28] Without loss of generality, let $r_1 \geq r_2$ and thus $v_1^* \leq v_2^*$. Since $r_i \leq \hat{v}$, by (CP), all types in $[r_i, v_i^*]$ should have the same priority in the allocation, where $v_i^* \geq \hat{v}$ is the upper bound of the linear part of G_i on $[r_i, \overline{v}]$. Then, the following corollary is a direct consequence of Theorem 3 and is therefore stated without proof.

Corollary 8. *For given r_1, r_2 with $r_1 \geq r_2$ and $r^M \leq r_i \leq \hat{v}$ for all $i = 1, 2$, the optimal mechanism takes on one of two forms:*

(a) If $\bar{J}(r_1) \geq \bar{J}(r_2)$ the optimal allocation is the one depicted in Figure 5a;

(b) If $\bar{J}(r_1) < \bar{J}(r_2)$ the optimal allocation is the one depicted in Figure 5b,

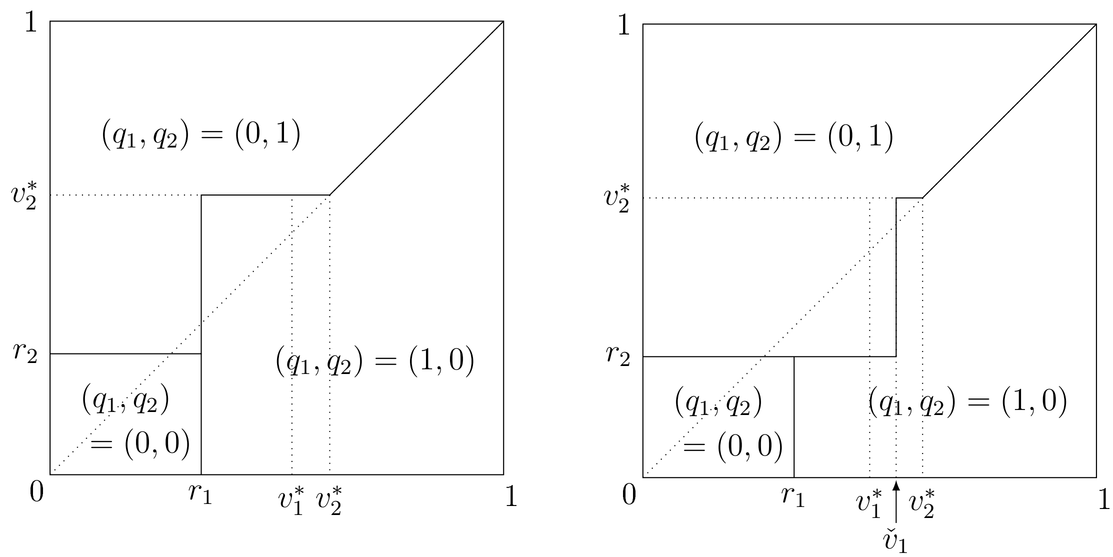
where $\bar{J}$ is as defined in (11).

The optimal mechanism assigns the object efficiently whenever any bidder has value higher than v_2^*; no bidder obtains the object if both bidders have values below their reserve prices; if only one bidder has value below his reserve price, then the other bidder obtains the object. If $v_1 \in [r_1, v_2^*]$ and $v_2 \in [r_2, v_2^*]$, then the optimal allocation depends on $\bar{J}(r_1)$ and $\bar{J}(r_2)$. If $\bar{J}(r_1) \geq \bar{J}(r_2)$, then bidder 1 receives the good as long as $v_1 \in [r_1, v_2^*]$.[29] (In this case, the optimal mechanism can be implemented via a standard auction with reserve price, followed by a sequence of two take-it-or-leave-it offers with prices being r_1 and r_2, should the object go unsold at the auction.) On the other hand, if $\bar{J}(r_1) < \bar{J}(r_2)$ and $v_2 \in [r_2, v_2^*]$,

[27] The proof of these statements is straightforward and thus omitted.

[28] We have already explained the first inequality. The second inequality must hold since the collusion-proofness constraint is not binding when $r_i > \hat{v}$ and J is strictly positive and increasing in $[\hat{v}, \overline{v}]$.

[29] Note that 1 obtains the object when $v_1 \in [v_1^*, v_2^*]$ and $v_2 \in [r_2, v_2^*]$ because $\bar{J}(v_1) = J(v_1) > \bar{J}(r_1) \geq \bar{J}(r_2) = \bar{J}(v_2)$.



(a) $\bar{J}(r_1) \geq \bar{J}(r_2)$

(b) $\bar{J}(r_1) < \bar{J}(r_2)$

Figure 5: Ex-post Allocation in the Optimal WCP Auction with $n = 2$ and $r_1 \geq r_2$

bidder 2 obtains the good if $v_1 < v_1^*$; instead, when $v_1 \in [v_1^*, v_2^*]$, the object goes to bidder 2 if $v_1 < \check{v}_1$ and goes to bidder 1 if $v_1 > \check{v}_1$, where $\check{v}_1 := \inf\{v_1 \in [v_1^*, v_2^*) | J(v_1) \geq \bar{J}(r_2)\}$.[30]

Example 1. To fix ideas, consider a single-peaked density with the peak at 1/2, which is given as $f(v) = 4v$ if $v \in [0, 1/2]$ and $f(v) = 4(1-v)$ if $v \in [1/2, 1]$. The optimal reserve prices are $r_1^* = 0.433$ and $r_2^* = 0.416$. These reserve prices induce cutoffs $v_1^* = 0.528, v_2^* = 0.535$. The average virtual values are $\bar{J}(r_1^*) = 0.197 > 0.184 = \bar{J}(r_2^*)$. Hence, an optimal mechanism allocates the object according to Figure 5a.

It is not difficult to see that the method used above extends to the case of more than two bidders, but the number of possible allocations one must consider grows large very quickly as the number of bidders increases. For instance, with three bidders, a given profile of reserve prices may give rise to six possible candidate optimal allocations.

Symmetric Auctions. There are many cases in which a seller, particularly government agency, may be compelled to treat the bidders in a nondiscriminatory manner. Theorem 3 allows a natural characterization of the optimal WCP auction even in this case. Formally, in addition to the assumptions that bidders are ex-ante symmetric and that the density is single-peaked, we impose a *symmetry* requirement that $Q_i = Q$ for all $i \in N$.[31]

[30] Note that $\check{v}_1$ is guaranteed to exist because J is strictly increasing.

[31] Deb and Pai (2013) study auctions where the allocation and payment rule cannot depend on the identity of bidders and show that almost any interim allocation can be implemented using anonymous auctions. In

Under the stated assumptions, for any reserve price $r \leq \hat{v}$ (which must be the same for all bidders), there is a value $v^*(r) \in [\hat{v}, \overline{v}]$ such that the concave closure of F on the interval $[r, \overline{v}]$ is linear in $[r, v^*(r)]$ and strictly concave in $[v^*(r), \overline{v}]$.

COROLLARY 9. *The allocation that solves [P] under the additional symmetry restriction is:*

$$q_i^*(v) = \begin{cases} \frac{1}{\#\{j \in N \mid v_j = \max_{k \in N} v_k\}} & \text{if } v_i = \max_{j \in N} v_j > v^*(r) \\ \frac{1}{\#\{j \in N \mid v_j \in [r, v^*(r)]\}} & \text{if } v_i \in [r, v^*(r)] \text{ and } \max_{j \in N} v_j \leq v^*(r) \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where r is a value in $(r^M, \hat{v}]$.

Proof. See Appendix C (page 44). ▮

Again, the characterization here is up to the choice of the reserve price. The auction allocates the object efficiently when bidders have high valuations, but allocates it randomly at a fixed price when bidders have low values. The optimal reserve price exceeds r^M since the region of efficient allocation $[v^*(r), \overline{v}]$ expands as r rises (and the seller benefits from this expansion). Formally, suppose the reserve price is raised from $r = r^M$ to $r^M + \varepsilon$. This entails only a second-order loss from withheld sale to the types in $[r^M, r^M + \varepsilon]$ since in that region virtual values are close to zero, but it results in the object being allocated efficiently among types $[v^*(r^M + \varepsilon), v^*(r^M)]$, which generates a first-order gain.

To conclude, we observe that the optimal symmetric collusion-proof auction given by (14) can be implemented by the following simple mechanism: letting $\overline{Q}$ $(= \frac{F(v^*)^n - F(r)^n}{n(F(v^*) - F(r))})$ denote the constant winning probability for type $v \in [r, v^*(r)]$, run a first or second-price auction with a minimum price R that solves $(v^*(r) - r)\overline{Q} = (v^*(r) - R)F(r)^{n-1}$; if the object goes unsold in the auction, then offer a posted price sale at price r.[32]

6 Strengthening the Notion of Collusion-Proofness

The weak notion of collusion-proofness presumes that a cartel will form if, and only if, all bidders benefit at least weakly from coordinating their bids. This provides a conservative test on the susceptibility of an auction to bidder collusion; if an auction fails to be weakly collusion-proof, there will be a consensus among bidders to form a cartel and manipulate

contrast, we require the expected final outcome to be nondiscriminatory for ex-ante identical bidders.

[32]Observe that the type $v^*(r)$ are indifferent between obtaining the object in the auction at the minimum price R and obtaining it in the posted-price sale at price r.

the auction. At the same time, because consensus (i.e., common knowledge about the profitability of colluding) is a strong requirement, there is still the possibility that even weakly collusion-proof auctions may be susceptible to collusion.

In this section, we show that the optimal weakly collusion-proof mechanism identified in the previous section can be made unsusceptible to collusion in a much stronger sense. To this end, we stack the deck against the seller by taking a quite permissive approach on how weak cartels form and behave. First, any informed bidder(s) as well as an uninformed mediator is allowed to propose a cartel manipulation. Second, the cartel formation need not be all-inclusive; so there can be partial or multiple cartels in operation. Also, bidders need not unanimously agree to form a cartel, in the sense that after some bidders reject a cartel proposal, the remaining bidders can form an alternative cartel. Further, if a bidder refuses to participate, the remaining bidders may punish the refusing bidder. We then show that the outcome of the optimal collusion-proof auction identified in the previous section can be implemented even if cartels can form and behave as outlined above, as long as cartel members plays only cartel-undominated strategies—a notion which is formalized in the next paragraph.

Take an auction $A \in \mathcal{A}^*$ and let $u_i^A(b \mid v_i) := v_i\xi_i(b) - \tau_i(b)$ for any bid profile $b \in \mathcal{B}$, $i \in N$ and $v_i \in \mathcal{V}_i$. For any potential cartel $C \subset N$, let $b_C = (b_i)_{i \in C}$ and $b_{N \setminus C} = (b_i)_{i \in N \setminus C}$ denote two arbitrary bid profiles for bidders within C and bidders outside C, respectively. Then, we say a bid profile b'_C **cartel-dominates** another profile b''_C at v_C if

$$u_i^A(b'_C, \tilde{b}_{N \setminus C}|v_i) \geq u_i^A(b''_C, \tilde{b}_{N \setminus C}|v_i), \forall \tilde{b}_{N \setminus C}, \forall i \in C$$

with strict inequality for at least one $i \in C$ and one $\tilde{b}_{N \setminus C}$. We say that a bid profile b_C is **cartel-undominated** at v_C if there is no profile b'_C that cartel-dominates it.[33]

We now describe a **cartel-game** and present our notion of robust collusion-proofness as a property concerning all equilibrium outcomes of the cartel game. A cartel game starts after the seller has announced auction A. All bidders and uninformed third parties are allowed to propose cartel agreements to all other bidders or subsets of them. Analogous to our earlier definition of an all-inclusive cartel agreement, an agreement specifies a mapping from reports to lotteries over bids for the participating bidders. However, the agreement in this case also specifies which agreement comes into force among accepting bidders depending on the set of accepting and rejecting bidders. With all available proposals, each bidder decides which one proposal, if any, to accept.[34] Decisions are simultaneous. If a bidder accepts a cartel

[33]Observe that this condition impose less restrictions on cartel behavior than requiring that every bidder refrains from playing weakly dominated strategies.

[34]It is not necessary to assume that a bidder observes the identity of the bidder proposing the collusive

proposal, then he commits to it. We allow bidders who have proposed a cartel to reject their own proposal in favor of a better one. We make no assumption on whether or not acceptance/rejection is observable and by whom. Subsequently, bidders play the auction A. As before, we maintain that cartels are enforceable and implemented by trustworthy third parties.

DEFINITION 5. *An auction A with (interim) equilibrium outcome $(Q_i, T_i)_{i\in N}$ is* ***robustly collusion-proof (or RCP)*** *if there exists no equilibrium outcome of a cartel-game following the announcement of auction A which is different from $(Q_i, T_i)_{i\in N}$ for at least one $i \in N$ and a positive measure of $v_i \in \mathcal{V}_i$, and where cartel-undominated strategies are played at any history.*

Finally, we now state the main result of this section, which shows that the optimal allocation rule identified in section 3, coupled with the canonical payment rule from Myerson (1981) gives rise to an RCP mechanism.

THEOREM 4. *The direct mechanism (q^*, t^*) where q^* is the optimal allocation rule (12) given in Theorem 3 and, for all $i \in N$,*

$$t_i^*(v) = q_i^*(v)v_i - \int_{\underline{v}_i}^{v_i} q_i^*(s_i, v_{-i})ds_i,$$

is an RCP mechanism.

Proof. See Appendix D (page 46). ∎

7 Conclusion

The current paper analyzes weak cartels in auctions. Unlike the seminal work by McAfee and McMillan (1992), we explicitly consider the interim incentives of bidders to participate in a cartel. This perspective leads to a different characterization of when auctions are susceptible to a weak cartel, how a cartel would operate if it is active, and how the auctioneer should respond to a weak cartel.

We show that a large class of auction rules, called winner-payable, are susceptible to a weak cartel if and only if they seek to implement non-constant allocation for any interval of susceptible types—types for which the concave closure of value distribution is linear. This characterization stands in sharp contrast to the existing theory of MM. While the latter suggests that a (first-price sealed-bid) auction is susceptible to a cartel whenever a

agreement.

bidder's value distribution satisfies a nondecreasing hazard rate, a condition satisfied by most standard distribution functions, the condition we identify for vulnerability to a cartel is much stronger, and not necessarily satisfied by all standard distribution functions. In particular, standard auctions as well as classical revenue-maximizing auctions are never susceptible to weak cartels if bidders' distribution functions are strictly concave. Furthermore, we use our characterization to identify optimal weakly collusion proof auctions. Our result suggests that an extension of the classical Myerson approach is optimal in the presence of a cartel.

Our analysis of a weak cartel focuses on the informational issue associated with a profitable cartel manipulation—i.e., how a cartel overcomes informational asymmetry among its members—but abstracts from the enforcement issues associated with carrying out the manipulation. In practice, the ease with which a cartel can enforce its agreement will also influence the extent to which an auction outcome can be manipulated. For instance, it is well known that an open ascending auction or second-price auction allows a cartel member to carry out its punishment against a deviator more easily than a first-price auction, and the difference could very well make the former more vulnerable than the latter to a cartel.[35] This aspect deserves further inquiry.

[35]See Robinson (1985) and Marshall and Marx (2007).

A Strong Cartels

We begin by introducing the notion of strong cartel manipulation.

Definition 6. *A direct mechanism* $\tilde{M}_A = (\tilde{q}, \tilde{t})$ *is a* ***strong cartel manipulation*** *of* A *if there exists a cartel agreement* α *and side payment rules* $(\mu_1, ..., \mu_n) : \mathcal{V} \to \Re^n$ *satisfying* $\mathbb{E}[\sum_i \mu_i(v)] \leq 0$ *such that*

$$\sum_{i \in N} \tilde{q}_i(v) = \mathbb{E}_{\alpha(v)}[\sum_{i \in N} \xi_i(b)] \quad and \quad \tilde{t}_i(v) = \mathbb{E}_{\alpha(v)}[\tau_i(b)] + \mu_i(v). \tag{15}$$

Comparing (15) with (4) reveals the way in which a strong cartel relaxes the limitation of a weak cartel. First, it allows side payments to be exchanged among bidders as long as they satisfy ex ante budget balancing (second term of (15)). Second, the strong cartel can implement any allocation rule that can be attained via some manipulation of bids *and* possible reallocation after the initial assignment (the first term of (15)). Note that (15) requires the total probability of assignment to be attainable via some profile of bids that the cartel coordinates on. In other words, the cartel is free to reallocate the good to its chosen member once the good is assigned by the seller.

As in the case of a weak cartel manipulation, it is without loss to assume that $\tilde{M}_A$ is incentive compatible, i.e. satisfies (IC). Given the definition of strong cartel manipulation, the definition of weakly collusion-proofness in the presence of strong cartels obtains naturally from the one for weak cartels.

Definition 7. *Given an auction* A*, its collusion-free equilibrium outcome* M_A *is* ***weakly collusion-proof in the presence of a strong cartel*** *(or WCP-S) if there exists* no *strong cartel manipulation* $\tilde{M}_A$ *of* A *satisfying* (IC) *and* $(C-IR)$*.*

We shall be interested in a strong cartel under a behavioral restriction—namely that a cartel never pays off bidders whose valuations are their reserve prices.

Definition 8. *We say that a cartel manipulation* $\tilde{M}_A$ *satisfies the* ***entry-exclusion constraint*** *if*

$$(EEC) \qquad\qquad U^{\tilde{M}_A}(r_i) = 0, \forall i \in N.$$

We refer to MM for a formal justification of (EEC) as a constraint on strong cartel behavior. In short, the constraint would arise in a model in which a large group of non-serious fringe bidders, all with no or small value for the object, could mimic serious bidders and request to join the cartel. In that case, if bidders with no chance of winning the auction were offered a positive compensation, profits for all cartel members would shrink to zero as a result of entry of a large number of bidders into the cartel.

The main result of this section is that, given (EEC), a strong cartel is reduced to a weak cartel—namely, a cartel can never collude more profitably with the help of side payments and reallocation than without, as long as it is either unable or unwilling to pay off the bidders with values below the reserve price.

Corollary 10. *Assume the an auction $A \in A^*$ satisfies the conditions (a) and (b) in the statement of Theorem 2. Then, an outcome M_A of A is WCP in the presence of a weak cartel* if and only if *it is WCP in the presence of a strong cartel that satisfies* (EEC)*.*

Proof. Given (b), any (weak) cartel manipulation must satisfy (EEC). Hence, if a M_A is WCP in the presence of a strong cartel that satisfies (EEC), then it is WCP in the presence of a weak cartel. To prove the converse, suppose M_A is WCP in the presence of a weak cartel. Then, it must satisfy (CP) by Theorem 1. Then, the proof of Theorem 2 shows that given (CP) (in addition to (b)), no strong cartel manipulation satisfying (EEC) satisfies both (IC) and $(C-IR)$. ▮

This result shows that side payments are of no help in relaxing incentive constraints for bidders, as long as the side payments are not used to cross-subsidize noncompetitive bidders. The key reason is that informational asymmetry among cartel member restricts the way the gains from collusion can be redistributed among bidder types, even with the help of side payments. The well-known payoff equivalence suggests that, once the allocation is fixed, high valuation bidders cannot be made better off unless low valuation bidders are equally made better off.

Theorem 2 of MM asserts that the ex ante optimal collusion under a weak cartel continues to be the optimal manipulation for a strong cartel satisfying (EEC). While the current result has the same flavor, there are several notable differences. First, as emphasized before, MM do not consider bidders' interim incentives to participate in a cartel, so the role side payments may play in offering participation incentives is different. Second, their result focuses on the first-price sealed-bid auction, whereas the current result applies to all winner-payable auctions. In fact, the nontrivial aspect of the proof pertains to the generality of the class of auctions considered.

B Proofs for Section 3

Proof of Lemma 1: Let (Q,T) be the interim rule that corresponds to $M = (q,t)$. Let us define $v_i^0 = \inf\{v_i \in \mathcal{V}_i \,|\, Q_i(v_i) > 0\}$ and

$$B_i(v_i) = \begin{cases} \frac{T_i(v_i)}{Q_i(v_i)} & \text{if } v_i > v_i^0 \\ v_i^0 & \text{if } v_i \le v_i^0. \end{cases}$$

Given that M satisfies (IC) and (IR), it is straightforward to see that B_i is nondecreasing over $(v_i^0, \overline{v}_i]$, $B_i(v_i) \in [v_i^0, \overline{v}_i], \forall v_i \in [v_i^0, \overline{v}_i]$, and $v_i^0 = \lim_{v_i \searrow v_i^0} B_i(v_i)$.[36] We now construct a winner-payable auction $A = (\mathcal{B}, \xi, \tau)$ by augmenting the message space so that for each $i \in N$, $\mathcal{B}_i = \mathcal{V}_i \cup \{b_i^0\}$, where b_i^0 is the non-participation option. For any message profile $b \in \mathcal{B}$, let

$$(\xi_i(b), \tau_i(b)) = \begin{cases} (q_i(v), q_i(v)B_i(v_i)) & \text{if } b = v \text{ for some } v \in \mathcal{V} \\ (1, b_i) & \text{if } b_i \in \{v_i^0, \overline{v}_i\} \text{ and } b_{-i} = b_{-i}^0 \\ (0,0) & \text{otherwise,} \end{cases}$$

where $b_{-i}^0 = (b_j^0)_{j \neq i}$. Then, for any $b = v \in \mathcal{V}$ with $\xi_i(b) > 0$, we have $\frac{\tau_i(b)}{\xi_i(b)} = B_i(v_i) \in [v_i^0, \overline{v}_i]$. Also, for $b_i \in \{v_i^0, \overline{v}_i\}$, $\xi_i(b_i, b_{-i}^0) = 1$ and $\tau_i(b_i, b_{-i}^0) = b_i$. The winner-payability of A can be verified by setting $\underline{b}^i = (v_i^0, b_{-i}^0)$ and $\overline{b}^i = (\overline{v}_i, b_{-i}^0)$.

Observe now that truthtelling is an equilibrium strategy in A. If all bidders other than i report their value truthfully, then it can be checked that the interim winning probability and payment bidder i obtains from reporting $v_i' \in \mathcal{V}_i$ is given by $Q_i(v_i')$ and $T_i(v_i')$, respectively. If he reports b_i^0, then his payoff is equal to 0. Thus, it is bidder i's best response to be truthful, since M satisfies (IC) and (IR). Finally, note that if all bidders employ the truthtelling strategy in A, then the resulting interim outcome is the same as (Q, T). ▌

C Proofs for Section 4

Proof of Theorem 1: Fix an equilibrium outcome $M_A = (q, t)$ of an auction $A \in \mathcal{A}^*$ and let (Q, T) denote its interim outcome. Suppose for a contradiction that M_A is WCP but Q_k is not constant in some interval $(a', b') \subset (r_k, \overline{v}_k]$ for some $k \in N$, where G_k is linear. Let (a, b) be the maximal (connected) interval in $[r_k, \overline{v}_k]$ containing (a', b') on which G_k is linear. Note that $F_k(s) = G_k(s)$ at $s = a, b$.

[36] To see the monotonicity of B_i suppose to the contrary that there are two types v_i and $\tilde{v}_i > v_i$ such that $Q_i(\tilde{v}_i) \ge Q_i(v_i) > 0$ but $B_i(\tilde{v}_i) < B_i(v_i)$. Then, $U_i^M(v_i) = Q_i(v_i)(v_i - B_i(v_i)) < Q_i(\tilde{v}_i)(v_i - B_i(\tilde{v}_i))$ so v_i finds it profitable to deviate to $\tilde{v}_i$'s strategy.

Let us define $\tilde{Q} = (\tilde{Q}_1, \cdots, \tilde{Q}_n)$ as follows:

$$\tilde{Q}_i(v_i) = \begin{cases} \bar{p} & \text{if } i = k \text{ and } v_i \in (a, b) \\ Q_i(v_i) & \text{otherwise} \end{cases}, \tag{16}$$

where $\bar{p}$ is defined to satisfy

$$\bar{p}(F_k(b) - F_k(a)) = \int_a^b Q_k(s) dF_k(s). \tag{17}$$

Observe first that $\tilde{Q}$ satisfies (M). For this, we only need to check that $Q_k(a) \leq \bar{p} = \frac{\int_a^b Q_k(s)dF_k(s)}{F_k(b)-F_k(a)} \leq Q_k(b)$, which clearly holds since Q_k is nondecreasing.

Since we need to ensure that $\tilde{Q}$ admits an ex-post allocation rule, we invoke the following result (for the proof see Mierendorff (2011) or Che et al. (2013)).

Lemma 2. *For any interim rule $(Q_i)_{i \in N}$, there exists an ex-post allocation rule that has Q as an interim allocation rule if and only if*

$$(B) \qquad \sum_{i \in N} \int_{v_i}^{\overline{v}_i} Q_i(s) dF_i(s) \leq 1 - \prod_{i \in N} F_i(v_i), \forall v = (v_i)_{i \in N} \in \mathcal{V}.$$

The, the next claim shows that, in addition to (M), $\tilde{Q}$ also satisfies (B).

Claim 1. *The interim allocation rule $\tilde{Q}$ satisfies (B).*

Proof. Since Q satisfies (B), it suffices to show that for all $v = (v_1, \cdots, v_n) \in \mathcal{V}$,

$$\sum_{i \in N} \int_{v_i}^{\overline{v}_i} \tilde{Q}_i(s) dF_i(s) \leq \sum_{i \in N} \int_{v_i}^{\overline{v}_i} Q_i(s) dF_i(s),$$

which, given (16), will hold if for all $v_k \in [\underline{v}_k, \overline{v}_k]$,

$$\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s) dF_k(s) \leq \int_{v_k}^{\overline{v}_k} Q_k(s) dF_k(s). \tag{18}$$

Note that (18) clearly holds for $v_k \geq b$ since $\tilde{Q}_k(s) = Q_k(s), \forall s \in [b, \overline{v}_k]$. Let us pick $v_k \in [a, b)$ and then we obtain as desired

$$\begin{aligned} \int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s) dF_k(s) &= \int_{v_k}^{b} \bar{p} dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s) dF_k(s) \\ &= \left[\frac{F_k(b) - F_k(v_k)}{F_k(b) - F_k(a)} \right] \int_a^b Q_k(s) dF_k(s) + \int_b^{\overline{v}_k} Q_k(s) dF_k(s) \\ &\leq \int_{v_k}^{b} Q_k(s) dF_k(s) + \int_b^{\overline{v}_k} Q_k(s) dF_k(s) = \int_{v_k}^{\overline{v}_k} Q_k(s) dF_k(s), \end{aligned} \tag{19}$$

where the second equality follows from the definition of $\bar{p}$, and the inequality from the fact that $Q_k(\cdot)$ is nondecreasing and thus

$$\int_a^b \frac{Q_k(s)}{F_k(b)-F_k(a)}dF_k(s) \leq \int_{v_k}^b \frac{Q_k(s)}{F_k(b)-F_k(v_k)}dF_k(s).$$

Also, for $v_k < a$, we have

$$\begin{aligned}\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s)dF_k(s) &= \int_{v_k}^{a} Q_k(s)dF_k(s) + \int_{a}^{\overline{v}_k} \tilde{Q}_k(s)dF_k(s)\\ &\leq \int_{v_k}^{a} Q_k(s)dF_k(s) + \int_{a}^{\overline{v}_k} Q_k(s)dF_k(s) = \int_{v_k}^{\overline{v}_k} Q_k(s)dF_k(s),\end{aligned}$$

where the inequality follows from (19). ∎

By Claim 1 and Lemma 2, $\tilde{Q}$ admits an ex post allocation rule $\hat{q}$ that has $\tilde{Q}$ as an interim allocation rule. Next, we use (Env) to construct an interim payment rule $\tilde{T}$ satisfying $\tilde{T}_i(r_i) = T_i(r_i)$ for all $i \in N$. Given this, we construct an (ex post) payment rule $\tilde{t}$ defined by

$$\tilde{t}_i(v) = \tilde{q}_i(v)\frac{\tilde{T}_i(v_i)}{\tilde{Q}_i(v_i)}.$$

Clearly, $E_{v_{-i}}[\tilde{t}_i(v)] = \tilde{T}_i(v_i)$. Then, a direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ has $(\tilde{Q}, \tilde{T})$ as the interim rule. By construction, $\tilde{M}_A$ satisfies (IC). We next show that it satisfies $(C - IR)$.

Claim 2. *$\tilde{M}_A$ interim Pareto dominates the original equilibrium payoff of the auction rule A (i.e. satisfies $(C - IR)$).*

Proof. First, it is clear that all bidders other than k will have their payoffs unaffected. Moreover, bidder k's payoff will only be affected when his value is above a. To show that $U_k^{\tilde{M}_A}(v_k) \geq U_k^{M_A}(v_k)$ for all $v_k \in [a, \overline{v}_k]$, with strict inequality for some v_k, it suffices to show that $U_k^{\tilde{M}_A}(b) \geq U_k^{M_A}(b)$, since $U_k^{\tilde{M}_A}$ is linear in $[a, b]$ while $U_k^{M_A}$ is convex but not linear, and since $\tilde{Q}_k(v_k) = Q_k(v_k)$ for all $v_k \in (b, \overline{v}_k]$ so $\tilde{U}_k^{M_A}$ and $U_k^{M_A}$ have the same slope beyond b.

To do so, we let $\hat{\mathcal{V}}_k \subset [r_k, \overline{v}_k]$ denote the (countable) set of points at which Q_k is discontinuous (i.e., jumps up). Given the nondecreasing Q_k and the concave closure G_k of F_k over the interval $[a, b] \in [r_k, \overline{v}_k]$, we obtain

$$\begin{aligned}&\int_a^b Q_k(s)(f_k(s) - g_k(s))ds\\ =&Q_k(s)(F_k(s) - G_k(s))\Big|_{a+}^{b-} - \sum_{v\in\hat{\mathcal{V}}_k\cap(a,b)} Q_k(s)(F_k(s) - G_k(s))\Big|_{v-}^{v+} - \int_a^b Q_k'(s)(F_k(s) - G_k(s))ds\end{aligned}$$

$$= \sum_{v \in \hat{\mathcal{V}}_k \cap (a,b)} (Q_k(v+) - Q_k(v-))(G_k(v) - F_k(v)) + \int_a^b Q_k'(s)(G_k(s) - F_k(s))ds \geq 0,$$

where $v-$ and $v+$ denote the left and right limit, respectively. Here the second equality follows from the fact that $F_k(s) = G_k(s)$ at $s = a, b$ and F_k and G_k are continuous, while the inequality from the fact that Q_k is nondecreasing and $G_k(s) \geq F_k(s), \forall s$. By the above inequality and the fact that g_k is constant over the interval $[a, b]$, we obtain

$$\begin{aligned} \int_a^b Q_k(s) f_k(s) ds &\geq \int_a^b Q_k(s) g_k(s) ds \\ &= \left(\int_a^b Q_k(s) ds \right) \left(\frac{G_k(b) - G_k(a)}{b - a} \right) = \left(\int_a^b Q_k(s) ds \right) \left(\frac{F_k(b) - F_k(a)}{b - a} \right), \end{aligned}$$

which yields

$$\bar{p} = \frac{\int_a^b Q_k(s) f_k(s) ds}{F_k(b) - F_k(a)} \geq \frac{\int_a^b Q_k(s) ds}{b - a}.$$

Thus, we obtain

$$U_k^{\tilde{M}_A}(b) - U_k^{\tilde{M}_A}(a) = \bar{p}(b - a) \geq \int_a^b Q_k(s) ds = U_k^{M_A}(b) - U_k^{M_A}(a).$$

or $U_k^{\tilde{M}_A}(b) \geq U_k^{M_A}(b)$ since $U_k^{\tilde{M}_A}(a) = U_k^{M_A}(a)$. ∎

Given Claim 2, the desired contradiction will follow if we show that $\tilde{M}_A$ can be implemented via a weak cartel manipulation. To this end, let $\tilde{B}_i(v_i) := \frac{\tilde{T}_i(v)}{\tilde{Q}_i(v_i)}$ if $v_i \in [r_i, \overline{v}_i]$ and $\tilde{B}_i(v_i) := 0$ otherwise.[37] We then exploit the winner-payability property to establish the following result.

Claim 3. *Given the winner-payability of A, for any given $v_i \in [r_i, \overline{v}_i]$, there exists $z_i(v_i) \in [0, 1]$, such that*

$$z_i(v_i)\tau_i(\underline{b}^i) + (1 - z_i(v_i))\tau_i(\overline{b}^i) = \tilde{B}_i(v_i). \tag{20}$$

Proof. First, we show that

$$B_i(r_i) \leq \tilde{B}_i(v_i) \leq B_i(\overline{v}_i), \forall v_i \in [r_i, \overline{v}_i], \forall i, \tag{21}$$

where $B_i(v_i) = \frac{T_i(v_i)}{Q_i(v_i)}$ for $v_i \in [r_i, \overline{v}_i]$. This is immediate if $i \neq k$ or if $i = k$ and $v_k \in [\underline{v}_k, a]$ since in those cases, $B_i(v_i) = \tilde{B}_i(v_i)$ and B_i is nondecreasing.

[37] Note that $r_i = \inf\{v_i \in \mathcal{V}_i \,|\, \tilde{Q}_i(v_i) > 0\} = \inf\{v_i \in \mathcal{V}_i \,|\, Q_i(v_i) > 0\}$.

Consider now $i = k$ and any $v_k \in (a, \overline{v}_k]$. The first inequality of (21) holds trivially. To prove the latter inequality, it suffices to show that $\tilde{B}_i(\overline{v}_i) \leq B_i(\overline{v}_i)$, since $\tilde{B}_i(\cdot)$ is nondecreasing. This inequality holds trivially if $\overline{v}_k = b$ since $B_k(b) \geq B_k(a) = \tilde{B}_k(a) = \tilde{B}_k(b)$. If $\overline{v}_k > b$, then $Q_k(\overline{v}_k) = \tilde{Q}_k(\overline{v}_k)$ and also

$$T(\overline{v}_k) - \tilde{T}(\overline{v}_k) = \overline{v}_k Q_k(\overline{v}_k) - \overline{v}_k \tilde{Q}_k(\overline{v}_k) + U_k^{\tilde{M}_A}(\overline{v}_k) - U_k^{M_A}(\overline{v}_k) = U_k^{\tilde{M}_A}(\overline{v}_k) - U_k^{M_A}(\overline{v}_k) \geq 0.$$

This implies $B_i(\overline{v}_i) \geq \tilde{B}_i(\overline{v}_i)$.

Next, we observe that for any $v_i \in [r_i, \overline{v}_i]$,

$$\inf\Big\{\frac{\tau_i(b)}{\xi_i(b)} \,|\, \xi_i(b) > 0, b \in \mathcal{B}\Big\} \leq B_i(v_i) \leq \sup\Big\{\frac{\tau_i(b)}{\xi_i(b)} \,|\, \xi_i(b) > 0, b \in \mathcal{B} \text{ and } \frac{\tau_i(b)}{\xi_i(b)} \leq \overline{v}_i\Big\}.$$

By definition, $\tau_i(\underline{b}^i)$ and $\tau_i(\overline{b}^i)$ equal respectively the first and the last terms in the above inequalities. Combining this with (21) means that for each $v_i \in [r_i, \overline{v}_i]$, $\tilde{B}_i(v_i) \in [\tau_i(\underline{b}^i), \tau_i(\overline{b}^i)]$, which guarantees the existence of $z_i(v_i)$ as in (20). ▮

It remains to show that $\tilde{M}_A$ is a cartel manipulation. To this end, we construct a cartel agreement α that implements $\tilde{M}_A$ in the sense of (4). First, the agreement $\alpha(v)$ assigns probability one to a bid profile $(b_1^0, \cdots, b_n^0)$ if $v_i \geq r_i$ for all i. Otherwise, $\alpha(v)$ assigns probability $\tilde{q}_i(v) z_i(v_i)$ to a bid profile $\underline{b}^i$ and probability $\tilde{q}_i(v)(1 - z_i(v_i))$ to a bid profile $\overline{b}^i$, for each i, and probability $1 - \sum_i \tilde{q}_i(v)$ to a bid profile $(b_1^0, \cdots, b_n^0)$. Under this cartel agreement, given profile $v \in \mathcal{V}$ of (reported) values, bidder i obtains the object with probability $\tilde{q}_i(v)$ and pays $\tilde{q}_i(v)\tilde{B}_i(v_i)$ in expectation. Hence, for each $v \in \mathcal{V}$,

$$\tilde{q}_i(v) = \mathbb{E}_{\alpha(v)}[\xi_i(b)] \text{ and } \tilde{t}_i(v) = \tilde{q}_i(v)\tilde{B}_i(v_i) = \mathbb{E}_{\alpha(v)}[\tau_i(b)],$$

as it remained to be shown. ▮

Proof of Corollary 1: Fix a bidder k for whom G_k is linear on some interval (a, b) with $b > r$ and $a \geq \underline{v}$. We show that in any standard auction, the winning probability of bidder k is non-constant in the interval $(\max\{a, r\}, b)$, which will imply by Theorem 1 that the auction is not WCP. Consider first the second-price and English auctions where each bidder bids his value in the undominated strategy. The interim winning probability of bidder k with $v_k \in (\max\{a, r\}, b)$ is equal to $Q_k(v_k) = \prod_{i \neq k} F_i(v_k)$, which is strictly increasing in the interval $(\max\{a, r\}, b)$.

Consider next the first-price auction (or Dutch auction since the two auctions are strategically equivalent). Note first that in undominated strategy equilibrium, (i) no bidder bids more than his value and (ii) no bidder puts an atom at any bid B if B wins with positive probability. Letting β_i denote bidder i's equilibrium strategy, note also that β_i is nondecreasing. Given (i), we must have $Q_k(v_k) > 0$ for all $v_k \in (\max\{a, r\}, b)$ since he can always bid some amount $B \in (\max\{a, r\}, v_k)$ and enjoy a positive payoff. Next, by (ii), there must be some $v_k \in (\max\{a, r\}, b)$ such that $\beta_k(v_k) < \beta_k(b)$ since otherwise $\beta_k(b)$ would be an atom bid. For such v_k, we must have $Q_k(v_k) < Q_k(b)$ so Q_k is non-constant in $(\max\{a, r\}, b)$. To see why, suppose to the contrary that $Q_k(v_k) = Q_k(b)$, which implies that no one else is submitting any bid between $\beta_k(v_k)$ and $\beta_k(b)$. Then, bidder k with value b can profitably deviate to lower his bid below $\beta_k(b)$ but above $\beta_k(v_k)$, a contradiction. ▮

Proof of Theorem 2: Recall $\mathcal{V}_i^0 = \{v \in [r_i, \overline{v}_i] | g_i'(v) = 0\}$ is the set of susceptible types, and let $\mathcal{V}_i^- := [r_i, \overline{v}_i] \backslash \mathcal{V}_i^0$ be the remaining set of types above r_i in which g_i is strictly decreasing. We let $\mathcal{V}_i^D \subset \mathcal{V}_i$ denote the set of types at which g_i drops discontinuously.

Consider an auction A which induces an equilibrium whose interim allocation probability satisfies $Q_i'(v) = 0$ for all $v \in \mathcal{V}_i^0$ and and the conditions (a) and (b). We prove that A is unsusceptible to collusion.

Suppose for contradiction that there is a weak cartel manipulation $\tilde{M}_A = (\tilde{q}, \tilde{t})$ implementing an interim Pareto improvement. Since, by definition of r_i, we have $\tau_i(b) \geq \xi_i(b)\underline{v}_i, \forall i \in N, \forall b \in \mathcal{B}_i$ $\tilde{M}_A$ is a weak manipulation of A, for each $v \leq r_i$,

$$U_i^{\tilde{M}_A}(v) \leq \max_{b \in \mathcal{B}_i} \xi_i(b)v - \tau_i(b) \leq \max_{b \in \mathcal{B}_i} \xi_i(b)v - \xi_i(b)r_i \leq 0,$$

so $(C - IR)$ implies that $U_i^{\tilde{M}_A}(v) = 0$, for $v \leq r_i$. That $\tilde{M}_A$ Pareto dominates M_A implies $U_i^{M_A}(v) = 0$, for $v \leq r_i$. Then, interim Pareto domination implies that

$$X_i(v_i) := U_i^{\tilde{M}_A}(v_i) - U_i^{M_A}(v_i) = \int_{r_i}^{v_i} (\tilde{Q}_i(s) - Q_i(s))ds \geq 0, \forall i, v_i. \tag{22}$$

Next, by the condition (b), we have

$$\sum_{i \in N} \int_{r_i}^{\overline{v}_i} Q_i(v_i) f_i(v_i) dv_i = \mathbb{E}[\sum_{i \in N} q_i(v)] = 1 - \prod_{i \in N} F_i(r_i).$$

It follows from this equality and Lemma 2 that

$$\sum_{i \in N} \int_{r_i}^{\overline{v}_i} \tilde{Q}_i(v_i) f_i(v_i) dv_i \leq 1 - \prod_{i \in N} F_i(r_i) = \sum_{i \in N} \int_{r_i}^{\overline{v}_i} Q_i(v_i) f_i(v_i) dv_i,$$

or

$$\sum_{i \in N} \int_{r_i}^{\overline{v}_i} (\tilde{Q}_i(v_i) - Q_i(v_i)) f_i(v_i) dv_i \leq 0. \tag{23}$$

Meanwhile,

$$
\begin{aligned}
&\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))f_i(v_i)dv_i\\
=&\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))g_i(v_i)dv_i-\Big(\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))[g_i(v_i)-f_i(v_i)]dv_i\Big)\\
=&\sum_{i\in N}\left(X_i(\overline{v}_i)g_i(\overline{v}_i)-\sum_{v\in\mathcal{V}_i^D}[X_i(v)g_i(v)]_{v^-}^{v^+}-\int_{r_i}^{\overline{v}_i}X_i(v_i)g_i{}'(v_i)dv_i\right)\\
&\quad+\sum_{i\in N}\sum_{v\in\mathcal{V}_i^{D'}}\Big[(G_i(v)-F_i(v))(\tilde{Q}_i(v)-Q_i(v))\Big]_{v^-}^{v^+}+\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(G_i(v_i)-F_i(v_i))[\tilde{Q}_i'(v_i)-Q_i'(v_i)]dv_i\\
=&\sum_{i\in N}\left(X_i(\overline{v}_i)g_i(\overline{v}_i)-\sum_{v\in\mathcal{V}_i^D}X_i(v)(g_i(v^+)-g_i(v^-))-\int_{r_i}^{\overline{v}_i}X_i(v_i)g_i{}'(v_i)dv_i\right)\\
&\quad+\sum_{i\in N}\sum_{v\in\mathcal{V}_i^{D'}}(G_i(v)-F_i(v))\Big[\tilde{Q}_i(v^+)-\tilde{Q}_i(v^+)-(Q_i(v^+)-Q_i(v^-))\Big]\\
&\qquad+\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(G_i(v_i)-F_i(v_i))[\tilde{Q}_i'(v_i)-Q_i'(v_i)]dv_i\\
\geq&0,
\end{aligned}
\tag{24}
$$

where $\mathcal{V}_i^{D'}$ is the set of values at which either $\tilde{Q}_i$ or Q_i jumps up. The first equality follows from the integration by parts. The second equality holds since X_i, G_i and F_i are continuous. The inequality holds since, for each $i\in N$, $X_i(v)\geq 0$, $g_i'(v)\leq 0$ whenever it is well defined, and $g_i(v^+)-g_i(v^-)<0$ for each $v\in\mathcal{V}_i^D$, and, whenever $G_i(v_i)>F_i(v_i)$, $Q_i(v^+)=Q_i(v^-)$, $\tilde{Q}_i(v^+)\geq\tilde{Q}_i(v^-)$, and $Q_i'(v_i)=0\leq\tilde{Q}_i'(v_i)$ (by the monotonicity of $\tilde{Q}_i$).

The last inequality combined with (23) means that the inequality must hold as equality, which in turn implies that $X_i(\overline{v}_i)=0$, and $X_i(v)=0$ for a.e. $v\in\mathcal{V}_i^-$ for each $i\in N$.[38]

We now prove that $U_i^{\tilde{M}_A}(v)-U_i^{M_A}(v)=\int_{r_i}^{v}(\tilde{Q}_i(s)-Q_i(s))ds=X_i(v)\leq 0$ for all $v\geq r_i$, for all $i\in N$, which will have established the desired contradiction. Suppose to the contrary that there exists v' such that $X_i(v')>0$. Recall that $X_i(r_i)=X_i(\overline{v}_i)=0$ and that X_i is continuous, and differentiable on $(r_i,\overline{v}_i)$. By the mean value theorem, there exists $v^1\in(r_i,v')$ such that $X_i(v^1)>0$ and that $X_i'(v^1)=\tilde{Q}_i(v^1)-Q_i(v^1)>0$ and $v^2\in(v',\overline{v}_i)$ such that $X_i(v^2)>0$ and that $X_i'(v^2)=\tilde{Q}_i(v^2)-Q_i(v^2)<0$. It follows that there exist $v''\in(v^1,v^2)$ such that $X_i(v'')>0$, and $X_i'(v)=\tilde{Q}_i(v)-Q_i(v)$ falls in v at $v=v''$, meaning either $\tilde{Q}_i(v)-Q_i(v)$ jumps down at $v=v''$ or $\tilde{Q}_i'(v'')-Q_i'(v'')<0$. In either case, since $\tilde{Q}_i$

[38]If $X_i(v)>0$ for some $v\in\mathcal{V}_i^D$, or for a positive measure set of v's in $\mathcal{V}_i^-$, then the inequality in (24) becomes strict, a contradiction to (23).

is nondecreasing, $Q_i(v)$ must increase in v at $v = v''$. This means that $v'' \in \mathcal{V}_i^-$ by the construction of Q_i. But then the above observation implies that $X_i(v'') = 0$, a contradiction. We thus conclude that $U_i^{\tilde{M}_A}(v) - U_i^{M_A}(v) = X_i(v) \leq 0$ for all v, and i. ▮

D Proofs for Section 5

Proof of Theorem 3: We first establish a two lemmas, Lemma 3 and 4. To do so, some notation is first required. Given any vector $r = (r_1, \ldots, r_n) \in \mathcal{V}$, we write the allocation rule in (12) as $q^*(\cdot; r)$ to make its dependence on r explicit. Next, we define

$$t_i^*(v;r) = q_i^*(v;r)v_i - \int_{\underline{v}_i}^{v_i} q_i^*(s_i, v_{-i}; r) ds_i. \tag{25}$$

Then, from now on, we let $M^*(r)$ denote a direct mechanism $(q^*(\cdot; r), t^*(\cdot; r))$. It is straightforward to see that $M^*(r)$ is dominant-strategy implementable. For any $r = (r_i)_{i \in N}$, let $[P; r]$ be the same optimization program as $[P]$, except that it *ignores* the constraint $r_i = \inf\{\frac{t_i^*(v;r)}{q_i^*(v;r)} \mid q_i^*(v;r) > 0\}$, which we will refer to as the constraint (R) henceforth.

LEMMA 3. *For any $r = (r_i)_{i \in N}$ with $r_i \geq J^{-1}(0), \forall i \in N$, the mechanism $M^*(r)$ solves $[P; r]$.*

Proof. We first prove that $q^*(\cdot; r)$ maximizes the objective function of $[P; r]$. To do so, rewrite the objective function by incorporating the collusion-proofness constraint into it: For each $i \in N$ and $k \in K_i$, define $Q_i^k = Q_i(v_i)$ and let $J_i^k := \frac{\int_{a_i^k}^{b_i^k} J_i(s) dF_i(s)}{F_i(b_i^k) - F_i(a_i^k)}$ if $v_i \in I_i^k$. Then, express the seller's expected revenue as

$$\begin{aligned}
&\sum_{i \in N} \int_{r_i}^{\overline{v}_i} J_i(v_i) Q_i(v_i) dF_i(v_i) \\
=& \sum_{i \in N} \int_{v_i \in [r_i, \overline{v}_i] \setminus \mathcal{V}_i^0} J_i(v_i) Q_i(v_i) dF_i(v_i) + \sum_{i \in N} \sum_{k \in K_i} \int_{a_i^k}^{b_i^k} J_i(v_i) Q_i(v_i) dF_i(v_i) \\
=& \sum_{i \in N} \int_{v_i \in [r_i, \overline{v}_i] \setminus \mathcal{V}_i^0} J_i(v_i) Q_i(v_i) dF_i(v_i) + \sum_{i \in N} \sum_{k \in K_i} Q_i^k \int_{a_i^k}^{b_i^k} J_i(v_i) dF_i(v_i) \\
=& \sum_{i \in N} \int_{v_i \in [r_i, \overline{v}_i] \setminus \mathcal{V}_i^0} J_i(v_i) Q_i(v_i) dF_i(v_i) + \sum_{i \in N} \sum_{k \in K_i} J_i^k Q_i^k (F_i(b_i^k) - F_i(b_i^k)) \\
=& \sum_{i \in N} \int_{r_i}^{\overline{v}_i} \bar{J}_i(v_i) Q_i(v_i) dF_i(v_i) \\
=& \mathbb{E}[\sum_{i \in N} \bar{J}_i(v_i) 1_{\{v_i \geq r_i\}} q_i(v_i, v_{-i})].
\end{aligned}$$

The expression within the expectation operator above is maximized by the allocation rule $q_i^*(\cdot;r)$ for each realization $v=(v_i)_{i\in N}$.

It is clear that since $\bar{J}_i$ is (weakly) increasing, the interim allocation rule resulting from $q^*(\cdot;r)$ satisfies (M). Also, (Env) is easily satisfied since $M^*(r)$ is dominant-strategy implementable. Lastly, the constraint (CP) is satisfied because, since the fact that $\bar{J}_i$ is constant over I_i^k implies all types in the interval I_i^k receive the object with a constant probability under $q_i^*(\cdot;r)$ for each $i\in N$. We thus conclude that $M^*(r)$ solves $[P;r]$. ▌

However, there is no guarantee that the mechanism $M^*(r)$ satisfies the constraint (R). The following result shows that starting from $M^*(r)$ it is always possible to satisfy (R) without reducing the seller's revenue.

LEMMA 4. *For any $r=(r_i)_{i\in N}\in\mathcal{V}$, there exists $\hat{r}\geq r$ such that $M^*(\hat{r})$ satisfies (R) and yields a (weakly) higher revenue for the seller than $M^*(r)$.*

Proof. As a first step, we prove the following claim.

CLAIM 4. *For any $\tilde{r}_i\geq r_i$, $\mathcal{V}_i^0(\tilde{r}_i)\subset\mathcal{V}_i^0(r_i)$.*

Proof. Consider any interval $I=[a,b]\subset\mathcal{V}_i^0(\tilde{r}_i)$ on which $G_i(\cdot;\tilde{r}_i)$ is linear. Then, for each $v_i\in I$, there is some $s\in[0,1]$ and $v_i',v_i''\in[\tilde{r}_i,\overline{v}_i]$ such that $G_i(v_i;\tilde{r}_i)=sF_i(v_i')+(1-s)F_i(v_i'')$. Since $\tilde{r}_i\geq r_i$ and thus $v_i',v_i''\in[r_i,\overline{v}_i]$, we have $G_i(v_i;r_i)\geq sF_i(v_i')+(1-s)F_i(v_i'')=G_i(v_i;\tilde{r}_i)$ by definition of $G_i(v_i;r_i)$. Thus, we have $G_i(v_i;r_i)\geq G_i(v_i;\tilde{r}_i)$ for all $v_i\in I$. This implies that $G_i(\cdot;r_i)$ is also linear over the interval I since, if not, it must be the case that over some subinterval of I, $G_i(\cdot,r_i)$ is strictly concave and $G_i(\cdot,r_i)=F_i(\cdot)>G_i(\cdot;\tilde{r}_i)$, which cannot happen due to the fact that $G_i(\cdot;\tilde{r}_i)$ is the concave envelope of F_i. Thus, we have shown that $I\subset\mathcal{V}_i^0(r_i)$ so $\mathcal{V}_i^0(\tilde{r}_i)\subset\mathcal{V}_i^0(r_i)$. ▌

For any $r=(r_i)_{i\in N}\in\mathcal{V}$, let $Q^*(\cdot;r)$ denote the interim allocation rule corresponding to $q^*(\cdot;r)$, and define $r_i^*(r):=\inf\{v_i\in\mathcal{V}_i\,|\,Q_i^*(v_i;r)>0\}$. (If $Q_i^*(\cdot,r)\equiv 0$, then let $r_i^*(r)=\overline{v}_i$.) Note that by construction of $q^*(\cdot;r)$, we have $r_i^*(r)\geq r_i,\forall i\in N$. Let $\pi^*(r)$ denote the seller's revenue that is generated by the mechanism $M^*(r)$.

Now fix any $r=(r_i)_{i\in N}$ and denote $r^1=r$. Define $r^2\in\mathcal{V}$ such that $r_i^2=r_i^*(r^1)$ for each $i\in N$. Then, we must have $\pi^*(r^2)\geq\pi^*(r^1)$. To see this, note that $q^*(\cdot;r^1)$ satisfies all the constraints of $[P;r^2]$, in particular (CP) since, for each $i\in N$, $Q_i^*(v_i;r^1)=0,\forall v_i\leq r_i^2$ and also since $Q_i^*(\cdot;r^1)$ is constant in each interval belonging to $\mathcal{V}_i^0(r_i^2)$, which is because $\mathcal{V}_i^0(r_i^2)\subset\mathcal{V}_i^0(r_i^1)$ by Claim 4 and the fact that $r_i^2=r_i^*(r^1)\geq r_i^1$. Thus, $M^*(r^1)$ cannot yield a higher seller's revenue than $M^*(r^2)$, which is a solution of $[P;r^2]$. Define recursively $r^n\in\mathcal{V}$ for all $n\geq 2$ such that $r_i^n=r_i^*(r^{n-1})$ for each $i\in N$. By following the same reasoning as above, we have $\pi^*(r^n) \geq \pi^*(r^{n-1})$ for all $n \geq 2$. Also, the sequence $(r^n)_{n\in\mathbb{N}}$ is (weakly) increasing in the set $\mathcal{V}$, and thus has a limit $\hat{r} \in \mathcal{V}$ such that $\hat{r}_i = r_i^*(\hat{r})$. Then, we have $\pi^*(r) = \pi^*(r^1) \leq \pi^*(r^2) \leq \cdots \leq \pi^*(\hat{r})$.

It remains to show that $M^*(\hat{r})$ satisfies (R). Note first that for each $v_i > \hat{r}_i$, we have some v_{-i} with $q_i^*(v_i, v_{-i}; \hat{r}) > 0$ since $Q_i^*(v_i; \hat{r}) > 0$. For such profile $v = (v_i, v_{-i})$, we have $t_i^*(v; \hat{r}) \leq q_i^*(v; \hat{r})v_i$ or $\frac{t_i^*(v;\hat{r})}{q_i^*(v;\hat{r})} \leq v_i$. Since this is true for all $v_i > \hat{r}_i$, we have $\inf\{\frac{t_i^*(v;\hat{r})}{q_i^*(v;\hat{r})} \,|\, q_i^*(v; \hat{r}) > 0\} \leq \hat{r}_i$. The desired result will follow if it is shown that this inequality cannot be strict. To do so, note first that for any $v_i < \hat{r}_i$, we have $q_i^*(v_i, v_{-i}; \hat{r}) = 0, \forall v_{-i}$. Also, for any $v_i \geq \hat{r}_i$,

$$t_i^*(v_i, v_{-i}; \hat{r}) \geq q_i^*(v_i, v_{-i}; \hat{r}) - (v_i - \hat{r}_i)q_i^*(v_i, v_{-i}; \hat{r}) = \hat{r}_i q_i^*(v_i, v_{-i}; \hat{r}), \forall v_{-i}$$

where the inequality holds since $q_i^*(\cdot, v_{-i}; \hat{r})$ is nondecreasing. ▌

We are now ready to prove Theorem 3. Consider any profile of reserve prices $\tilde{r} = (\tilde{r}_i)_{i\in N}$ that results from solving $[P]$. Then, the optimal revenue cannot be greater than that from $M^*(\tilde{r})$ since $M^*(\tilde{r})$ solves $[P; \tilde{r}]$ according to Lemma 3. Then, by Lemma 4, one can find a profile r such that $M^*(r)$ satisfies all the constraints of $[P]$ and yields no less revenue for the seller than $M^*(\tilde{r})$ does, which means that $M^*(r)$ is a solution of $[P]$. The proof that $r_i \geq J_i^{-1}(0), \forall i \in N$ at the optimum of $[P]$ is straightforward and hence omitted. ▌

Proof of Corollary 6: We first observe that

$$\int_{r_i}^{\overline{v}_i} J_i(v_i)dF_i(v_i) = (1 - F_i(r_i))r_i,$$

which can be readily verified using the definition of J_i and integration-by-parts. Thus, for any $v_i \in \mathcal{V}_i^0 = [r_i, \overline{v}_i]$, we have $\bar{J}_i(v_i) = r_i$. Then, the allocation rule in (12) requires allocating the object to bidder i if $v_i \geq r_i = \bar{J}_i(v_i) > \max\{r_j \,|\, v_j \geq r_j = \bar{J}_j(v_j) \text{ and } j \neq i\}$. This means that bidder i must always be given the priority to receive the object over bidder j if $r_i > r_j$. In case $r_i = r_j$, the priority can be given to either of bidder i and j. (Note that in the statement of Theorem 3 bidders with equal virtual values obtain the object with the same probability; it is without loss to treat them asymmetrically as we do here). Let such priority rule be denoted by a permutation function $\pi : N \to N$ satisfying that $\pi(j) < \pi(i)$ if $r_i < r_j$. The interim allocation rule that results from this priority rule is then given as $\prod_{j:\pi(j)<\pi(i)} F_j(r_j)$ for each bidder i with $v_i \geq r_i$. Also, the seller's expected revenue under this interim allocation rule coincides with the expression within the square bracket of (13), which must then be maximized by choosing $r = (r_i)_{i\in N}$ optimally. ▌

Proof of Corollary 9: First, by the symmetry of auction rule, we must have $r_i = r$ for all i and some $r \geq r^M$. Let us consider the case where $r \leq \hat{v}$ (while we will see below that $r \leq \hat{v}$

is required at the optimum). There is a value $v^*(r) \geq \hat{v}$ such that G is linear in $[r, v^*(r)]$ while it is strictly concave elsewhere, which implies that $\bar{J}(v)$ defined in (11) is constant for $v \in [r, v^*(r)]$ and strictly increasing for $v > v^*(r)$. Using this, it is straightforward to see that the allocation q_i^* in (12) coincides with (14).

We now show that $r^M < r \leq \hat{v}$ at the optimum. We first argue that $r \leq \hat{v}$. If $r > \hat{v}$, then there is no range where G is linear, which means that the corresponding optimal rule given by (12) is the one which allocates the object efficiently among the bidders whose values are greater than r. Clearly, this mechanism is revenue-dominated by a mechanism where $r' = \hat{v}$ and the object is efficiently allocated to the bidders whose values are greater than r', since $\hat{v} > r^M$ so the extra revenue can be generated from selling to bidders with values in $[\hat{v}, r]$.

We next show that $r > r^M$. Since we already know that $r \geq r^M$ at the optimum, we need to argue that $r \neq r^M$ at the optimum. Note first that the interim allocation rule is given by

$$Q^*(v) = \begin{cases} F(v)^{n-1} & \text{if } v > v^* \\ \frac{F(v^*)^n - F(r)^n}{n(F(v^*) - F(r))} = \frac{\sum_{k=0}^{n-1} F(v^*)^{n-1-k} F(r)^k}{n} & \text{if } v \in [r, v^*] \\ 0 & \text{otherwise.} \end{cases}$$

The seller's revenue from each bidder can then be written as

$$n \int_r^{\bar{v}} J(v) Q^*(v) f(v) dv = \left(\sum_{k=0}^{n-1} F(v^*)^{n-1-k} F(r)^k \right) \int_r^{v^*} J(v) f(v) dv + n \int_{v^*}^{\bar{v}} F(v)^{n-1} J(v) f(v) dv.$$

Keeping in mind that v^* is a function of r, we differentiate the above expressions with r, set $r = r^M$, and use $J(r^M) = 0$ to obtain

$$\underbrace{\left(\sum_{k=0}^{n-2} (n-1-k) F(v^*)^{n-2-k} F(r^M)^k f(v^*) \left(\frac{dv^*}{dr} \right) + \sum_{k=1}^{n-1} F(v^*)^{n-1-k} F(r^M)^{k-1} f(r^M) \right)}_{=A} \int_{r^M}^{v^*} J(v) f(v) dv$$
$$+ J(v^*) \underbrace{\left(\sum_{k=0}^{n-1} F(v^*)^{n-1-k} F(r^M)^k - n F(v^*)^{n-1} \right) f(v^*) \left(\frac{dv^*}{dr} \right)}_{=B}.$$

It is straightforward to check that

$$A \geq \left(\sum_{k=0}^{n-2} (n-1-k) F(v^*)^{n-2-k} F(r^M)^k \right) f(v^*) \left(\frac{dv^*}{dr} \right) = \frac{-B}{F(v^*) - F(r)}.$$

Thus the above expressions is no less than

$$\left(J(v^*) - \frac{\int_{r^M}^{v^*} J(v) f(v) dv}{F(v^*) - F(r^M)} \right) B > 0,$$

where the strict inequality holds since $v^* > r^M$ and $\frac{dv^*}{dr} > 0$ imply $B > 0$. Thus, it is profitable for the seller to raise r above r^M. ▌

E Proof for Section 6

Proof of Theorem 4: Since the RCP mechanism we will construct below is a direct mechanism, we henceforth focus on the case in which the seller offers a direct mechanism M. Let $\tilde{M} = (\tilde{q}, \tilde{t})$ denote an equilibrium outcome that results from a cartel game following announcement of M.

As a first step we show there is a lower bound for the payoff that each bidder obtains in $\tilde{M}$ if we assume that all bidders play cartel-undominated strategies (on and off the equilibrium path). To this end, fix any bidder i, and let $\pi^i = \{C^1, \ldots, C^k\}$ denote an arbitrary partition of $N \backslash \{i\}$, with the interpretation that each group $C^\ell, \ell = 1, \ldots, k$, forms a cartel, in case bidder i chooses not to join any cartel. Let Π^i denote the set of all such partitions. Finally, for any cartel C, let $\Omega(v_C)$ be the set of cartel-undominated strategies at v_C.

LEMMA 5. *In any equilibrium outcome $\tilde{M}$, the interim payoff of each bidder i with value v_i must be at least*

$$\overline{U}^M(v_i) := \sup_{v_i'} \mathbb{E}_{v_{-i}} \left[\inf \left\{ u_i^M(v_i', v_{C^1}', \ldots, v_{C^k}' | v_i) \middle| v_{C^\ell}' \in \Omega(v_{C^\ell}) \ \forall C^\ell \in \pi^i \ and \ \pi^i \in \Pi^i \right\} \right]. \tag{26}$$

Proof. For any type profile (v_i, v_{-i}) and bidder i's report v_i', define

$$\overline{u}_i^M(v_i'|v_i, v_{-i}) = \inf \left\{ u_i^M(v_i', v_{C^1}', \ldots, v_{C^k}' | v_i) \middle| v_{C^\ell}' \in \Omega(v_{C^\ell}) \ \forall C^\ell \in \pi^i \text{ and } \pi^i \in \Pi^i \right\}.$$

Let H_i denote the set of all on-path histories where bidder i has just received collusive proposals (this includes the histories in which no proposal has been made to him). For each $h_i \in H_i$, let $\tau_i(h_i)$ denote the probability with which h_i arises at equilibrium. Let $\mu_i(h_i) \in \Delta(\mathcal{V}_{-i})$ denote the bidder i's updated belief (under Bayes rule) given that he has observed a (private) history h_i.

We now argue that at any history $h_i \in H_i$, the expected payoff of bidder i with value v_i is at least

$$\sup_{v_i' \in \mathcal{V}_i} \mathbb{E}_{\mu_i(h_i)}[\overline{u}_i^M(v_i'|v_i, v_{-i})]. \tag{27}$$

Note first that this is the lowest payoff bidder i can get when, following h_i, he does not become member of any cartel, no matter what cartels other bidders will form, given that all cartels employ cartel-undominated strategies and bidder i with belief $\mu_i(h_i)$ optimally responds to that. Hence, his equilibrium payoff after history h_i, following which he does not join any cartel, cannot fall below (27). The same is true after history h_i', following which he joins some cartel, because his payoff from deviating and rejecting all cartel proposals must be at least (27).

Thus, bidder i's interim payoff in the cartel game is at least

$$\begin{aligned}\mathbb{E}_{\tau_i}[\sup_{v_i'\in\mathcal{V}_i}\mathbb{E}_{\mu_i(h_i)}[\overline{u}_i^M(v_i'|v_i,v_{-i})]] &\geq \sup_{v_i'\in\mathcal{V}_i}\mathbb{E}_{\tau_i}[\mathbb{E}_{\mu_i(h_i)}[\overline{u}_i^M(v_i'|v_i,v_{-i})]] \\ &= \sup_{v_i'\in\mathcal{V}_i}\mathbb{E}_{v_{-i}}[\overline{u}_i^M(v_i'|v_i,v_{-i})] = \overline{U}_i^M(v_i),\end{aligned}$$

where the first equality follows from the fact that $\mathbb{E}_{\tau_i}[\mathbb{E}_{\mu_i(h_i)}[\cdot]] = \mathbb{E}_{v_{-i}}[\cdot]$ and the second equality from the definition of $\overline{u}_i^M$ and $\overline{U}_i^M$. ∎

Next, we observe that if bidder i with value v_i reports truthfully, and others report any arbitrary v_{-i}, then he earns the ex-post payoff equal to

$$u_i^M(v_i,v_{-i}|v_i) = v_i q_i^*(v_i,v_{-i}) - t_i^*(v_i,v_{-i}) = \int_{\underline{v}_i}^{v_i} q_i^*(s_i,v_{-i})ds_i. \tag{28}$$

It follows from this that u_i^M is (weakly) decreasing in v_{-i} as q_i^* is.

Next, define

$$\tilde{\mathcal{V}}^i := \{v\in\mathcal{V}\,|\,\text{either (a) } v_i\in\text{int}(\mathcal{V}_i^0) \text{ or (b) } v_i\notin\mathcal{V}_i^0 \text{ and } \bar{J}_i(v_i)\neq\bar{J}_j(v_j),\forall j\neq i,\forall v_j\in\mathcal{V}_j^0\},$$

where int$(\cdot)$ denotes the interior of a set so $\text{int}(\mathcal{V}_i^0) = \cup_{k\in K_i}(a_i^k,b_i^k)$. Given that $\bar{J}_i$ is strictly increasing over $\mathcal{V}_i\backslash\mathcal{V}_i^0$, it is straightforward to see that $\tilde{\mathcal{V}}^i$ has a full measure (i.e., its measure is equal to 1). Define $\tilde{\mathcal{V}} = \cap_{i\in N}\tilde{\mathcal{V}}^i$ and note that $\tilde{\mathcal{V}}$ also has a full measure since it is a finite intersection of full-measure sets. We prove the following claim.

Claim 5. *For any i, any partition $\pi^i = \{C^1,\dots,C^k\}$ of $N\backslash\{i\}$, and any $v\in\tilde{\mathcal{V}}$,*

$$u_i^M(v_i,v_{C^1}',...,v_{C^k}'|v_i) \geq u_i^M(v_i,v_{-i}|v_i),\forall v_{C^\ell}'\in\Omega(v_{C^\ell}),\ell=1,\cdots,k. \tag{29}$$

Proof. Fix any $v\in\tilde{\mathcal{V}}$. We first show that for any $C\subsetneq N$ and $i\in N\backslash C$,

$$u_i^M(v_C',v_{N\backslash C}|v_i) \geq v_i^M(v_C,v_{N\backslash C}|v_i),\forall v_C'\in\Omega(v_C). \tag{30}$$

Let $C' = \arg\max_{i\in C}\bar{J}_i(v_i')$. Observe first that for any $i\in N\backslash C$, the allocation rule $q_i^*(v_C',v_{N\backslash C})$, and thus the ex-post payoff $u_i^M(v_C',v_{N\backslash C})$, depends on v_C' only through $v_{C'}'$. Clearly, if $v_{C'}'\leq v_{C'}$, then (30) is immediately implied by the fact that $u_i^M(v_i,v_{-i}|v_i)$ is (weakly) decreasing in v_{-i}.

Thus, we assume from now that $v'_i > v_i$ for at least one $i \in C'$. To simplify notation, let $C_{-i} = C\backslash\{i\}$ and $C_{+i} = C \cup \{i\}$. Letting $v''_C = v_C \wedge v'_C$ (i.e. $v''_i = \min\{v'_i, v_i\}$ for all $i \in C$), we show that

$$u_i^M(v'_C, \tilde{v}_{N\backslash C}|v_i) \leq u_i^M(v''_C, \tilde{v}_{N\backslash C}|v_i), \forall i \in C, \forall \tilde{v}_{N\backslash C} \in \mathcal{V}_{N\backslash C}. \tag{31}$$

To do so, change the strategy of any bidder $j \in C$ from v'_j to v''_j and observe that

$$u_i^M(v'_C, \tilde{v}_{N\backslash C}|v_i) \leq u_i^M(v''_j, v'_{C_{-j}}, \tilde{v}_{N\backslash C}|v_i) \tag{32}$$

since the dominant-strategy incentive compatibility of M for bidder j means

$$u_j^M(v'_C, \tilde{v}_{N\backslash C}|v_j) \leq u_j^M(v''_j, v'_{C_{-j}}, \tilde{v}_{N\backslash C}|v_j),$$

and also since, for any $i \in C\backslash\{j\}$, the fact that u_i^M is decreasing in $v_j, j \neq i$ implies

$$u_i^M(v'_C, \tilde{v}_{N\backslash C}|v_i) \leq u_i^M(v''_j, v'_{C_{-j}}, \tilde{v}_{N\backslash C}|v_i).$$

Now start from the strategy profile $(v''_j, v'_{C_{-j}})$ and change the strategy of another bidder $j' \in C\backslash\{j\}$ from $v'_{j'}$ to $v''_{j'}$, which (weakly) increases the payoffs of bidders in C in a way analogous to (32). The inequality (31) will then follow from repeating the same argument one by one for all bidders in C. In order for v'_C not to be cartel dominated by v''_C, (31) must hold with equality for all $i \in C$.

To prove (30), we first establish that

$$u_i^M(v'_C, v_{N\backslash C}|v_i) = u_i^M(v''_C, v_{N\backslash C}|v_i), \forall i \in N\backslash C, \tag{33}$$

which is equivalent to

$$0 = \int_{\underline{v}_i}^{v_i} \left(q_i^*(s_i, v'_C, v_{N\backslash C_{+i}}) - q_i^*(s_i, v''_C, v_{N\backslash C_{+i}})\right) ds_i, \forall i \in N\backslash C.$$

This equality will hold if the integrand is equal to zero for a.e. $s_i \in (\underline{v}_i, v_i)$.[39] Suppose for a contradiction that for some $i \in N\backslash C$, the integrand is negative in an interval $(\underline{s}_i, \overline{s}_i) \subset (\underline{v}_i, v_i)$, i.e., $q_i^*(s_i, v'_C, v_{N\backslash C_{+i}}) < q_i^*(s_i, v''_C, v_{N\backslash C_{+i}}), \forall s_i \in (\underline{s}_i, \overline{s}_i)$.[40] Then, for all $s_i \in (\underline{s}_i, \overline{s}_i)$, there is some $h \in C'$ such that $q_h^*(s_i, v'_C, v_{N\backslash C_{+i}}) > q_h^*(s_i, v''_C, v_{N\backslash C_{+i}})$, since the values of bidders in

[39]Note that $q_i^*(s_i, v'_C, v_{N\backslash C_{+i}}) \leq q_i^*(s_i, v''_C, v_{N\backslash C_{+i}})$ since $v'_C \geq v''_C$ and q_i^* is decreasing in v_{-i}.

[40]Note that this inequality holds only if

$$\max\{\max_{j\in C} \bar{J}_j(v''_j), \max_{j\in N\backslash C_{+i}} \bar{J}_j(v_j)\} \leq \bar{J}_i(s_i) \leq \max\{\max_{j\in C} \bar{J}_j(v'_j), \max_{j\in N\backslash C_{+i}} \bar{J}_j(v_j)\},$$

which results in an interval of s_i's.

$N\backslash C$ do not change across the two profiles while the values of bidders in C strictly increase at least for some of them. Clearly, we must also have $v_h = v''_h < v'_h$ and $\bar{J}_h(v_h) < \bar{J}_h(v'_h)$. Using this, we show that (31) cannot hold as equality for the two profiles, $(s_i, v'_C, v_{N\backslash C_{+i}})$ and $(s_i, v''_C, v_{N\backslash C_{+i}})$, which will lead to the desired contradiction.

To do so, use (25), (28), and the fact that $v''_h = v_h$ to write

$$\begin{aligned}
&u_h^M(s_i, v''_C, v_{N\backslash C_{+i}}|v_h) - u_h^M(s_i, v'_C, v_{N\backslash C_{+i}}|v_h) \\
&= \int_{\underline{v}_h}^{v_h} q_h^*(s_i, s_h, v''_{C_{-h}}, v_{N\backslash C_{+i}}) ds_h \\
&\qquad - \left[(v_h - v'_h) q_h^*(s_i, v'_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) + \int_{\underline{v}_h}^{v'_h} q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) ds_h \right] \\
&= \int_{\underline{v}_h}^{v_h} \left(q_h^*(s_i, s_h, v''_{C_{-h}}, v_{N\backslash C_{+i}}) - q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) \right) ds_h \qquad (34) \\
&\qquad + \int_{v_h}^{v'_h} \left(q_h^*(s_i, v'_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) - q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) \right) ds_h. \qquad (35)
\end{aligned}$$

The integrands in (34) and (35) are both nonnegative due to the fact that q_h^* is decreasing in v_{-h} and increasing in v_h. As it is argued above that (31) must hold as equality, the payoff difference $u_h^M(s_i, v''_C, v_{N\backslash C_{+i}}|v_h) - u_h^M(s_i, v'_C, v_{N\backslash C_{+i}}|v_h)$ must be equal to zero, which means that (35) must also be equal to zero, implying $q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) = q_h^*(s_i, v'_h, v'_{C_{-h}}, v_{N\backslash C_{+i}})$ for all $s_h \in (v_h, v'_h)$. Since $q_h^*(s_i, v'_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) > q_h^*(s_i, v''_h, v''_{C_{-h}}, v_{N\backslash C_{+i}}) \geq q_h^*(s_i, v_h, v'_{C_{-h}}, v_{N\backslash C_{+i}})$, this in turn implies

$$q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) > q_h^*(s_i, v_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}), \forall s_h \in (v_h, v'_h]. \qquad (36)$$

To draw a contradiction, recall the assumption that $v \in \tilde{\mathcal{V}} \subset \tilde{\mathcal{V}}^h$. Consider first the case in which $v_h \in \text{int}(\mathcal{V}_h^0)$. Then, $\bar{J}_h(s_h) = \bar{J}_h(v_h)$ for all $s_h \in (v_h, v_h + \varepsilon)$ with some $\varepsilon > 0$, which implies that for all $s_h \in (v_h, v_h + \varepsilon)$, $q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) = q_h^*(s_i, v_h, v'_{C_{-h}}, v_{N\backslash C_{+i}})$, contradicting (36). Consider next the case in which $v_h \in \mathcal{V}_h \backslash \mathcal{V}_h^0$. We claim that $\bar{J}_h(v_h) = \bar{J}_i(s_i)$. If $\bar{J}_h(v_h) > \bar{J}_i(s_i)$, then $q_i^*(s_i, v''_C, v_{N\backslash C_{+i}}) = 0$, which contradicts with the fact that $q_i^*(s_i, v''_C, v_{N\backslash C_{+i}}) > 0$. If $\bar{J}_h(v_h) < \bar{J}_i(s_i)$, then $\bar{J}_h(s_h) < \bar{J}_i(s_i)$ for all $s_h \in [v_h, v_h + \varepsilon)$ with some $\varepsilon > 0$ and thus $q_h^*(s_i, s_h, v'_{C_{-h}}, v_{N\backslash C_{+i}}) = 0$ for all such s_h, which contradicts (36). Thus, we must have $\bar{J}_h(v_h) = \bar{J}_i(s_i)$. Given this, we cannot have $s_i \in \mathcal{V}_i^0$ since the fact that $v \in \tilde{\mathcal{V}} \subset \tilde{\mathcal{V}}^h$ and $v_h \in \mathcal{V}_h \backslash \mathcal{V}_h^0$ implies that $\bar{J}_h(v_h) \neq \bar{J}_j(v_j), \forall j \neq h, \forall v_j \in \mathcal{V}_j^0$. We have so far established that for all $s_i \in (\underline{s}_i, \overline{s}_i)$, $\bar{J}_i(s_i) = \bar{J}_h(v_h)$ and $s_i \in \mathcal{V}_i \backslash \mathcal{V}_i^0$, which cannot be true since $\bar{J}_i$ is strictly increasing over $\mathcal{V}_i \backslash \mathcal{V}_i^0$. Thus, the proof of (33) is complete.

The inequality (31) then follows immediately from (33) since u_i^M is decreasing in v_{-i} and $v''_C \leq v_C$ so $u_i^M(v''_C, v_{N\backslash C}|v_i) \geq u_i^M(v_C, v_{N\backslash C}|v_i)$, which establishes (30).

Now consider any partition $\pi^i = \{C^1, ..., C^k\} \in \Pi^i$. Repeatedly applying the argument used to establish (30) to the cartels C^1 through C^k, we obtain (29). ▌

Since $\tilde{\mathcal{V}}$ has a full measure, the fact that (29) holds for any $v \in \tilde{\mathcal{V}}$ implies that $\overline{U}_i^M(v_i) \geq \mathbb{E}_{v_{-i}}[u_i^M(v_i, v_{-i}|v_i)] = U_i^M(v_i)$ for a.e. $v_i \in \mathcal{V}_i$, which in fact means that $\overline{U}_i^M(v_i) \geq U_i^M(v_i)$ for all $v_i \in \mathcal{V}_i$, since the function $\overline{U}_i^M$, a value function of the optimization program given in (26), must be continuous.

In light of Lemma 5, for any arbitrary equilibrium $\tilde{M}$ of the cartel game, we must have $U_i^{\tilde{M}}(v_i) \geq \overline{U}_i^M(v_i) \geq U_i^M(v_i)$ for all $i \in N$ and $v_i \in \mathcal{V}_i$. The robust collusion-proofness of $M = (q^*, t^*)$ then immediately follows from the payoff equivalence, noting that because M is WCP (and therefore not interim dominated by $\tilde{M}$), we must have $U_i^{\tilde{M}}(v_i) = U_i^M(v_i)$ for all $i \in N$ and all $v_i \in \mathcal{V}_i$. ▌

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