Extremal $GI/GI/1$ Queues Given First Two Moments

Yan Chen

Joint Work with Ward Whitt
Department of Industrial Engineering and Operations Research

October 15, 2018
- Evaluate Quality of Queueing Approximations.
- Propose Tight Bounds for Complex Queueing Systems.
Motivation

- Evaluate Quality of Queueing Approximations.
- Propose Tight Bounds for Complex Queueing Systems.
Evaluate Quality of Queueing Approximations.
Propose Tight Bounds for Complex Queueing Systems.
Background

**GI/GI/1 Model:**

- **unlimited** waiting room and **FCFS** discipline.
- mean inter-arrival time and service time \((m_1 = 1, s_1 = \rho)\).
- fixing the first two moments \(m_2 = (1 + c_s^2)m_1^2, s_2 = (1 + c_s^2)s_1^2\).
- \(F\) inter-arrival time \([0, M_a]\), \(G\) service time \([0, M_s]\).

**Approximations and Bounds:**

Background

**GI/GI/1 Model:**
- unlimited waiting room and FCFS discipline.
- mean inter-arrival time and service time \((m_1 = 1, s_1 = \rho)\).
- fixing the first two moments \(m_2 = (1 + c_2^2)m_1^2, s_2 = (1 + c_2^2)s_1^2\).
- \(F\) inter-arrival time \([0, M_a]\), \(G\) service time \([0, M_s]\).

Approximations and Bounds:
**Background**

**GI/GI/1 Model:**
- **unlimited** waiting room and **FCFS** discipline.
- mean inter-arrival time and service time \((m_1 = 1, \ s_1 = \rho)\).
- fixing the first two moments \(m_2 = (1 + c_a^2)m_1^2, s_2 = (1 + c_s^2)s_1^2\).
- \(F\) inter-arrival time \([0, M_a]\), \(G\) service time \([0, M_s]\).

**Approximations and Bounds:**
**Background**

*GI/GI/1 Model:*

- unlimited waiting room and FCFS discipline.
- mean inter-arrival time and service time \((m_1 = 1, s_1 = \rho)\).
- fixing the first two moments \(m_2 = (1 + c_a^2)m_1^2, s_2 = (1 + c_s^2)s_1^2\).
- \(F\) inter-arrival time \([0, M_a]\), \(G\) service time \([0, M_s]\).

**Approximations and Bounds:**

### Background

**GI/GI/1 Model:**
- **unlimited** waiting room and FCFS discipline.
- mean inter-arrival time and service time \((m_1 = 1, s_1 = \rho)\).
- fixing the first two moments \(m_2 = (1 + c_a^2)m_1^2, s_2 = (1 + c_s^2)s_1^2\).
- \(F\) inter-arrival time \([0, M_a]\), \(G\) service time \([0, M_s]\).

**Approximations and Bounds:**
Background

**GI/GI/1 Model:**
- **unlimited** waiting room and FCFS discipline.
- mean inter-arrival time and service time \((m_1 = 1, s_1 = \rho)\).
- fixing the first two moments \(m_2 = (1 + c_a^2)m_1^2, s_2 = (1 + c_s^2)s_1^2\).
- \(F\) inter-arrival time \([0, M_a]\), \(G\) service time \([0, M_s]\).

**Approximations and Bounds:**
Background

GI/GI/1 Model:
- unlimited waiting room and FCFS discipline.
- mean inter-arrival time and service time \( (m_1 = 1, s_1 = \rho) \).
- fixing the first two moments \( m_2 = (1 + c_a^2)m_1^2, s_2 = (1 + c_s^2)s_1^2 \).
- \( F \) inter-arrival time \([0, M_a]\), \( G \) service time \([0, M_s]\).

Approximations and Bounds:
Background

- $\mathcal{P}_2(M) \equiv \mathcal{P}_2(m_1, c^2, M)$: cdf’s with support in the interval $[0, M]$ given mean $m_1$ and second moment $(c^2 + 1)m_1^2$.
- $\mathcal{P}_{a,2,k}(M_a) \times \mathcal{P}_{a,2,k}(M_s)$: k-point distribution by fixing mean, variance and bounded support.
- Steady-State Waiting Time:

\[
W \overset{d}{=} [W + V - U]^+,
\]

\[
w : \mathcal{P}_{a,2}(M_a) \times \mathcal{P}_{s,2}(M_s) \to \mathbb{R},
\]

where $0 < \rho < 1$ and

\[
w(F, G) \equiv E[W(F, G)].
\]
\begin{itemize}
    \item \( \mathcal{P}_2(M) \equiv \mathcal{P}_2(m_1, c^2, M) \): cdf’s with support in the interval \([0, M]\) given mean \( m_1 \) and second moment \((c^2 + 1)m_1^2\).
    
    \item \( \mathcal{P}_{a,2,k}(M_a) \times \mathcal{P}_{a,2,k}(M_s) \): \textbf{k-point distribution} by fixing mean, variance and bounded support.
\end{itemize}

\begin{itemize}
    \item Steady-State Waiting Time:
    \[
    W \overset{d}{=} [W + V - U]^+, \\
    \]
    
    \[
    w : \mathcal{P}_{a,2}(M_a) \times \mathcal{P}_{s,2}(M_s) \rightarrow \mathbb{R}, \\
    \]
    
    where \( 0 < \rho < 1 \) and
    \[
    w(F, G) \equiv E[W(F, G)].
    \]
\end{itemize}
Background

- $\mathcal{P}_2(M) \equiv \mathcal{P}_2(m_1, c^2, M)$: cdf’s with support in the interval $[0, M]$ given mean $m_1$ and second moment $(c^2 + 1)m_1^2$.
- $\mathcal{P}_{a,2,k}(M_a) \times \mathcal{P}_{a,2,k}(M_s)$: k-point distribution by fixing mean, variance and bounded support.
- Steady-State Waiting Time:

$$W \overset{d}{=} [W + V - U]^+,$$

$$w : \mathcal{P}_{a,2}(M_a) \times \mathcal{P}_{s,2}(M_s) \rightarrow \mathbb{R},$$

where $0 < \rho < 1$ and

$$w(F, G) \equiv E[W(F, G)].$$
Background

- $\mathcal{P}_2(M) \equiv \mathcal{P}_2(m_1, c^2, M)$: cdf’s with support in the interval $[0, M]$ given mean $m_1$ and second moment $(c^2 + 1)m_1^2$.
- $\mathcal{P}_{a,2,k}(M_a) \times \mathcal{P}_{a,2,k}(M_s)$: $k$-point distribution by fixing mean, variance and bounded support.
- **Steady-State** Waiting Time:

$$W \overset{d}{=} [W + V - U]^+,$$

$$w : \mathcal{P}_{a,2}(M_a) \times \mathcal{P}_{s,2}(M_s) \to \mathbb{R},$$

where $0 < \rho < 1$ and

$$w(F, G) \equiv E[W(F, G)].$$
Queueing Bounds:

- **Pollaczek-Khintchine formula:**
  \[
  E[W] = \frac{\tau \rho (1 + c_s^2)}{2(1 - \rho)} = \frac{\rho^2 (1 + c_s^2)}{2(1 - \rho)}.
  \]

- **Kingman (1962) bound:**
  \[
  E[W] \leq \frac{\rho^2 ([c_s^2 / \rho^2] + c_s^2)}{2(1 - \rho)}.
  \]

- **Daley (1977) bound:**
  \[
  E[W] \leq \frac{\rho^2 ([2 - \rho) c_s^2 / \rho] + c_s^2)}{2(1 - \rho)}.
  \]
Queueing Bounds:

- **Pollaczek-Khintchine formula:**

  \[
  E[W] = \frac{\tau \rho (1 + c_s^2)}{2(1 - \rho)} = \frac{\rho^2 (1 + c_s^2)}{2(1 - \rho)}.
  \]

- **Kingman (1962) bound:**

  \[
  E[W] \leq \frac{\rho^2 ([c_a^2 / \rho^2] + c_s^2)}{2(1 - \rho)},
  \]

- **Daley (1977) bound:**

  \[
  E[W] \leq \frac{\rho^2 ([2 - \rho] c_a^2 / \rho] + c_s^2)}{2(1 - \rho)}.
  \]
Queueing Bounds:

- Pollaczek-Khintchine formula:
  \[ E[W] = \frac{\tau \rho (1 + c_s^2)}{2(1 - \rho)} = \frac{\rho^2(1 + c_s^2)}{2(1 - \rho)}. \]

- Kingman (1962) bound:
  \[ E[W] \leq \frac{\rho^2([c_a^2/\rho^2] + c_s^2)}{2(1 - \rho)}, \]

- Daley (1977) bound:
  \[ E[W] \leq \frac{\rho^2([(2 - \rho)c_a^2/\rho] + c_s^2)}{2(1 - \rho)}. \]
Objectives

We want to answer a Long Standing Open Problem in Queueing Theory.

- Given any $G$, what is the extremal inter-arrival time dist $F^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- Given any $F$, what is the extremal service time dist $G^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- What are the extremal $F^*$ and $G^*$ leading to overall UB and LB of $\mathbb{E}[W(F/G/1)]$?
Objectives

We want to answer a **Long Standing Open Problem** in Queueing Theory.

- Given any $G$, what is the **extremal inter-arrival time dist** $F^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- Given any $F$, what is the extremal service time dist $G^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- What are the extremal $F^*$ and $G^*$ leading to overall UB and LB of $\mathbb{E}[W(F/G/1)]$?
We want to answer a Long Standing Open Problem in Queueing Theory.

- Given any $G$, what is the extremal inter-arrival time dist $F^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- Given any $F$, what is the extremal service time dist $G^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- What are the extremal $F^*$ and $G^*$ leading to overall UB and LB of $\mathbb{E}[W(F/G/1)]$?
Objectives

We want to answer a Long Standing Open Problem in Queueing Theory.

- Given any $G$, what is the extremal inter-arrival time dist $F^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- Given any $F$, what is the extremal service time dist $G^*$ attaining the UB and LB of $\mathbb{E}[W(F/G/1)]$?
- What are the extremal $F^*$ and $G^*$ leading to overall UB and LB of $\mathbb{E}[W(F/G/1)]$?
Objectives

We want to solve an optimization problem:

- **objectives**: \( \sup / \inf E[W(F, G)] \)
- **subject to**: \( F \in \mathcal{P}_{a,2}(M_a), \, G \in \mathcal{P}_{s,2}(M_s) \).

Challenges: a) Non tractable formulation b) Hard to know and Hard to guess.
Objectives

We want to solve an optimization problem:

- **objectives**: sup / inf $E[W(F, G)]$
- **subject to**: $F \in \mathcal{P}_{a,2}(M_a), G \in \mathcal{P}_{s,2}(M_s)$.

Challenges: a) **Non tractable formulation** b) **Hard to know and Hard to guess**.
Two-point Distribution

\[ \mathcal{P}_{a,2,2}(M_a) \times \mathcal{P}_{s,2,2}(M_s): \text{two-point dists given two moments:} \]

- \( F \in \mathcal{P}_{a,2,2} \): \( c_a^2/(c_a^2 + (b_a - 1)^2) \) at \( b_a \),
  \( (b_a - 1)^2/(c_a^2 + (b_a - 1)^2) \) at \( 1 - c_a^2/(b_a - 1) \)
where \( 1 + c_a^2 \leq b_a \leq M_a \).

- one-parameter family indexed by \( b_a \).

- \( F_u \): \( c_a^2/(c_a^2 + (r - 1)^2) \) at \( M_a \) and \( (r - 1)^2/(c_a^2 + (r - 1)^2) \) on
  \( 1 - c_a^2/(r - 1) \) for \( r = M_a \).

- \( F_0 \): \( c_a^2/(1 + c_a^2) \) at 0, \( 1/(c_a^2 + 1) \) at \( (m_2/m_1) = m_1(c_a^2 + 1) \) for
  \( r = 1 + c_a^2 \).

We can define \( G_0, G_u \) by the same argument.
Two-point Distribution

\[ \mathcal{P}_{a,2,2}(M_a) \times \mathcal{P}_{s,2,2}(M_s) : \text{two-point dists given two moments:} \]

- \( F \in \mathcal{P}_{a,2,2} : \frac{c_a^2}{(c_a^2 + (b_a - 1)^2)} \) at \( b_a \),
- \( \frac{(b_a - 1)^2}{(c_a^2 + (b_a - 1)^2)} \) at \( 1 - \frac{c_a^2}{(b_a - 1)} \)

where \( 1 + c_a^2 \leq b_a \leq M_a \).

**one-parameter family indexed by \( b_a \)**

- \( F_u : \frac{c_a^2}{(c_a^2 + (r - 1)^2)} \) at \( M_a \) and \( \frac{(r - 1)^2}{(c_a^2 + (r - 1)^2)} \) on \( 1 - \frac{c_a^2}{(r - 1)} \) for \( r = M_a \).
- \( F_0 : \frac{c_a^2}{1 + c_a^2} \) at \( 0 \), \( \frac{1}{(c_a^2 + 1)} \) at \( \frac{m_2}{m_1} = m_1(c_a^2 + 1) \) for \( r = 1 + c_a^2 \).

We can define \( G_0, G_u \) by the same argument.
Two-point Distribution

\( \mathcal{P}_{a,2,2}(M_a) \times \mathcal{P}_{s,2,2}(M_s) \): two-point dists given two moments:

- **\( F \in \mathcal{P}_{a,2,2} \):**
  \[ c_a^2 / (c_a^2 + (b_a - 1)^2) \text{ at } b_a, \]
  \[ (b_a - 1)^2 / (c_a^2 + (b_a - 1)^2) \text{ at } 1 - c_a^2 / (b_a - 1) \]

where \( 1 + c_a^2 \leq b_a \leq M_a \).

  one-parameter family indexed by \( b_a \)

- **\( F_u \):**
  \[ c_a^2 / (c_a^2 + (r - 1)^2) \text{ at } M_a \]
  \[ (r - 1)^2 / (c_a^2 + (r - 1)^2) \text{ on } 1 - c_a^2 / (r - 1) \text{ for } r = M_a. \]

- **\( F_0 \):**
  \[ c_a^2 / (1 + c_a^2) \text{ at } 0, 1 / (c_a^2 + 1) \text{ at } (m_2/m_1) = m_1(c_a^2 + 1) \text{ for } r = 1 + c_a^2. \]

We can define \( G_0, G_u \) by the same argument.
Two-point Distribution

\(\mathcal{P}_{a,2,2}(M_a) \times \mathcal{P}_{s,2,2}(M_s)\): **two-point** dists given two moments:

- \(F \in \mathcal{P}_{a,2,2}: \frac{c_a^2}{(c_a^2 + (b_a - 1)^2)}\) at \(b_a\),
  \(\frac{(b_a - 1)^2}{(c_a^2 + (b_a - 1)^2)}\) at \(1 - \frac{c_a^2}{(b_a - 1)}\)

  where \(1 + c_a^2 \leq b_a \leq M_a\).

  **one-parameter family indexed by** \(b_a\)

- \(F_u: \frac{c_a^2}{(c_a^2 + (r - 1)^2)}\) at \(M_a\) and \(\frac{(r - 1)^2}{(c_a^2 + (r - 1)^2)}\) on \(1 - \frac{c_a^2}{(r - 1)}\) for \(r = M_a\).

- \(F_0: \frac{c_a^2}{1 + c_a^2}\) at 0, \(\frac{1}{(c_a^2 + 1)}\) at \(\frac{m_2}{m_1} = m_1(c_a^2 + 1)\) for \(r = 1 + c_a^2\).

We can define \(G_0, G_u\) by the same argument.
$\mathcal{P}_{a,2,2}(M_a) \times \mathcal{P}_{s,2,2}(M_s)$: two-point dists given two moments:

- $F \in \mathcal{P}_{a,2,2}$: $c_a^2/(c_a^2 + (b_a - 1)^2)$ at $b_a$, 
  $(b_a - 1)^2/(c_a^2 + (b_a - 1)^2)$ at $1 - c_a^2/(b_a - 1)$
where $1 + c_a^2 \leq b_a \leq M_a$.

  one-parameter family indexed by $b_a$

- $F_u$: $c_a^2/(c_a^2 + (r - 1)^2)$ at $M_a$ and $(r - 1)^2/(c_a^2 + (r - 1)^2)$ on $1 - c_a^2/(r - 1)$ for $r = M_a$.
- $F_0$: $c_a^2/(1 + c_a^2)$ at $0$, $1/(c_a^2 + 1)$ at $(m_2/m_1) = m_1(c_a^2 + 1)$ for $r = 1 + c_a^2$.

We can define $G_0, G_u$ by the same argument.
Main Results

(a) Given any parameter vector \((1, c_a^2, \rho, c_s^2)\) and a bounded interval of support \([0, M_s]\) for the service-time cdf \(G\), where \(M_s \geq \rho(c_s^2 + 1)\), the pair \((F_0, G_u)\) attains the tight UB of the steady-state mean \(E[W]\), while a pair \((F_0, G_{u,n})\) attains the tight UB of the transient mean \(E[W_n]\), where \(G_{u,n}\) is a two-point distribution with \(G_{u,n} \Rightarrow G_u\) as \(n \to \infty\).
(b) For the unbounded interval of support $[0, \infty)$, the tight UB of $E[W]$ is not attained directly, but is obtained asymptotically in the limit as $M_s \to \infty$ in part (a). The extremal service-time cdf $G_u$ is asymptotically deterministic with the given mean $\rho$ as $M_s \to \infty$, but that deterministic distribution does not have parameter $c_s^2$ if $c_s^2 \neq 0$. Moreover, the mean $E[W(F_0, G_u)]$ does not approach the mean in the associated extremal $F_0/D/1$ queue as $M_s \to \infty$, yet approach to $\lim_{M_s \to \infty} E[W(F_0, G_u^*)]$. 
Main Theoretic Results

**Theorem**

(a version of the classic moment problem) Let $\phi : \mathcal{C} \to \mathbb{R}$ be a continuous function, where $\mathcal{C}$ is a compact subset of $[0, M]$. Assume that $\mathcal{P}_n$ is not empty. Then there exists $P^* \in \mathcal{P}_{n,n+1}$ such that

$$
\sup \left\{ \int_0^M \phi \, dP : P \in \mathcal{P}_n \right\} = \sup \left\{ \int_0^M \phi \, dP : P \in \mathcal{P}_{n,n+1} \right\} = \sum_{k=1}^{n+1} \phi(t_k) P^*\left(\{t_k\}\right),
$$

where $\{t_k : 1 \leq k \leq n + 1\}$ is the support of $P^*$. 

Main Theoretic Results

**Theorem**

(reduction to a three-point distribution) Consider the class of GI/GI/1 queues with interarrival times \( \{U_n\} \) distributed as \( U \) with cdf \( F \in \mathcal{P}_{a,2} \) and service times \( \{V_n\} \) distributed as \( V \) with cdf \( G \in \mathcal{P}_{s,2} \) where \( 0 < \rho < 1 \) and the sets \( \mathcal{P}_{a,2} \) and \( \mathcal{P}_{s,2} \) are nonempty. The function \( w: \mathcal{P}_{a,2} \times \mathcal{P}_{s,2} \to \mathbb{R} \) is continuous. Hence, the following suprema are attained as indicated:

(a) For any specified \( G \in \mathcal{P}_{s,2} \), there exists \( F^*(G) \in \mathcal{P}_{a,2,3}(M_a) \) such that

\[
\begin{align*}
\sup_{F \in \mathcal{P}_{a,2}} w(F, G) &= \sup \{ w(F, G) : F \in \mathcal{P}_{a,2} \} = \sup \{ w(F, G) : F \in \mathcal{P}_{a,2,3}(M_a) \} = w(F^*(G), G).
\end{align*}
\]
Main Theoretic Results

Theorem

(reduction to a three-point distribution) Consider the class of GI/GI/1 queues with interarrival times \( \{U_n\} \) distributed as \( U \) with cdf \( F \in \mathcal{P}_{a,2} \) and service times \( \{V_n\} \) distributed as \( V \) with cdf \( G \in \mathcal{P}_{s,2} \) where \( 0 < \rho < 1 \) and the sets \( \mathcal{P}_{a,2} \) and \( \mathcal{P}_{s,2} \) are nonempty. The function \( w : \mathcal{P}_{a,2} \times \mathcal{P}_{s,2} \to \mathbb{R} \) is continuous. Hence, the following suprema are attained as indicated:

(a) For any specified \( G \in \mathcal{P}_{s,2} \), there exists \( F^*(G) \in \mathcal{P}_{a,2,3}(M_a) \) such that

\[
\begin{align*}
\left( w^*_a \right)(G) & \equiv \sup \{ w(F, G) : F \in \mathcal{P}_{a,2} \} \\
& = \sup \{ w(F, G) : F \in \mathcal{P}_{a,2,3}(M_a) \} = w(F^*(G), G).
\end{align*}
\]
### Theorem

*(continued)*

(b) For any specified \( F \in \mathcal{P}_{a,2} \), there exists \( G^*(F) \in \mathcal{P}_{s,2,3}(M_s) \) such that

\[
\begin{align*}
    w_s^\uparrow(F) & \equiv \sup \{ w(F, G) : G \in \mathcal{P}_{s,2}(M_s) \} \\
                       & = \sup \{ w(F, G) : G \in \mathcal{P}_{s,2,3}(M_s) \} = w(F, G^*(F)).
\end{align*}
\]

(c) There exists \((F^{**}, G^{**})\) in \( \mathcal{P}_{a,2,3}(M_a) \times \mathcal{P}_{s,2,3}(M_s) \) such that

\[
\begin{align*}
    w^\uparrow & \equiv \sup \{ w(F, G) : F \in \mathcal{P}_{a,2}, G \in \mathcal{P}_{s,2}(M_s) \} \\
                     & = \sup \{ w(F, G) : F \in \mathcal{P}_{a,2,3}(M_a), G \in \mathcal{P}_{s,2,3}(M_s) \} \\
                     & = w(F^{**}, G^{**}) = w^\uparrow_a(G^{**}) = w_s^\uparrow(F^{**}).
\end{align*}
\]
Ideas of Proof for Case (a)

- Start with $W \overset{d}{=} [W + V_G - U_F]^+$. 
  
- $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for $\phi$ expressed as the double integral
  \[
  \phi(u) = \int_0^{\infty} \int_0^{\infty} (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.
  \]

- Optimize over $F$: $\sup_{F \in \mathcal{P}_{a,2}} \int_0^{M_a} \phi(u) dF$. It can be written as
  \[
  \sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}.
  \]

- Update $W_1 = (W + V - U_{F_1})^+$. $E[W_1] \geq E[W]$?

- Repeat to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $E[W_2] \geq E[W_1]$?
Ideas of Proof for Case (a)

- Start with $W \overset{d}{=} [W + V_G - U_F]^+$. 
- $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for $\phi$ expressed as the double integral 

$$
\phi(u) \equiv \int_0^\infty \int_0^\infty (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.
$$

- Optimize over $F$: $\sup_{F \in \mathcal{P}_{a,2}} \int_0^{M_a} \phi(u) dF$. It can be written as 

$$
\sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}.
$$

- Update $W_1 = (W + V - U_{F_1})^+$. $E[W_1] \geq E[W]$ ?
- Repeat to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $E[W_2] \geq E[W_1]$ ?
Ideas of Proof for Case (a)

- Start with $W \overset{d}{=} [W + V_G - U_F]^+$.
- $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for $\phi$ expressed as the double integral
  $$\phi(u) \equiv \int_0^{\infty} \int_0^{\infty} (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.$$  
- Optimize over $F$: $\sup_{F \in \mathcal{P}_{a,2}} \int_0^{M_a} \phi(u) dF$. It can be written as
  $$\sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}.$$  
- Update $W_1 = (W + V - U_F)^+$. $E[W_1] \geq E[W]$?
- Repeat to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $E[W_2] \geq E[W_1]$?
Ideas of Proof for Case (a)

- Start with $W \overset{d}{=} [W + V_G - U_F]^+$.

- $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u)dF$ for $\phi$ expressed as the double integral
  
  $$\phi(u) \equiv \int_0^\infty \int_0^\infty (x + v - u)^+ dG(v)dH(x), \quad 0 \leq u \leq M_a.$$ 

- Optimize over $F$: $\sup_{F \in \mathcal{P}_{a,2}} \int_0^{M_a} \phi(u)dF$. It can be written as
  
  $$\sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}.$$ 

- Update $W_1 = (W + V - U_{F_1})^+$. $E[W_1] \geq E[W]?$

- Repeat to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $E[W_2] \geq E[W_1]?$
Ideas of Proof for Case (a)

- Start with $W \overset{d}{=} [W + V_G - U_F]^+$. 

- $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for $\phi$ expressed as the double integral

  $$
  \phi(u) \equiv \int_0^\infty \int_0^\infty (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.
  $$

- Optimize over $F$: $\sup_{F \in \mathcal{P}_{a,2}} \int_0^{M_a} \phi(u) dF$. It can be written as

  $\sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}$. 

- Update $W_1 = (W + V - U_{F_1})^+$. $E[W_1] \geq E[W]$?

- Repeat to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $E[W_2] \geq E[W_1]$?
Ideas of Proof for Case (a)

- Suppose fixed point exists:

\[(W(F^*, G) + V - U_{F^*})^+ \overset{d}{=} W(F^*, G).\]

With the property

\[E[(W(F^*, G) + V - U_{F^*})^+] = \sup \{E[(W(F, G) + V - U_F)^+] : F \in \mathcal{P}_{a,2}\}.\]

- Show \(F^* \in \mathcal{P}_{a,2,3}.\)
Ideas of Proof for Case (a)

- Suppose **fixed point** exists:

\[
(W(F^*, G) + V - U_{F^*})^+ \overset{d}{=} W(F^*, G). 
\]

With the property

\[
E[(W(F^*, G) + V - U_{F^*})^+]
= \sup \{E[(W(F, G) + V - U_F)^+] : F \in \mathcal{P}_{a,2}\}.
\]

- **Show** \(F^* \in \mathcal{P}_{a,2,3}\).
Step 1: Characterize the Fixed Point.

- a) Define $\eta(F)$ as the set of all $F_1$ as a cdf of $U$ that attains the supremum of the function $\zeta : \mathcal{P}_{a,2} \rightarrow \mathbb{R}$ defined by

$$\zeta(F) \equiv \sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}.$$ 

- b) Let $\mathcal{P}^*_a$ be the subset of all fixed points of the map $\eta : \mathcal{P}_{a,2} \rightarrow 2^{\mathcal{P}_{a,2}}$ defined above, i.e.,

$$\mathcal{P}^*_a = \{ F \in \mathcal{P}_{a,2} : F \in \eta(F) \}.$$ 

We will prove that the set $\mathcal{P}^*_a$ is nonempty and always contains an element of $\mathcal{P}^*_a$. 
Sketch of Proof

Step 1: Characterize the Fixed Point.

a) Define $\eta(F)$ as the set of all $F_1$ as a cdf of $U$ that attains the supremum of the function $\zeta : \mathcal{P}_{a,2} \rightarrow \mathbb{R}$ defined by

$$\zeta(F) \equiv \sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2} \}.$$ 

b) Let $\mathcal{P}_{a,2}^*$ be the subset of all fixed points of the map $\eta : \mathcal{P}_{a,2} \rightarrow 2^{\mathcal{P}_{a,2}}$ defined above, i.e.,

$$\mathcal{P}_{a,2}^* \equiv \{ F \in \mathcal{P}_{a,2} : F \in \eta(F) \}.$$ 

We will prove that the set $\mathcal{P}_{a,2}^*$ is nonempty and always contains an element of $\mathcal{P}_{a,2,3}^*$. 
Step 2. Existence by Berge and Kakutani.

- a) Apply the Berge Maximum Theorem: map $\eta$ defined as the set of maximizers has a closed graph:
  i) upper hemicontinuous property, ii) $\phi(.)$ is continuous and bounded and iii) Closed Graph Theorem over compact domain.

- b) Apply the Kakutani Fixed Point Theorem.

- c) Restrict probability space with finite support (Need first asymptotic argument).
Sketch of Proof

Step 2. Existence by Berge and Kakutani.

a) Apply the Berge Maximum Theorem: map \( \eta \) defined as the set of maximizers has a closed graph:
   i) upper hemicontinuous property, ii) \( \phi(.) \) is continuous and bounded and iii) Closed Graph Theorem over compact domain.

b) Apply the Kakutani Fixed Point Theorem.

c) Restrict probability space with finite support (Need first asymptotic argument).
Step 2. Existence by Berge and Kakutani.

a) Apply the Berge Maximum Theorem: map $\eta$ defined as the set of maximizers has a closed graph:
   i) upper hemicontinuous property, ii) $\phi(.)$ is continuous and bounded and iii) Closed Graph Theorem over compact domain.

b) Apply the Kakutani Fixed Point Theorem.

c) Restrict probability space with finite support (Need first asymptotic argument).
Sketch of Proof

Step 3. Completing the proof of the existence of a fixed-point.
  a) Apply Kakutani’s Fixed Point Theorem

**Theorem**

(Kakutani’s fixed point theorem) If $S$ is a non-empty compact and convex subset of some Euclidean space $\mathbb{R}^d$ and $\psi : S \rightarrow 2^S$ is a set-valued function with a closed graph such that $\psi(x)$ is non-empty and convex for all $x \in S$, then the map $\psi$ has a fixed point, i.e., there exists $x \in S$ such that $x \in \psi(x)$.

b) Fixed Point Exists

**Lemma**

(fixed point in the proof of Theorem 2) There exists a fixed point in $P_{a,2}$ of the map $\eta : P_{a,2} \rightarrow 2^{P_{a,2}}$. 
**Sketch of Proof**

**Step 3.** Completing the proof of the existence of a fixed-point.
- a) Apply Kakutani’s Fixed Point Theorem

**Theorem**

(Kakutani’s fixed point theorem) If \( S \) is a non-empty compact and convex subset of some Euclidean space \( \mathbb{R}^d \) and \( \psi : S \rightarrow 2^S \) is a set-valued function with a closed graph such that \( \psi(x) \) is non-empty and convex for all \( x \in S \), then the map \( \psi \) has a fixed point, i.e., there exists \( x \in S \) such that \( x \in \psi(x) \).

- b) Fixed Point Exists

**Lemma**

(fixed point in the proof of Theorem 2) There exists a fixed point in \( P_{a,2} \) of the the map \( \eta : P_{a,2} \rightarrow 2^{P_{a,2}} \).
Step 4. Uniqueness via Duality

- a) **Duality**: Focusing on $\phi$. The objective of the dual problem is to find the vector $\lambda^* \equiv (\lambda_0^*, \lambda_1^*, \lambda_2^*)$ that attains the infimum

$$
\gamma(m_1, m_2) \equiv \inf_{\lambda \equiv (\lambda_0, \lambda_1, \lambda_2)} \{\lambda_0 + \lambda_1 m_1 + \lambda_2 m_2\},
$$

where $m_i \equiv E[U^i], \ i = 1, 2$ and $\lambda_i$ are the decision variables (which are unconstrained), such that

$$
\psi(u) \equiv \lambda_0 + \lambda_1 u + \lambda_2 u^2 \geq \phi(u) \quad \text{for all} \quad u \in \mathcal{F}
$$

where $\mathcal{F}$ is the support of $F$. 

Thus, $\lambda^*$ provides the unique solution for the problem.
Sketch of Proof

Step 5. Uniqueness via Duality

- a) Restrict $K_n \cap P_{s,2}$ (Need second asymptotic argument).
- b) Apply non-degeneracy and strictly convexity lemmas:

  Lemma (non-degeneracy and uniqueness in LP) A standard LP has a unique optimal solution if and only if its dual has a non-degenerate optimal solution.

  Lemma (strictly convexity and decrease) If density of $V ightarrow W$ is strictly positive on $[0, M_a]$, as occurs when the cdf $G$ of $V$ is in $K_m$, then $\phi$ is strictly decreasing and strictly convex on $[0, M_a]$. 
Sketch of Proof

Step 5. Uniqueness via Duality

a) Restrict $K_n \cap \mathcal{P}_{s,2}$ (Need second asymptotic argument).

b) Apply non-degeneracy and strictly convexity lemmas:

Lemma

(non-degeneracy and uniqueness in LP) A standard LP has a unique optimal solution if and only if its dual has a non-degenerate optimal solution.

Lemma

(strictly convexity and decrease) If density of $V + W$ is strictly positive on $[0, M_a]$, as occurs when the cdf $G$ of $V$ is in $K_n$, then $\phi$ is strictly decreasing and strictly convex on $[0, M_a]$. 
Step 5. Uniqueness via Duality

a) Restrict $K_n \cap \mathcal{P}_{s,2}$ (Need second asymptotic argument).

b) Apply non-degeneracy and strictly convexity lemmas:

Lemma

(non-degeneracy and uniqueness in LP) A standard LP has a unique optimal solution if and only if its dual has a non-degenerate optimal solution.

Lemma

(strictly convexity and decrease) If density of $V + W$ is strictly positive on $[0, M_a]$, as occurs when the cdf $G$ of $V$ is in $K_n$, then $\phi$ is strictly decreasing and strictly convex on $[0, M_a]$. 
Step 5. Uniqueness via Duality

- a) Restrict $K_n \cap \mathcal{P}_{s,2}$ (Need second asymptotic argument).
- b) Apply non-degeneracy and strictly convexity lemmas:

**Lemma**

(non-degeneracy and uniqueness in LP) A standard LP has a unique optimal solution if and only if its dual has a non-degenerate optimal solution.

**Lemma**

(strictly convexity and decrease) If density of $V + W$ is strictly positive on $[0, M_a]$, as occurs when the cdf $G$ of $V$ is in $K_n$, then $\phi$ is strictly decreasing and strictly convex on $[0, M_a]$. 
Step 6. Asymptotical Arguments

a) Restrict to all cdf’s $F$ with support

$$S_{k+1} \equiv \{x_1, \ldots, x_{k+1} : 0 \leq x_1 < \cdots < x_{k+1} \leq x_u\}.$$ 

b) Complete the first asymptotical argument.

Tightness $\{F(k)\}$ and subsequence convergence:

$$E[W_{kj}] = E[(W_{kj} + V - U_{F(kj)})^+] \quad \text{for all} \quad j \geq 1$$

c) Complete the second asymptotical argument.

Denseness of $K_n \cap \mathcal{P}_{s,2}$:

Lemma

(a dense subset) The subset $K_n \cap \mathcal{P}_{s,2}$ is a dense subset of $\mathcal{P}_{s,2}$.
Step 6. Asymptotical Arguments

a) Restrict to all cdf’s $F$ with support

$$S_{k+1} \equiv \{x_1, \ldots, x_{k+1} : 0 \leq x_1 < \cdots < x_{k+1} \leq x_u\}.$$ 

b) Complete the first asymptotical argument.

Tightness $\{F(k)\}$ and subsequence convergence:

$$E[W_{k_j}] = E[(W_{k_j} + V - U_{F(k_j)})^+] \quad \text{for all} \quad j \geq 1$$

c) Complete the second asymptotical argument.

Denseness of $K_n \cap \mathcal{P}_{s,2}$:

Lemma

(a dense subset) The subset $K_n \cap \mathcal{P}_{s,2}$ is a dense subset of $\mathcal{P}_{s,2}$. 

Step 6. Asymptotical Arguments

a) Restrict to all cdf’s $F$ with support

$$S_{k+1} \equiv \{x_1, \ldots, x_{k+1} : 0 \leq x_1 < \cdots < x_{k+1} \leq x_u\}.$$ 

b) Complete the first asymptotical argument. Tightness $\{F(k)\}$ and subsequence convergence:

$$E[W_{kj}] = E[(W_{kj} + V - U_{F(kj)})^+] \quad \text{for all} \quad j \geq 1$$

c) Complete the second asymptotical argument. Denseness of $K_n \cap P_{s,2}$:

**Lemma**

(a dense subset) The subset $K_n \cap P_{s,2}$ is a dense subset of $P_{s,2}$. 
Sketch of Proof

Step 7. Solving the case (b).

- a) A Reverse-Time Representation:

\[ \zeta(G_1) \equiv \sup \{ E[(W_1 + (M_s - V) - U)^+] : G_V \in \mathcal{P}_{s,2} \} \]

- b) Uniqueness by Duality via using

**Lemma**

if \( F \in K_n, \phi_s(v) \) is strictly positive, strictly decreasing and strictly convex on \([0, M_s] \).

- c) Two asymptotic arguments as before.

Step 8. Obtain the case (c) by using the case (a) and (b).
Step 7. Solving the case (b).

- a) A Reverse-Time Representation:

\[ \zeta(G_1) \equiv \sup \{ E[(W_1 + (M_s - V) - U)^+] : G_V \in \mathcal{P}_{s,2} \} . \]

- b) Uniqueness by Duality via using

Lemma

*if* \( F \in K_n, \phi_s(v) is \phi_s is strictly positive, strictly decreasing and strictly convex on \([0, M_s]\).*

- c) Two asymptotic arguments as before.

Step 8. Obtain the case (c) by using the case (a) and (b).
Step 7. Solving the case (b).

- a) A Reverse-Time Representation:
  \[ \zeta(G_1) \equiv \sup \{ E[(W_1 + (M_s - V) - U)^+] : G_V \in \mathcal{P}_{s,2} \} \]

- b) Uniqueness by Duality via using

**Lemma**

If \( F \in K_n \), \( \phi_s(v) \) is \( \phi_s \) is strictly positive, strictly decreasing and strictly convex on \([0, M_s]\).

- c) Two asymptotic arguments as before.

Step 8. Obtain the case (c) by using the case (a) and (b).
Step 7. Solving the case (b).

- a) A Reverse-Time Representation:
  \[ \zeta(G_1) \equiv \sup \{ E[(W_1 + (M_s - V) - U)^+] : G_V \in \mathcal{P}_{s,2} \} \]

- b) Uniqueness by Duality via using

\[ \text{Lemma} \]

\[ \text{if } F \in K_n, \phi_s(v) \text{ is } \phi_s \text{ is strictly positive, strictly decreasing and strictly convex on } [0, M_s]. \]

- c) Two asymptotic arguments as before.

Step 8. Obtain the case (c) by using the case (a) and (b).
Upper Bound Inequality

Overall Upper Bound inequalities for unbounded support:

\[
E[W(F/G/1)] \leq E[W(F_0/G_{u^*}/1)] \text{ (Numerics and Simulation)} \\
\leq \frac{2(1 - \rho)\rho/(1 - \delta)c_d^2 + \rho^2 c_s^2}{2(1 - \rho)} \\
< \text{ Daley's Bound < Kingman's Bound.}
\]

\(\delta \in (0,1)\) and \(\delta = \exp(-(1 - \delta)/\rho)\).

Theorem

(an UB for \(E[W(F_0, G_{u^*})]\)) For the GI/GI/1 queue with parameter four-tuple \((1, c_d^2, \rho, c_s^2)\), if \(E[W(F_0, G_{u^*})]\) is UB, then

\[
E[W(F_0, G_{u^*})] \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_d^2 + \rho^2 c_s^2}{2(1 - \rho)}.
\]
Upper Bound Inequality

Overall Upper Bound inequalities for unbounded support:

\[ \mathbb{E}[W(F/G/1)] \leq \mathbb{E}[W(F_0/G_{u^*}/1)] \quad \text{(Numerics and Simulation)} \]
\[ \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2 c_s^2}{2(1 - \rho)} \]
\[ < \text{Daley’s Bound} < \text{Kingman’s Bound}. \]

\( \delta \in (0, 1) \) and \( \delta = \exp(-(1 - \delta)/\rho) \).

Theorem

(an UB for \( \mathbb{E}[W(F_0, G_{u^*})] \)) For the GI/GI/1 queue with parameter four-tuple \((1, c_a^2, \rho, c_s^2)\), if \( \mathbb{E}[W(F_0, G_{u^*})] \) is UB, then

\[ \mathbb{E}[W(F_0, G_{u^*})] \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2 c_s^2}{2(1 - \rho)}. \]
Upper Bound Inequality

Overall Upper Bound inequalities for unbounded support:

\[ \mathbb{E}[W(F/G/1)] \leq \mathbb{E}[W(F_0/G_{u^*}/1)] \quad \text{(Numerics and Simulation)} \]

\[ \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2 c_s^2}{2(1 - \rho)} \]

\[ < \text{Daley’s Bound} < \text{Kingman’s Bound}. \]

\( \delta \in (0, 1) \) and \( \delta = \exp(-(1 - \delta)/\rho) \).

Theorem

(an UB for \( E[W(F_0, G_{u^*})] \)) For the GI/GI/1 queue with parameter four-tuple \( (1, c_a^2, \rho, c_s^2) \), if \( E[W(F_0, G_{u^*})] \) is UB, then

\[ E[W(F_0, G_{u^*})] \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2 c_s^2}{2(1 - \rho)}. \]
Daley Decomposition

**Theorem**

*(Daley decomposition)* Consider the GI/GI/1 model with specified interarrival-time cdf $F \in \mathcal{P}_{a,2}(1, c_a^2)$ and unspecified service-time cdf $G \in \mathcal{P}_{s,2}(\rho, c_s^2, \mathcal{M}_s)$. As $\mathcal{M}_s \to \infty$,

$$E[W(F, G_{u*})] \equiv \lim_{\mathcal{M}_s \to \infty} E[W(F, G_u(M_s))]$$

$$= E[W(F, D_\rho)] + E[W(D_1, G_{u*})]$$

$$= E[W(F, D_\rho)] + \frac{\rho^2 c_s^2}{2(1 - \rho)}.$$
Apply Delay Decomposition:

\[
\lim_{M_s \to \infty} E[W(F, G_u)] = E[W(F, D)] + \lim_{M_s \to \infty} E[W(D, G_u)]
\]

= \( E[W(F, D)] + \frac{c_s^2}{2(1 - \rho)} \).

Find the best \( b(\rho) = 1 \):

\[
E[W(F, G)] \leq E[W(F_0, G_u)] \leq \frac{a_{LB}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{a(\rho)c_a^2 + b(\rho)c_s^2}{2(1 - \rho)}.
\]

Find the best \( a(\rho) \) by solving:

\[
a(\rho) \geq a_{LB}(\rho) = \inf_{c_s^2 > 0} \left\{ \frac{2(1 - \rho)}{c_s^2} \sup_{F \in P_{a,2}} E[W(F, D)] \right\}.
\]
Sketch of Proof

- **Apply Delay Decomposition:**
  \[
  \lim_{M_s \to \infty} E[W(F, G_u)] = E[W(F, D)] + \lim_{M_s \to \infty} E[W(D, G_u)] = E[W(F, D)] + \frac{c_s^2}{2(1 - \rho)}.
  \]

- **Find the best** \(b(\rho) = 1\):
  \[
  E[W(F, G)] \leq E[W(F_0, G_u)] \leq \frac{a_{\text{LB}}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{a(\rho)c_a^2 + b(\rho)c_s^2}{2(1 - \rho)}.
  \]

- **Find the best** \(a(\rho)\) by solving:
  \[
  a(\rho) \geq a_{\text{LB}}(\rho) = \inf_{c_s^2 > 0} \left\{ \frac{2(1 - \rho)}{c_s^2} \sup_{F \in P_{a,2}} E[W(F, D)] \right\}.
  \]
Apply Delay Decomposition:

\[
\lim_{M_s \to \infty} E[W(F, G_u)] = E[W(F, D)] + \lim_{M_s \to \infty} E[W(D, G_u)] \\
= E[W(F, D)] + \frac{c_s^2}{2(1 - \rho)}.
\]

Find the best \( b(\rho) = 1 \):

\[
E[W(F, G)] \leq E[W(F_0, G_u)] \leq \frac{a_{LB}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{a(\rho)c_a^2 + b(\rho)c_s^2}{2(1 - \rho)}.
\]

Find the best \( a(\rho) \) by solving:

\[
a(\rho) \geq a_{LB}(\rho) = \inf_{c_a^2 > 0} \left\{ \frac{2(1 - \rho)}{c_a^2} \sup_{F \in P_{a,2}} E[W(F, D)] \right\}.
\]
Sketch of Proof

- Convert $F_0/GI/1$ to $D/\tilde{G}/1$ by applying

**Lemma**

*(inter-arrival time reduction)* For the $F_0/GI/1$ model with service time $V$ having mean $\rho$ and scv $c_s^2$, the mean steady-state waiting time can be expressed as

\[
\mathbb{E}[W(F_0(p)/GI/1)] = \mathbb{E}[W(D(1/p)/RS(V,p)/1)] + \rho(1 - p)/p
\]

\[
= \mathbb{E}[W(D(1/p)/RS(V,p)/1)] + \rho(1 - p)/p
\]

\[
= \mathbb{E}[W(D(1/p)/RS(V,p)/1)] + \rho c_s^2.
\]

- Link to $D/M/1$:

\[
(1 + c_s^2)^{-1} \mathbb{E}[W(D, RS(V, p))] \leq \mathbb{E}[W(D, M)] = \delta \rho/(1 - \delta).
\]
Sketch of Proof

- Convert $F_0/GI/1$ to $D/\tilde{G}/1$ by applying

**Lemma**

*(inter-arrival time reduction)* For the $F_0/GI/1$ model with service time $V$ having mean $\rho$ and scv $c_s^2$, the mean steady-state waiting time can be expressed as

$$
\mathbb{E}[W(F_0(p)/GI/1)] = \mathbb{E}[W(D(1/p)/RS(V, p)/1)] + \rho(1 - p)/p
$$

$$
= \mathbb{E}[W(D(1/p)/RS(V, p)/1)] + \rho(1 - p)/p
$$

$$
= \mathbb{E}[W(D(1/p)/RS(V, p)/1)] + \rho c_s^2.
$$

- Link to $D/M/1$:

$$(1 + c_a^2)^{-1}E[W(D, RS(V, p))] \leq E[W(D, M)] = \delta \rho/(1 - \delta).$$
Sketch of Proof

- Combine previous results:

\[
a_{LB}(\rho) = \inf_{c_a^2 > 0} \frac{2(1 - \rho) \sup_{F \in \mathcal{P}_{a,2}} E[W(F, D)]}{c_a^2} \leq \inf_{c_a^2 > 0} \frac{2(1 - \rho) E[W(F_0, D)]}{c_a^2} \leq \inf_{c_a^2 > 0} \{2\rho(1 - \rho) + \frac{(1 + c_a^2)\delta \rho}{(1 - \delta)^2(1 - \rho)}\} \leq \frac{\rho(2 - 2\rho)}{1 - \delta} (\text{as } c_a^2 \to \infty).\]

So \( a_{LB}(\rho) \leq \rho(2 - 2\rho)/(1 - \delta) \) and

\[
E[W(F_0, G_{u^*})] \leq \frac{a_{LB}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2 c_s^2}{2(1 - \rho)}.
\]
Sketch of Proof

- Combine previous results:

\[
\begin{align*}
  a_{LB}(\rho) & = \inf_{c_a^2 > 0} \left( \frac{2(1 - \rho) \sup_{F \in \mathcal{P}_{a,2}} E[W(F, D)]}{c_a^2} \right) \\
  & = \inf_{c_a^2 > 0} \left( \frac{2(1 - \rho)}{c_a^2} E[W(F_0, D)] \right) \\
  \leq & \inf_{c_a^2 > 0} \{2\rho(1 - \rho) + \frac{(1 + c_a^2)\delta \rho}{(1 - \delta)^2(1 - \rho)} \} \\
  \rightarrow & \frac{\rho(2 - 2\rho)}{1 - \delta} \text{ (as } c_a^2 \rightarrow \infty). \\
\end{align*}
\]

So \( a_{LB}(\rho) \leq \frac{\rho(2 - 2\rho)}{1 - \delta} \) and

\[
E[W(F_0, G_{u^*})] \leq \frac{a_{LB}(\rho) c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2 c_s^2}{2(1 - \rho)}.
\]
Sketch of Proof

Combine previous results:

$$a_{LB}(\rho) = \inf_{c_a^2 > 0} \frac{2(1 - \rho) \sup_{F \in \mathcal{P}_{a,2}} E[W(F, D)]}{c_a^2}$$

$$= \inf_{c_a^2 > 0} \frac{2(1 - \rho)E[W(F_0, D)]}{c_a^2}$$

$$\leq \inf_{c_a^2 > 0} \{2\rho(1 - \rho) + \frac{(1 + c_a^2)\delta\rho}{(1 - \delta)2(1 - \rho)} \}$$

$$\rightarrow \frac{\rho(2 - 2\rho)}{1 - \delta} \text{(as } c_a^2 \rightarrow \infty) .$$

So $$a_{LB}(\rho) \leq \frac{\rho(2 - 2\rho)}{1 - \delta}$$ and

$$E[W(F_0, G_{u^*})] \leq \frac{a_{LB}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2c_s^2}{2(1 - \rho)}.$$
Sketch of Proof

Combine previous results:

\[
a_{LB}(\rho) = \inf_{c_a^2 > 0} \frac{2(1 - \rho) \sup_{F \in \mathcal{P}_{a,2}} E[W(F, D)]}{c_a^2}
\]

\[
= \inf_{c_a^2 > 0} \frac{2(1 - \rho)E[W(F_0, D)]}{c_a^2}
\]

\[
\leq \inf_{c_a^2 > 0} \left\{ 2\rho(1 - \rho) + \frac{(1 + c_a^2)\delta \rho}{(1 - \delta)2(1 - \rho)} \right\}
\]

\[
\rightarrow \frac{\rho(2 - 2\rho)}{1 - \delta} \text{ (as } c_a^2 \rightarrow \infty \text{).}
\]

So \( a_{LB}(\rho) \leq \rho(2 - 2\rho)/(1 - \delta) \) and

\[
E[W(F_0, G_{u^*})] \leq \frac{a_{LB}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2c_s^2}{2(1 - \rho)}.
\]
Combine previous results:

\[ a_{LB}(\rho) = \inf_{c_a^2 > 0} \frac{2(1 - \rho) \sup_{F \in P_{a,2}} E[W(F, D)]}{c_a^2} \]

\[ = \inf_{c_a^2 > 0} \frac{2(1 - \rho) E[W(F_0, D)]}{c_a^2} \]

\[ \leq \inf_{c_a^2 > 0} \left\{ 2\rho(1 - \rho) + \frac{(1 + c_a^2)\delta\rho/(1 - \delta)2(1 - \rho)}{c_a^2} \right\} \]

\[ \rightarrow \frac{\rho(2 - 2\rho)}{1 - \delta} \text{ (as } c_a^2 \rightarrow \infty). \]

So \( a_{LB}(\rho) \leq \rho(2 - 2\rho)/(1 - \delta) \) and

\[ E[W(F_0, G_{u^*})] \leq \frac{a_{LB}(\rho)c_a^2 + c_s^2}{2(1 - \rho)} \leq \frac{2(1 - \rho)\rho/(1 - \delta)c_a^2 + \rho^2c_s^2}{2(1 - \rho)}. \]
Summary for Extremal $GI/GI/1$ Queues

Table: A comparison of the bounds and approximations for the scaled steady-state mean $(1 - \rho)E[W]/\rho^2$ in the $GI/GI/1$ model as a function of $\rho$ for the case $c_a^2 = c_s^2 = 4.0$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Tight LB</th>
<th>HTA</th>
<th>Tight UB</th>
<th>New UB</th>
<th>$\delta$</th>
<th>MRE</th>
<th>Daley</th>
<th>Kingman</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.000</td>
<td>4.000</td>
<td>37.989</td>
<td>38.002</td>
<td>0.000</td>
<td>0.0%</td>
<td>40.000</td>
<td>202.000</td>
</tr>
<tr>
<td>0.20</td>
<td>0.000</td>
<td>4.000</td>
<td>18.080</td>
<td>18.112</td>
<td>0.007</td>
<td>0.2%</td>
<td>20.000</td>
<td>52.000</td>
</tr>
<tr>
<td>0.30</td>
<td>0.000</td>
<td>4.000</td>
<td>11.661</td>
<td>11.731</td>
<td>0.041</td>
<td>0.6%</td>
<td>13.333</td>
<td>24.222</td>
</tr>
<tr>
<td>0.40</td>
<td>0.000</td>
<td>4.000</td>
<td>8.641</td>
<td>8.722</td>
<td>0.107</td>
<td>0.9%</td>
<td>10.000</td>
<td>14.500</td>
</tr>
<tr>
<td>0.50</td>
<td>0.500</td>
<td>4.000</td>
<td>6.941</td>
<td>7.020</td>
<td>0.203</td>
<td>1.1%</td>
<td>8.000</td>
<td>10.000</td>
</tr>
<tr>
<td>0.60</td>
<td>1.111</td>
<td>4.000</td>
<td>5.884</td>
<td>5.946</td>
<td>0.324</td>
<td>1.1%</td>
<td>6.667</td>
<td>7.556</td>
</tr>
<tr>
<td>0.70</td>
<td>1.480</td>
<td>4.000</td>
<td>5.168</td>
<td>5.216</td>
<td>0.467</td>
<td>0.9%</td>
<td>5.714</td>
<td>6.082</td>
</tr>
<tr>
<td>0.80</td>
<td>1.719</td>
<td>4.000</td>
<td>4.662</td>
<td>4.693</td>
<td>0.629</td>
<td>0.7%</td>
<td>5.000</td>
<td>5.125</td>
</tr>
<tr>
<td>0.90</td>
<td>1.883</td>
<td>4.000</td>
<td>4.287</td>
<td>4.302</td>
<td>0.807</td>
<td>0.4%</td>
<td>4.444</td>
<td>4.469</td>
</tr>
<tr>
<td>0.95</td>
<td>1.946</td>
<td>4.000</td>
<td>4.134</td>
<td>4.142</td>
<td>0.902</td>
<td>0.2%</td>
<td>4.211</td>
<td>4.216</td>
</tr>
<tr>
<td>0.98</td>
<td>1.979</td>
<td>4.000</td>
<td>4.052</td>
<td>4.055</td>
<td>0.960</td>
<td>0.1%</td>
<td>4.082</td>
<td>4.082</td>
</tr>
<tr>
<td>0.99</td>
<td>1.990</td>
<td>4.000</td>
<td>4.025</td>
<td>4.027</td>
<td>0.980</td>
<td>0.0%</td>
<td>4.040</td>
<td>4.041</td>
</tr>
</tbody>
</table>
Derive the Upper Bound by **numerical** and **simulation** search:

\[ E[W(F/G/1)] \leq E[W(F_0/G_{u^*}/1)]. \]

- Obtain accurate values of \( E[W(F_0/G_{u^*}/1)] \) by algorithms.
- To be continued...... on Wednesday.
New Objective

- Derive the Upper Bound by **numerical** and **simulation** search:

\[ \mathbb{E}[W(F/G/1)] \leq \mathbb{E}[W(F_0/G_{u*}/1)]. \]

- Obtain **accurate values** of \( \mathbb{E}[W(F_0/G_{u*}/1)] \) by algorithms.

- To be continued...... on Wednesday.
New Objective

- Derive the Upper Bound by numerical and simulation search:
  \[ \mathbb{E}[W(F/G/1)] \leq \mathbb{E}[W(F_0/G_{u^*}/1)]. \]

- Obtain accurate values of \( \mathbb{E}[W(F_0/G_{u^*}/1)] \) by algorithms.

- To be continued...... on Wednesday.