New problems on minimizing movements††

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In this paper I intend to deepen the idea of minimizing movement which has been presented in the conference [10]. Such idea seems to be suitable to unify many problems in calculus of variations, differential equations, geometric measure theory: among others, steepest descent methods, heat equation, mean curvature flow, monotone operators, various evolution problems, etc. This paper is completely self-contained and may be read independently of the papers quoted in the bibliography; nevertheless, we remark that the main definitions of this paper may be considered as slight generalizations of the definitions given in [10] and that the paper [10] has been inspired mainly by the paper [1].

Minimizing movements are tied in various ways to penalization methods, Γ-convergence, singular perturbation, geometric measure theory, etc., hence the bibliographic indications will be unavoidably partial and far from being complete. In many cases the reader can surely find many other interesting references, as well as many interesting examples, problems, conjectures suggested by his own experience, which could be more interesting and expressive than those presented in this paper. One could think of finding general hypotheses on $F$ and $S$ such that the set of minimizing movements $MM(F,S)$ or the set of generalized minimizing movements $GMM(F,S)$ are nonvoid, or finite or such that their elements can be characterized by some differential equation, and/or some other meaningful condition.

I believe that the idea of minimizing movement is the natural meeting point of many problems of analysis, geometry, mathematical physics and numerical analysis, and its development will require the contribution of many researchers with different backgrounds.

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Notation. In this paper we shall indicate by $\mathbb{Z}$ the set of signed integers, by $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ the extended real line, by $[a,b]$ (for $-\infty < a < b < +\infty$) the open interval $\{x \in \mathbb{R}: a < x < b\}$ and by $[x] = \max\{z \in \mathbb{Z}: z < x\}$ the integral part of $x$. Furthermore, $S$ will denote a topological space and, for any pair of metric spaces $M$ and $M'$, $Lip(M, M')$ (shortened to $Lip(M)$ if $M' = \mathbb{R}$) denotes the set of $M'$-valued Lipschitz continuous functions on $M$. If $u \in Lip(M, M')$ we denote by $lip(u, M, M')$ the Lipschitz constant of $u$. If $M' = \mathbb{R}$, then we


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shorten $\text{lip}(u, M)$ for $\text{lip}(u, M, R)$; when $M$ is unambiguously determined by the context, we also write $\text{lip}(u)$ for $\text{lip}(u, M)$.

1. Minimizing movements definitions and examples in $R^n$

Let us define the minimizing movements $MM(F, S)$.

**Definition 1.1** – Let $F : ]1, +\infty[ \times \mathbb{Z} \times S \times S \rightarrow \bar{R}$ and $u : \mathbb{R} \rightarrow S$; we say that $u$ is a minimizing movement associated to $F$, $S$, and we write $u \in MM(F, S)$, if there exists $w : ]1, +\infty[ \rightarrow \mathbb{Z}$ such that for any $t \in \mathbb{R}$

$$\lim_{\lambda \to +\infty} w(\lambda, [\lambda t]) = u(t)$$

and for any $\lambda \in ]1, +\infty[$, $k \in \mathbb{Z}$

$$F(\lambda, k, w(\lambda, k + 1), w(\lambda, k)) = \min_{s \in S} F(\lambda, k, s, w(\lambda, k)).$$

One can consider the following examples.

**Example 1.1** – Let be $S = \mathbb{R}^n$, $f \in \text{Lip}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and let $\xi \in \mathbb{R}^n$ be given. Set

$$F(\lambda, k, x, y) = \begin{cases} |x - \xi|^2 & \text{if } k \leq 0, \\ f(x) + \lambda|x - y|^2 & \text{if } k > 0; \end{cases}$$

then, $u \in MM(F, S)$ if and only if $u \in \text{Lip}(\mathbb{R}, \mathbb{R}^n)$ and $u$ solves

$$\begin{cases} u(t) = \xi & \forall t \leq 0 \\ \frac{2}{dt} = -\nabla f(u) & \text{in } [0, +\infty[. \end{cases}$$

**Example 1.2** – Let be $S = \mathbb{R}^n$, $f \in \text{Lip}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and let $\xi \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, $1 < \beta \leq 2$, be given. Set

$$F(\lambda, k, x, y) = \begin{cases} |x - \xi|^{\beta} & \text{if } k \leq 0, \\ \lambda f(x) + (\lambda|x - y|)^{\beta} & \text{if } k > 0; \end{cases}$$

then, $u \in MM(F, S)$ if and only if $u \in \text{Lip}(\mathbb{R}, \mathbb{R}^n)$ and $u$ solves

$$\begin{cases} u(t) = \xi & \forall t \leq 0 \\ \beta \frac{du}{dt} = -\left|\frac{du}{dt}\right|^{2-\beta} \nabla f(u) & \text{in } [0, +\infty[. \end{cases}$$

**Example 1.3** – Let be $S = \mathbb{R}^n$, $f \in \text{Lip}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ with $|\nabla f(x)| \neq 0$ for any $x$, and let $\xi \in \mathbb{R}^n$ be given. Set

$$F(\lambda, k, x, y) = |x - \xi|^2$$

if $k \leq 0,$
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\[ F(\lambda, k, x, y) = \begin{cases} f(x) & \text{if } k > 0, \; \lambda|x - y| \leq 1, \\ \infty & \text{if } k > 0, \; \lambda|x - y| > 1; \end{cases} \]

then, \( u \in MM(F, S) \) if and only if \( u \in Lip(\mathbb{R}, \mathbb{R}^n) \) and \( u \) solves

\[
\begin{align*}
\frac{du}{dt} &= -\frac{\nabla f(u)}{||\nabla f(u)||} & \forall t \leq 0 \\
\left\{ \begin{array}{l}
u(t) = \xi \\
u(0) = 0
\end{array} \right. & \text{in } ]0, +\infty[.
\end{align*}
\]

**Remark 1.1** - If we replace in example 1.1 the hypothesis \( f \in Lip(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) \) by the weaker condition \( f \in Lip(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) \), then neither the existence nor the uniqueness are guaranteed in \( MM(F, \mathbb{R}^n) \). We can only say that any \( u \in MM(F, \mathbb{R}^n) \) is a Lipschitz function satisfying the Cauchy problem (1.1), but the opposite implication does not hold. Indeed, if \( n = 1, \xi = 0 \) and \( f(x) = -|\sin x|^{3/2} \), then \( u \equiv 0 \) is a solution of (1.1) which does not belong to \( MM(F, \mathbb{R}) \).

**Remark 1.2** - An example showing that \( MM(F, \mathbb{R}^n) \) in example 1.1 may be empty if we only assume that \( f \in Lip(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) \) is the following: let \( n = 1, \xi = 0 \) and \( F \) as in example 1.1, with

\[
f(x) = \begin{cases} -|\sin x|^{3/2} + \sin^4 x \sin \frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases}
\]

Then, \( MM(F, S) = \emptyset \).

**Remark 1.3** - Finally, if we only assume that \( f \) is a Lipschitz continuous function everywhere differentiable in \( \mathbb{R}^n \) whose derivatives are discontinuous in some point and if we define \( F \) as in example 1.1, then it can happen that functions \( u \in MM(F, \mathbb{R}^n) \) do not solve the Cauchy problem (1.1). Indeed, let \( n = 1, \xi = 0 \) and let \( F \) be as in example 1.1, with

\[
f(x) = \begin{cases} x + \sin^2 x \sin \frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases}
\]

Then \( u = 0 \) belongs to \( MM(F, \mathbb{R}) \) and does not satisfy (1.1).

Remark 1.2 and many other cases where \( MM(F, S) = \emptyset \) motivate the definition of generalized minimizing movements \( GMM(F, S) \).

**Definition 1.2** - Let be \( F :]1, +\infty[ \times \mathbb{Z} \times S^2 \to \overline{\mathbb{R}} \) and \( u : \mathbb{R} \to S \); we say that \( u \) is a generalized minimizing movement associated to \( F, S \), and we write
$u \in GMM(F,S)$, if there exist a sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ such that $\lim \lambda_i = +\infty$ and a sequence $\{w_i\}_{i \in \mathbb{N}}$ of functions $w_i : \mathbb{Z} \to S$ such that for any $t \in \mathbb{R}$

$$\lim_{i \to +\infty} w_i([\lambda_i t]) = u(t)$$

and for any $i \in \mathbb{N}$, $k \in \mathbb{Z}$

$$F(\lambda_i, k, w_i(k+1), w_i(k)) = \min_{s \in S} F(\lambda_i, k, s, w_i(k)).$$

**Remark 1.4** - It seems to me that generalized minimizing movements $GMM$ could give some good formalization of the heuristic idea of curve of maximal slope, and that it would be interesting to compare it with other definitions already proposed (see e.g. [3], [12], [13]). All these definitions agree in the case $S = \mathbb{R}^n$ and $f \in Lip(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$.

**Remark 1.5** - Notice that in the case considered in remark 1.2 there are two elements in $GMM(F,S)$, namely the two solutions of the Cauchy problem $u' = -\nabla f(u)$, $u(0) = 0$ that are different from 0 for any $t > 0$.

**Remark 1.6** - It might seem restrictive to consider only functions $u \in GMM(F,S)$ defined on the whole real line. However it is easy include in our definitions the case of functions $u : [a, b] \to S$, where $a, b \in \mathbb{R}$, with $a < b$. For instance, if $S = \mathbb{R}^n$ and $f \in Lip(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then all the solutions defined in $[a, b]$ of the equation

$$2\frac{d u}{d t} = -\nabla f(u)$$

are restrictions to $[a, b]$ of functions $u \in MM(F, \mathbb{R}^n)$, where

$$F(\lambda, k, x, y) = \begin{cases} 0 & \text{if } k < [\lambda a]; \\ f(x) + \lambda |x - y|^2 & \text{if } [\lambda a] \leq k \leq [\lambda b]; \\ 0 & \text{if } k > [\lambda b]. \end{cases}$$

It is also interesting to consider the following

**Example 1.4** - Let be $S = \mathbb{R}^n$, $f \in Lip(\mathbb{R}^n)$ and let $\xi \in \mathbb{R}^n$, $\beta \in ]1, +\infty[$ be given. Set

$$F(\lambda, k, x, y) = \begin{cases} |x - \xi|^{\beta} & \text{if } k \leq 0, \\ f(x) + \lambda^{\beta-1}|x - y|^{\beta} & \text{if } k > 0, \end{cases}$$

then $GMM(F,S) \neq \emptyset$ and if $u \in GMM(F,S)$ then $u \in Lip(\mathbb{R}, \mathbb{R}^n)$. If moreover $f \in Lip(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ then each $u \in GMM(F,S)$ solves

$$\begin{cases} u(t) = \xi & \forall t \leq 0, \\ \beta \frac{d u}{d t} = -\left|\frac{d u}{d t}\right|^{2-\beta} \nabla f(u) & \text{in } [0, +\infty[. \end{cases}$$

(1.4)
for \( \beta \leq 2 \), and

\[
\begin{cases}
  u(t) = \xi & \forall t \leq 0 \\
  \beta \left| \frac{du}{dt} \right|^{\beta-2} \frac{du}{dt} = -\nabla f(u) & \text{in } [0, +\infty[ \\
\end{cases}
\]

for \( \beta \geq 2 \).

**Remark 1.7** - In example 1.4 we remarked that if \( f \in Lip(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) \), then each \( u \in GMM(F, S) \) solves equation (1.4)-(1.5). On the other hand, by remark 1.1, there could exist solutions of (1.4)-(1.5) which don’t belong neither to \( MM(F, S) \) nor to \( GMM(F, S) \). If we want to gather all the solutions of (1.4), (1.5) as minimizing movements, we may consider conjectures of the following type.

**Conjecture 1.1** - Let \( f \in Lip(\mathbb{R}^n) \cap C^1(\mathbb{R}^n), \xi \in \mathbb{R}^n \). Then, \( u \) is a solution of (1.1) if and only if there exists \( \rho : \mathbb{R}^{n+1} \to \mathbb{R} \) such that

(a) \( \rho(\cdot, \lambda) \in Lip(\mathbb{R}^n) \) for any \( \lambda > 0 \) and

\[
\lim_{\lambda \to +\infty} Lip(f - \rho(\cdot, \lambda)) = 0;
\]

(b) \( u \in GMM(\Phi, \mathbb{R}^n) \), where \( \Phi(\lambda, k, x, y) \) is defined by

\[
\Phi(\lambda, k, x, y) = \begin{cases}
  |x - \xi|^2 & \text{if } k \leq 0, \\
  \varphi(x, \lambda) + \lambda|x - y|^2 & \text{if } k > 0.
\end{cases}
\]

**Remark 1.8** - It would also be interesting to consider non-autonomous equations, taking a function \( f \) depending on \( n + 1 \) variables, and the functionals

\[
F(\lambda, k, x, y) = \begin{cases}
  |x - \xi|^2 & \text{if } k \leq 0, \\
  \lambda \int_0^{k+1} f\left( x, \frac{\tau}{\lambda} \right) d\tau + (\lambda|x - y|)\beta & \text{if } k > 0.
\end{cases}
\]

**Remark 1.9** - Instead of regularity conditions such as Lipschitz continuity or differentiability, in the previous examples lower semicontinuity conditions and convexity or quasiconvexity hypotheses could be introduced, or bounds of the following type:

\[
\inf_{x \in \mathbb{R}^n} \frac{f(x)}{1 + |x|^\beta} > -\infty.
\]

The same type of condition will likely play an important rôle in passing from \( \mathbb{R}^n \) to infinite dimensional spaces. In this paper we shall not explore the general, abstract problems which arise in such spaces, limiting ourselves to point out in
the next chapters some meaningful example of minimizing movement arising in
the theory of partial differential equations and in geometric measure theory.

2. Minimizing movements and PDE

Examples 1.1, 1.2, 1.3, 1.4 show that the definitions of $MM(F,S)$ and
$GMM(F,S)$ cover in the case $S = \mathbb{R}^n$ some problems of steepest descent. Many
interesting and difficult problems can be formulated if $S$ is a space of functions
and $F$ is an integral functional, giving problems of gradient flow type. For the
sake of simplicity we consider in this chapter only spaces of functions defined on
$\mathbb{R}^n$ and unless otherwise stated the integrals are intended on the whole of $\mathbb{R}^n$.

We begin with an example related to the heat equation.

**Example 2.1** - Let be $S = H^{1,2}(\mathbb{R}^n)$ and $\varphi \in H^{1,2}(\mathbb{R}^n)$ be given; set

$$
F(\lambda, k, f, g) = \begin{cases}
|f - \varphi|^2 dx & \text{if } k \leq 0, \\
|\nabla f|^2 + \lambda |f - g|^2 dx & \text{if } k > 0;
\end{cases}
$$

then, $u \in MM(F,S)$ if and only if $u : \mathbb{R} \rightarrow S$ is continuous, $u(t) = \varphi$ for any $t \leq 0$, and, setting $v(x,t) = u(t)(x)$, $v$ solves

$$
\frac{\partial v}{\partial t} = \Delta_x v \quad \text{in } \mathbb{R}^n \times ]0, +\infty[.
$$

Besides the heat equation, other evolution equations might be considered (see e.g. [12], [22]), and it is likely true that in many cases the existence of functions $u$ in $MM(F,S)$ or in $GMM(F,S)$ can be obtained in more general hypotheses than those already considered in the literature.

**Conjecture 2.1** - Let $S = H^{1,2}(\mathbb{R}^n)$ and $\varphi \in H^{1,2}(\mathbb{R}^n)$, $1 < \beta \leq 2$ be given; set

$$
F(\lambda, k, f, g) = \begin{cases}
|f - \varphi|^2 dx & \text{if } k \leq 0, \\
\lambda \int |\nabla f|^2 + (\lambda |f - g|)^\beta dx & \text{if } k > 0;
\end{cases}
$$

then, $u \in MM(F,S)$ if and only if $u : \mathbb{R} \rightarrow S$ is continuous, $u(t) = \varphi$ for any $t \leq 0$ and, setting $v(x,t) = u(t)(x)$, $v$ solves

$$
\frac{\beta}{2} \frac{\partial v}{\partial t} = \left|\frac{\partial v}{\partial t}\right|^{2-\beta} \Delta_x v \quad \text{in } \mathbb{R}^n \times ]0, +\infty[.
$$

In a different direction, example 2.1 may be generalized as follows.
CONJECTURE 2.2 - Let $S = H^{1,2}(\mathbb{R}^n)$, $\varphi \in H^{1,2}(\mathbb{R}^n)$ and $a_1, \ldots, a_n \in L^\infty(\mathbb{R}^n)$ be given; set

$$F(\lambda, k, f, g) = \begin{cases} 
\int |f - \varphi|^2 dx & \text{if } k \leq 0, \\
\int |\nabla f|^2 - 2 \sum_{i=1}^n a_i \frac{\partial g}{\partial x_i} f + \lambda |f - g|^2 dx & \text{if } k > 0;
\end{cases}$$

then, $MM(F, S) \neq \emptyset$, and $u \in MM(F, S)$ if and only if $u : \mathbb{R} \to S$ is continuous, $u(t) = \varphi$ for any $t \leq 0$ and, setting $v(x, t) = u(t)(x)$, $v$ solves

$$\frac{\partial v}{\partial t} = \Delta_x v + \sum_{i=1}^n a_i \frac{\partial v}{\partial x_i} \quad \text{in } \mathbb{R}^n \times [0, +\infty[.$$

The following conjecture concerns a "moving obstacle" problem.

CONJECTURE 2.3 - Let $S = H^{1,2}(\mathbb{R}^n)$ and $\varphi \in H^{1,2}(\mathbb{R}^n)$ be given; set

$$F(\lambda, k, f, g) = \begin{cases} 
\int |f - \varphi|^2 dx & \text{if } k \leq 0, \\
\int |\nabla f|^2 + \lambda |f - g|^2 dx & \text{if } f \geq g, \\
+\infty & \text{otherwise};
\end{cases}$$

then, $u \in MM(F, S)$ if and only if $u : \mathbb{R} \to S$ is continuous, $u(t) = \varphi$ for any $t \leq 0$, and, setting $v(x, t) = u(t)(x)$, $v$ solves

$$2 \frac{\partial v}{\partial t} = \Delta_x v + |\Delta_x v| \quad \text{in } \mathbb{R}^n \times [0, +\infty[.$$

REMARK 2.1 - In discussing the previous conjectures it will likely be convenient to study first the existence and the uniqueness of an element $u \in MM(F, S)$ and in a second time its regularity, in order to clarify the sense which can be given to the differential equations stated therein. The following example with a "fixed obstacle" seems to be closer to known techniques (see e.g. [3], [12]).

EXAMPLE 2.2 - Let be $S = H^{1,2}(\mathbb{R}^n)$, $\psi \in C^\infty_0(\mathbb{R}^n)$ and $\varphi \in H^{1,2}(\mathbb{R}^n)$ be given, with $\varphi \geq \psi$; set

$$F(\lambda, k, f, g) = \int |f - \varphi|^2 dx \quad \text{if } k \leq 0,$$
and, for $k > 0$, 

$$F(\lambda, k, f, g) = \begin{cases} 
\int |\nabla f|^2 + \lambda|f - g|^2 \, dx & \text{if } f \geq \psi, \\
+\infty & \text{otherwise}; 
\end{cases}$$

then $MM(F, S) \neq \emptyset$.

The introduction of $MM(F, S)$ and $GM(F, S)$ leads also to interesting problems related to hyperbolic equations, as in the following

**Problem 2.1**. For $S = H^{1,2}(\mathbb{R}^n)$, and 

$$F(\lambda, k, f, g) = \begin{cases} 
\int |f - \varphi|^2 \, dx & \text{if } k \leq 0, \\
\int \frac{1}{\log \lambda} |\nabla f|^2 - 2 \sum_{i=1}^{n} a_i \frac{\partial g}{\partial x_i} f + \lambda|f - g|^2 \, dx & \text{if } k > 0 
\end{cases}$$

find conditions on $a_i$ and $\varphi$ in order to obtain $MM(F, S) \neq \emptyset$ (or at least $GM(F, S) \neq \emptyset$) and $u \in MM(F, S)$ (or $u \in GM(F, S)$) if and only if $u(t) = \varphi$ for any $t \leq 0$ and, setting $v(x, t) = u(t)(x)$,

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{n} a_i(x) \frac{\partial v}{\partial x_i}.$$

**Remark 2.2**. Problem 2.1 is inspired to the idea that by a suitable choice of $F$ it is possible to combine time discretization with the vanishing viscosity method (see e.g. [23]). The possibility of such combinations is one of the best features of the minimizing movements. A similar approximation idea will be considered in conjecture 2.4.

**Remark 2.3** - In the previous problem, as well as in conjecture 2.2, it would also be interesting to consider time-dependent coefficients $a_i \in L^{\infty}(\mathbb{R}^{n+1})$ and the functionals

$$F(\lambda, k, f, g) = \begin{cases} 
\int |f - \varphi|^2 \, dx & \text{if } k \leq 0, \\
\int_{k}^{k+1} d\tau \int |\nabla f|^2 - 2 \sum_{i=1}^{n} a_i \left( x, \frac{\tau}{\lambda} \right) \frac{\partial g}{\partial x_i} f + \lambda|f - g|^2 \, dx & \text{if } k > 0, 
\end{cases}$$

$$F(\lambda, k, f, g) = \begin{cases} 
\int |f - \varphi|^2 \, dx & \text{if } k \leq 0, \\
\int_{k}^{k+1} d\tau \int \frac{1}{\log \lambda} |\nabla f|^2 - 2 \sum_{i=1}^{n} a_i \left( x, \frac{\tau}{\lambda} \right) \frac{\partial g}{\partial x_i} f + \\
+\lambda|f - g|^2 \, dx & \text{if } k > 0.
\end{cases}$$
Remark 2.4 – After studying the equation
\[
\frac{\partial v}{\partial t} = \sum_{i=1}^{n} a_i(x) \frac{\partial v}{\partial x_i},
\]
systems of first order equations could be studied as well, even for time-dependent coefficients.

In the context of minimizing movements it is possible to consider various examples of boundary value problems of very general type, so that, beside example 2.1 concerning the heat equation, we may consider the following problem.

Problem 2.2 – For \( S = H^{1.2}(\mathbb{R}^n) \), \( E \subset \mathbb{R}^{n+1} \), setting \( E_t = \{ x \in \mathbb{R}^n : (x,t) \in E \} \) and

\[
F(\lambda, k, f, g) = \int_k^{k+1} d\tau \left( \int_{\mathbb{R}^n} \lambda |f - \varphi|^2 + |\nabla f|^2 - 2f \varphi(x, \tau) dx + (\log \lambda) \int_{\mathbb{R}^n \setminus E_{\tau, \lambda}} |f|^2 dx \right)
\]
find conditions on \( \varphi \) and \( E \) in order to obtain \( \text{MM}(F, S) \neq \emptyset \), or at least \( G\text{MM}(F, S) \neq \emptyset \).

Remark 2.5 – Setting \( v(x, t) = u(t)(x) \) for \( u \in \text{MM}(F, S) \), the term \( \log \lambda \int_{\mathbb{R}^n \setminus E_{\tau, \lambda}} |f|^2 dx \) in the previous problem leads, in the regular cases (for instance \( \varphi \in C_0^\infty(\mathbb{R}^{n+1}) \) and \( E = \{(x, t) : t > |x|^2 \} \)), to the equation \( \partial v/\partial t = \Delta v + \varphi \) in the open set \( E \) with the boundary condition \( v(x, t) = 0 \) for any \((x, t) \in \partial E\).

Remark 2.6 – An interesting connection between \( \Gamma \)-convergence, time discretization, limits of parabolic equations and mean curvature flow is the following conjecture, related to the \( \Gamma \)-convergence result in [24], and the problems considered in the papers [2], [4], [5], [8], [9], [14], [15], [16], [21]. The function \( F \) of conjecture 2.4 arises from a time discretization of the equation
\[
\frac{\partial u}{\partial \tau} = \Delta u + \mu(u - u^3)
\]
and a limit as \( \mu \to +\infty \). The utility of such a connection has been suggested in [17].

Conjecture 2.4 – Let \( S = L^2(\mathbb{R}^{n+1}) \) and \( \varphi \in \text{Lip}(\mathbb{R}^n) \) be given; set
\[
F(\lambda, k, f, g) = \int_\mathbb{R} dy \int_{\mathbb{R}^n} |f(x, y) - \varphi(x, y) \exp\{-|x|^2 - y^2\}|^2 dx \quad \text{if } k \leq 0,
\]
and, for \( k > 0 \),
\[
F(\lambda, k, f, g) = \int_\mathbb{R} dy \int_{\mathbb{R}^n} |\nabla_x f|^2 + \log \lambda(|f|^2 - 1)^2 + \lambda |f - g|^2 dx
\]
if \( f(y) \in H^{1,2}(\mathbb{R}^n) \) for any \( y \in \mathbb{R} \), \( F(\lambda, k, f, g) = +\infty \) otherwise.

Then, \( MM(F, S) \neq \emptyset \).

**Remark 2.7** - In the previous conjecture, it would also be interesting to replace \( \log \lambda \) by \( \lambda^\beta \), for some positive \( \beta \) and \( (1 - |f|^2)^2 \) by \( (1 - \cos f) \) or by any other function satisfying the hypotheses of the Modica-Mortola theorem (see [24]). In the study of flow of periodic surfaces by mean curvature it would also be interesting the replacement of \( L^2(\mathbb{R}^{n+1}) \) by a space of periodic functions (see [9]).

### 3. Minimizing movements of surfaces and of rectifiable currents

Since we are going to consider possible applications of minimizing movements in geometric measure theory, where the notion of Hausdorff measure play a central rôle, we recall its definition, for the reader's convenience.

**Definition 3.1** - Let \( E \subset \mathbb{R}^n \). For any \( \alpha > 0 \) we define

\[
\mathcal{H}^\alpha(E) = \frac{[\Gamma(1/2)]^\alpha}{2^\alpha \Gamma(1 + \alpha/2)} \sup_{\delta > 0} \left\{ \sum_{i=1}^{\infty} \text{diam}^\alpha(E_i) : E \subset \bigcup_{i \in I} E_i, \text{diam}(E_i) < \delta \right\},
\]

where \( \Gamma(r) = \int_0^\infty e^{-rt} \frac{1}{r} \, dr \) is the Euler function. We also define \( \mathcal{H}^0(E) \) to be the cardinality of \( E \) if \( E \) is a finite set, and \( +\infty \) otherwise. The first problem related to flow by mean curvature is the following.

**Conjecture 3.1** - Let \( S \) be the space of all open convex bounded subsets of \( \mathbb{R}^n \), endowed with the metric \( d(C_1, C_2) = \mathcal{H}^n(C_1 \triangle C_2) \); given \( \alpha \in [0, +\infty[ \) and \( C \in S \), set

\[
F(\lambda, k, C_1, C) = \begin{cases} 
\mathcal{H}^n(C_1 \triangle C) & \text{if } k \leq 0, \\
\mathcal{H}^{n-1}(\partial C_1) + \lambda^\alpha \int_{C_1 \triangle C} \left[ \text{dist}(x, \partial C) \right]^\alpha d\mathcal{H}^n(x) & \text{if } k > 0;
\end{cases}
\]

then, there exists a unique \( u \in MM(F, S) \) and, in the case \( \alpha = 1 \), \( \partial u(t) \) moves along its mean curvature.

The preceding conjecture is connected with some results of [1], [18] and [19]. The following conjecture deals with mean curvature flow as well.

**Conjecture 3.2** - Let be

\[
S = \{ E \subset \mathbb{R}^n : \mathcal{H}^n(E) < +\infty, \ x \in E \Leftrightarrow \lim_{\rho \to 0^+} \rho^{-n} \mathcal{H}^n(B_\rho(x) \setminus E) = 0 \},
\]

where \( B_\rho(x) \) denotes the open ball centered at \( x \) with radius \( \rho \). We endow \( S \) with the distance \( d(E_1, E_2) = \mathcal{H}^n(E_1 \triangle E_2) \); given \( \alpha \in ]0, +\infty[, L \in S \) such that
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\[ \mathcal{H}^{n-1}(\partial L) < +\infty \quad \text{and} \quad g \in L^1(\mathbb{R}^n) \cap L^n(\mathbb{R}^n), \]

set

\[
F(\lambda, k, E_1, E_2) = \begin{cases} 
\mathcal{H}^n(E_1 \triangle L) & \text{if } k \leq 0, \\
\mathcal{H}^{n-1}(\partial E_1) + \int_{E_1} g(x) \, dx + \lambda \int_{E_1 \triangle E_2} [\text{dist}(x, \partial E_2)]^\alpha \, dx & \text{if } k > 0,
\end{cases}
\]

then, \( GMM(F, S) \neq \emptyset. \)

REMARK 3.1 - Conjecture 3.2 is interesting even under stronger regularity hypotheses on \( L \) and \( g \); remark also that the presence of \( g \) makes nontrivial the case \( n = 1 \) as well. If \( g \equiv 0, \alpha = 1 \) and \( P(L) < +\infty \) (where \( P(L) \) denotes the perimeter of \( L \), see [6], [7], [11]), conjecture 3.2 is tied to some results of [1]. Even for \( n = 1 \), it is not clear whether conjecture 3.2 holds without the assumption \( P(L) < +\infty \). If one does not want to use explicitly the notion of perimeter, one can consider the hypothesis (equivalent if \( n = 1 \), slightly more restrictive if \( n > 1 \), \( \mathcal{H}^{n-1}(\partial L) < +\infty \). Very close to the one dimensional case is the case where \( g \) and \( L \) present spherical symmetry. The case of spherical symmetry is also interesting in connection with conjecture 2.4 (see [4], [19], [25]).

Problems relative to mean curvature flow on subsets \( K \subset \mathbb{R}^n \) are taken into account in the following problem (see also [2], [20], [21]).

PROBLEM 3.1 - Given \( s \geq 1 \) and a compact set \( K \subset \mathbb{R}^n \) such that \( \mathcal{H}^s(K) < +\infty \), let

\[
S = \left\{ E \subset K : x \in E \iff \lim_{\rho \to 0^+} \rho^{-s} \mathcal{H}^s(K \cap B_\rho(x) \setminus E) = 0 \right\},
\]

endowed with the distance \( d(E_1, E_2) = \mathcal{H}^s(E_1 \triangle E_2) \); given a set \( L \in S \) we define

\[
F(\lambda, k, E_1, E_2) = \mathcal{H}^s(E_1 \triangle L)
\]

if \( k \leq 0 \) and

\[
F(\lambda, k, E_1, E_2) = \mathcal{H}^{s-1}(\partial E_1 \cap \partial (K \setminus E_1)) + \int_{E_1} g \, d\mathcal{H}^s + \lambda \int_{E_1 \triangle E_2} \text{dist}(x, \partial E_2) \, d\mathcal{H}^s
\]

if \( k > 0 \). Our problem is the following: find conditions on \( K, L, g \) such that \( GMM(F, S) \neq \emptyset. \)

In order to pass from the study of subsets of \( \mathbb{R}^n \) to very general problems concerning currents in a metric space, we shall introduce some concepts which may be considered as a wide generalization of some definitions in [16], and are a development of some ideas in [26].

DEFINITION 3.2 - Let \( M \) be a metric space, let \( k \geq 1 \) be an integer and let \( \mathcal{B}(M), \mathcal{B}^{\infty}(M) \) be the class of Borel subsets of \( M \) and the space of bounded real valued functions \( f : M \to \mathbb{R} \) respectively. We say that \( G : \mathcal{B}^{\infty}(M) \times (\text{Lip}(M))^k \to \mathbb{R} \) belongs to \( GMT_k(M) \) (the set of geometric measure theory
functionals of dimension $k$ relative to the space $M$) if the following conditions are satisfied:

(a) there exist a finite Borel measure $\mu$ in $M$ and $\alpha : \{\text{Lip}(M)\}^k \to \mathcal{B}^\infty(M)$ such that

$$G(f_0; f_1, \ldots, f_k) = \int_M f_0 \alpha(f_1, \ldots, f_k) \, d\mu \quad \forall f_0 \in \mathcal{B}^\infty(M),$$

$$\forall f_1, \ldots, f_k \in \text{Lip}(M);$$

(b) for any choice of $f_1, \ldots, f_k \in \text{Lip}(M)$ the following inequality holds

$$|\alpha(f_1, \ldots, f_k)(x)| \leq \prod_{i=1}^k \text{lip}(f_i), \quad \forall x \in M.$$

It is easy to introduce the concept of boundary of $H \in \text{GMT}_{k+1}(M)$ as follows.

**Definition 3.3** - Let $H \in \text{GMT}_{k+1}(M)$; we say that $G$ is the boundary of $H$, and we write $G = \partial H$, if $G \in \text{GMT}_k(M)$ and

$$H(1; f_0, \ldots, f_k) = G(f_0; f_1, \ldots, f_k) \quad \forall f_0 \in \text{Lip}(M) \cap \mathcal{B}^\infty(M),$$

$$\forall f_1, \ldots, f_k \in \text{Lip}(M).$$

**Remark 3.2** - It is also possible to give a notion of mass of a functional $G \in \text{GMT}_k(M)$, taking into account that from definition 3.2 and Radon-Nikodym theorem it easily follows that for any $G \in \text{GMT}_k(M)$ there exists a pair $(\mu, \alpha)$ verifying (a), (b) which minimizes $\mu(M)$. Denoting by $(\bar{\mu}, \bar{\alpha})$ such a minimizing pair, for any $B \in \mathcal{B}(M)$ and for any other pair $(\mu, \alpha)$ verifying (a), (b) in definition 3.2, $\mu(B) \geq \bar{\mu}(B)$ holds, hence $\bar{\mu}(B)$ is determined by $M$ and $G$ and is said mass of $G$ and denoted $\|G\|_M$, or briefly $\|G\|$ if there is no ambiguity.

**Remark 3.3** - Notice that, given two metric spaces $M, M'$ having the same elements and equivalent but different distance functions, the sets $\text{GMT}_k(M)$ and $\text{GMT}_k(M')$ coincide, but the masses $\|G\|_M$ and $\|G\|_{M'}$ may be different.

We now give the definition of push forward of a functional $G \in \text{GMT}_k(M)$ (cf. [16, 4.1.7, 4.1.14]).

**Definition 3.4** - Let $M, M'$ be metric spaces, $\varphi : M \to M'$ be a Lipschitz continuous function, $G \in \text{GMT}_k(M)$, $H \in \text{GMT}_k(M')$. We write $H = \varphi_#(G)$ if

$$H(f_0; f_1, \ldots, f_k) = G(f_0 \circ \varphi; f_1 \circ \varphi, \ldots, f_k \circ \varphi) \quad \forall f_0 \in \mathcal{B}^\infty(M),$$

$$\forall f_1, \ldots, f_k \in \text{Lip}(M').$$

Very interesting particular cases of $G \in \text{GMT}_k(M)$ are the integral rectifiable currents (see [16, 4.1.24, for the case $M = \mathbb{R}^n$], defined as follows.

**Definition 3.5** - We say that $G \in \text{GMT}_k(M)$ belongs to $\text{IRC}_k(M)$, the set of integral rectifiable $k$-currents in $M$, if there exist a Borel set $B \subset \mathbb{R}^k$ and
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a Lipschitz function \( \varphi : B \to M \) such that \( \mathcal{H}^k(B) < +\infty \) and \( G = \varphi_#([|B|]) \),

where \([|B|] \in G M T_k(\mathbb{R}^k)\) is defined by

\[
[|B|](f_0; f_1, \ldots, f_k) = \int_B f_0(x) J(f)(x) \, dx \quad \forall f_0 \in B^\infty(\mathbb{R}^k),
\]

\( f_1, \ldots, f_k \in \text{Lip}(\mathbb{R}^k) \)

and \( J(f) \) is the Jacobian of the map \( f = (f_1, \ldots, f_k) \).

We are now in a position to formulate in a general form a minimizing movement problem concerning integral rectifiable currents which may be considered as a wide generalization of classical problems of mean curvature flow.

**Problem 3.1** - Let \( S = IR C_k(M) \) endowed of the following topology: we say that \( C \subset S \) is closed if, for any sequence \( \{G_h\}_{h \in \mathbb{N}} \subset C \) and any \( G \in S \), the condition

\[
\lim_{h \to +\infty} G_h(f_0; f_1, \ldots, f_k) = G(f_0; f_1, \ldots, f_k)
\]

for any \( f_0 \in \text{Lip}(M) \cap B^\infty(\mathbb{R}^k); f_1, \ldots, f_k \in \text{Lip}(M) \) implies that \( G \in C \).

Given any \( \xi \in IR C_k(M) \) we define \( F(\lambda, k, G, H) = \|G - \xi\|_M \) if \( k \leq 0 \) and

\[
\|G\|_M + \lambda \inf_{\Theta} \left\{ \int_M \text{dist}(x, \text{supp}(H)) \, d\|\Theta\| : \Theta \in IR C_{k+1}(M), \partial\Theta = G - H \right\}
\]

if \( k > 0 \). Our problem is the following: find conditions on \( \xi, M \), such that \( MM(F, S) \neq \emptyset \), or \( G M M(F, S) \neq \emptyset \) and, in the affirmative case, study the properties of the elements of \( MM(F, S) \) or \( G M M(F, S) \).

**Remark 3.4** - Finally, we remark that the theory presented in this paper contains the stationary case, in the sense that a point \( x \in S \) is \( MM(F, S) \)-stationary, or \( G M M(F, S) \)-stationary, if the constant function \( u \equiv x \) belongs to \( MM(F, S) \) or to \( G M M(F, S) \). Obviously, it would be interesting to compare, in the examples presented in this paper as well as in other examples to be formulated, the notions of \( MM \)-stationary point and \( G M M \)-stationary point with other notions of stationary point given elsewhere.

**Bibliography**


