Local vs Non-local Forward Equations for Option Pricing

Rama Cont[*] Yu Gu[†]

Abstract

When the underlying asset is a continuous martingale, call option prices solve the Dupire equation, a forward parabolic PDE in the maturity and strike variables. By contrast, when the underlying asset is described by a discontinuous semimartingale, call prices solve a partial integro-differential equation (PIDE), containing a non-local integral term. We show that the two classes of equations share no common solution: a given set of option prices is either generated from a continuous martingale ("diffusion") model or from a model with jumps, but not both. In particular, our result shows that Dupire's inversion formula for reconstructing local volatility from option prices does not apply to option prices generated from models with jumps.

1 Introduction

Since the seminal work of Black, Scholes and Merton [3, 16] partial differential equations (PDE) have been used as a way of characterizing and efficiently computing option prices. In the Black-Scholes-Merton model and various extensions of this model which retain the Markov property of the risk factors, option prices can be characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. The use of backward PDEs for option pricing has been extended to cover options with path-dependent and early exercise features, as well as to multifactor models (see e.g. [1]). When the underlying asset exhibit jumps, option prices can be computed by solving an analogous partial integro-differential equation (PIDE) [2, 9].

A second important step was taken by Dupire [10, 11] who showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = S_t\sigma(t, S_t)dW_t,$$

prices of call options (at a given date t_0) solve a *forward* PDE

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K) \tag{1.1}$$

on $[t_0,\infty)\times(0,\infty)$ in the strike and maturity variables, with the initial condition

$$\forall K>0 \quad C_{t_0}(t_0,K) = (S_{t_0}-K)_+.$$

[*]Imperial College London, r.cont@imperial.ac.uk

[†]Columbia University, yg2254@columbia.edu

This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Moreover it has been widely used as a tool for *model calibration*: the local volatility function given by the *Dupire formula* [11]

$$\sigma(t,K) := \sqrt{2\left(\frac{C_T(t,K)+r(t)KC_K(t,K)}{K^2C_{KK}(t,K)}\right)}, \tag{1.2}$$

provides a candidate solution to the inverse problem of extracting information on volatility from observed option prices [6].

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [2, 5, 8, 14, 4]. Most of these constructions use the Markov property of the underlying process in a crucial way (see however [15]). As noted by Dupire [12], the forward PDE holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t\delta_t dW_t,$$

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t,S)$ given by

$$\sigma(t,S) = \sqrt{E\{\delta_t^2|S_t = S\}}.$$

Recently Bentata and Cont [7] have derived a forward partial integro-differential equation for prices of call options in a model where the dynamics of the underlying asset under the pricing measure is described by a discontinuous semimartingale, extending Dupire forward equation to a large class of non-Markovian models with jumps.

In particular for the jump diffusion model, when the underlying process is given by

$$S_T = S_{t_0} + \int_{t_0}^T S_{t-}r_t dt + \int_{t_0}^T S_{t-}\tilde{\sigma}_t dW_t + \int_{t_0}^T \int_{-\infty}^{\infty} S_{t-}(e^y-1)\tilde{N}(dtdy),$$

with the compensated Poisson measure $\tilde{N}(dtdy) = N(dtdy) - \nu(dy)dt$, the forward equation becomes

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\tilde{\sigma}(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2} + \int_0^\infty y\frac{\partial^2 C_{t_0}}{\partial K^2}(T,dy)\phi(\ln(\frac{K}{y})) \tag{1.3}$$

with the same initial condition $C_{t_0}(t_0,K) = (S_{t_0}-K)_+$, where ϕ is the double exponential tail of the lévy measure ν defined as

$$\phi(x) = \begin{cases} \int_x^\infty (e^z - e^x)\nu(dz) & \text{if } x > 0, \\ \int_{-\infty}^x (e^x - e^z)\nu(dz) & \text{if } x < 0. \end{cases} \tag{1.4}$$

A natural question arises about whether the two classes of equations, i.e., (1.1) and (1.3), could generate the same option prices. If not, what would happen when we apply Dupire's formula to option prices generated from model with jumps? In practice, there is no clue that whether the price process is continuous or not, so given the market call option price surface as a function of maturity and strike $C(T, K)$, people apply Dupire's formula (1.2) to recover the local volatility surface $\sigma(t, K)$, and expect that the corresponding local volatility model to reproduce the market price, but is it always feasible?

In this paper, we show that there exists no common solution to (1.1) and (1.3), which means the sets of possible call option prices generated by the two different kinds of models, i.e., the local volatility model and the jump diffusion model, don't overlap with each other. It turns out that the appearance of the non-local structure in (1.3), i.e., the integral term, brings significant difference to the equation from the local PDE, and we can show the recovered local volatility surface from model with jumps blows up in the short time maturities.

The following of the paper is organized as follows. In Section 2 we make assumptions about the parameters appeared in the equations and present some technical lemmas, then we prove the main theorem in Section 3.

2 Assumptions and lemmas

We first make assumptions about the parameters appeared in (1.1) and (1.3).

Assumption 1. *$r(t), \sigma(t, K)$ are bounded deterministic functions, σ is bounded from below by some positive constant, and*

$$|\sigma(t, K) - \sigma(s, K)| \leq M|t - s|^{\alpha}, \tag{2.1}$$
$$|K\sigma_K(t, K)| + |K^2\sigma_{KK}(t, K)| + |K^3\sigma_{KKK}(t, K)| \leq M \tag{2.2}$$

for some uniform constant M and Hölder-exponent $\alpha \in (0, 1)$.

Assumption 2. *$\nu \neq 0$ so there exists $x_0 \neq 0$ such that $\phi(x_0) > 0$ and*

$$\int_1^{\infty} e^y \nu(dy) < \infty. \tag{2.3}$$

The following are two technical lemmas.

Lemma 2.1. *$\phi(x)$ is continuous, $\int_{-1}^{1} \phi(x)dx < \infty$, and $\phi(x)$ is bounded when $|x| \geq 1$.*

Proof. By Fubini's theorem,

$$\int_0^1 \phi(x)dx = \int_1^{\infty} (e^z - e + 1)\nu(dz) + \int_0^1 (e^z z - e^z + 1)\nu(dz) < \infty.$$

In the same way, we can show $\int_{-1}^{0} \phi(x)dx < \infty$. In addition, $\phi(x)$ is finite, and increases in $(-\infty, 0)$ while decreases in $(0, \infty)$, therefore, $\phi(x)$ is bounded when $|x| \geq 1$.

To prove the continuity, take $x > 0$ for example. Assume $y > x$, we have:

$$\phi(x) - \phi(y) = \int_0^{\infty} \left[(e^z - e^x)1_{z>x} - (e^z - e^y)1_{z\geq y}\right] \nu(dz) = (I) + (II) + (III),$$

where

$$(I) = \int_0^\infty e^z 1_{x<z<y} \nu(dz),$$
$$(II) = \int_0^\infty (e^y - e^x) 1_{z>x} \nu(dz),$$
$$(III) = -\int_0^\infty e^y 1_{x<z<y} \nu(dz).$$

We see that $(I) + (II) + (III) \to 0$ as $y \to x$, which completes the proof. □

Lemma 2.2. *Let S_t satisfy*

$$dS_t = S_t(r(t)dt + \sigma(t, S_t)dW_t), \tag{2.4}$$

and $S_t = S_0 \exp(X_t)$ for some X_t. Assume the density function of X_t is $G_t(x)$, we have:

$$G_t(x) \leq \frac{M}{\sqrt{t}} \exp(-\frac{x^2}{Mt}) \tag{2.5}$$

for some positive constant M.

Proof. First X_t solves

$$dX_t = (r(t) - \frac{1}{2}\sigma^2(t, S_0 e^{X_t}))dt + \sigma(t, S_0 e^{X_t})dW_t \tag{2.6}$$

with $X_0 = 0$. Then we have $G_t(x) = \Gamma(t, 0, x)$ where $\Gamma(t, 0, x)$ is the solution to the PDE

$$\partial_t \Gamma = \frac{1}{2}\partial_{xx}\left(\sigma^2(t, S_0 e^x)\Gamma\right) + \frac{1}{2}\partial_x\left(\sigma^2(t, S_0 e^x)\Gamma\right) - r(t)\partial_x \Gamma \tag{2.7}$$

with initial condition $\Gamma(0, 0, x) = \delta(x)$. By the classical results of parabolic PDE [13, Page 24, (6.12)], we have

$$\Gamma(t, 0, x) \leq \frac{M}{\sqrt{t}} \exp(-\frac{x^2}{Mt}) \tag{2.8}$$

for some positive constant M. The proof is completed. □

3 Results and conclusions

In this section, we state and prove our main result. Without loss of generality, we assume $t_0 = 0$ and denote $C_{t_0}(T, K)$ as $C(T, K)$. The following is the main theorem.

Theorem 3.1. *There exists no common solution to (1.1) and (1.3).*

Proof. Suppose $C(T, K)$ is the solution to (1.1), we have

$$C(T, K) = \exp(-\int_0^T r(t)dt)\mathbb{E}\{(S_T - K)_+\},$$

where S_t satisfies $dS_t = S_t(r(t)dt + \sigma(t, S_t)dW_t)$. If we denote the density of S_t as $p_t(x)$, we have:

$$C(T, K) = \exp(-\int_0^T r(t)dt) \int_K^\infty (y - K)p_T(y)dy.$$

So

$$C_{KK}(T, K) = \exp(-\int_0^T r(t)dt)p_T(K).$$

If $C(T, K)$ is also a solution to (1.3), then we have:

$$\frac{1}{2}\sigma(T, K)^2K^2C_{KK}(T, K) = \frac{1}{2}\tilde{\sigma}(T, K)^2K^2C_{KK}(T, K) + \int_0^\infty yC_{KK}(T, y)\phi(\ln(\frac{K}{y}))dy. \tag{3.1}$$

Let $x = \ln(K/S_0)$, by change of variable $z = \ln(y/S_0)$, we have:

$$\begin{aligned}&\frac{1}{2}\left[\sigma(T, S_0e^x)^2 - \tilde{\sigma}(T, S_0e^x)^2\right] S_0^2e^{2x} \exp(-\int_0^T r(t)dt)p_T(S_0e^x)\\ =& \exp(-\int_0^T r(t)dt) \int_{-\infty}^\infty S_0e^z p_T(S_0e^z)\phi(x - z)S_0e^z dz.\end{aligned}$$

This leads to

$$\frac{1}{2}\left[\sigma(T, S_0e^x)^2 - \tilde{\sigma}(T, S_0e^x)^2\right] e^{2x}p_T(S_0e^x) = \int_{-\infty}^\infty e^{2z}p_T(S_0e^z)\phi(x - z)dz. \tag{3.2}$$

Let $S_T = S_0 \exp(X_T)$ and the density of X_T be $G_T(x)$, by elementary computation, we have:

$$p_T(x) = \frac{1}{x}G_T(\ln(\frac{x}{S_0})).$$

Now (3.2) becomes

$$\frac{1}{2}\left[\sigma(T, S_0e^x)^2 - \tilde{\sigma}(T, S_0e^x)^2\right] e^xG_T(x) = \int_{-\infty}^\infty e^zG_T(z)\phi(x - z)dz. \tag{3.3}$$

By assumption, ϕ is positive at some $x_0 \neq 0$. Let $x = x_0$ in (3.3), and when $T \to 0$, by Lemma 2.2 we have

$$\begin{aligned}LHS &= \frac{1}{2}\left[\sigma(T, S_0e^{x_0})^2 - \tilde{\sigma}(T, S_0e^{x_0})^2\right] e^{x_0}G_T(x_0)\\ &\leq \frac{1}{2}|\sigma(T, S_0e^{x_0})^2 - \tilde{\sigma}(T, S_0e^{x_0})^2|e^{x_0}\frac{M}{\sqrt{T}}\exp(-\frac{x_0^2}{MT}) \to 0.\end{aligned}$$

Next, we show in (3.3)

$$RHS = \int_{-\infty}^\infty e^zG_T(z)\phi(x_0 - z)dz \to \phi(x_0) \neq 0,$$

which produces contradiction and completes the proof.

Consider

$$\int_{\mathbb{R}} e^z G_T(z)\phi(x_0 - z)dz = \int_{\mathbb{R}} G_T(z)(e^z\phi(x_0 - z) - \phi(x_0))dz + \phi(x_0),$$

by Lemma 2.1, $e^z\phi(x_0 - z)$ as a function of z is continuous at 0, so $\forall\varepsilon > 0$, there exists $\delta > 0$, such that when $|z| < \delta$, we have:

$$|e^z\phi(x_0 - z) - \phi(x_0)| < \varepsilon.$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}} G_T(z)|e^z\phi(x_0 - z) - \phi(x_0)|dz \\
\leq &\int_{|z|<\delta} G_T(z)\varepsilon dz + |\phi(x_0)|\int_{|z|\geq\delta} G_T(z)dz + \int_{|z|\geq\delta} G_T(z)e^z|\phi(x_0 - z)|dz \\
:=&(I) + (II) + (III).
\end{aligned}$$

We see that $(I) \leq \varepsilon$, $(II) \to 0$ as $T \to 0$ by Lemma 2.2. For (III), we have:

$$\begin{aligned}
(III) \leq &M\exp(\frac{MT}{4})\int_{|z|\geq\delta/\sqrt{T}} \exp(-\frac{1}{M}(z - \frac{M\sqrt{T}}{2})^2)|\phi(x_0 - \sqrt{T}z)|dz \\
\leq &M\exp(\frac{MT}{4})\int_{|z|\geq\delta/\sqrt{T}, |x_0-\sqrt{T}z|\geq 1} \exp(-\frac{1}{M}(z - \frac{M\sqrt{T}}{2})^2)|\phi(x_0 - \sqrt{T}z)|dz \\
+ &M\exp(\frac{MT}{4})\int_{|z|\geq\delta/\sqrt{T}, |x_0-\sqrt{T}z|< 1} \exp(-\frac{1}{M}(z - \frac{M\sqrt{T}}{2})^2)|\phi(x_0 - \sqrt{T}z)|dz \\
\leq &M\exp(\frac{MT}{4})\int_{|z|\geq\delta/\sqrt{T}} \exp(-\frac{1}{M}(z - \frac{M\sqrt{T}}{2})^2)dz \\
+ &M\exp(\frac{MT}{4})\frac{1}{\sqrt{T}}\exp(-\frac{1}{M}(\frac{\delta}{\sqrt{T}} - \frac{M\sqrt{T}}{2})^2)\int_{|z|<1} |\phi(z)|dz,
\end{aligned}$$

where we used M for possibly different constants. First, let $T \to 0$, then $\varepsilon \to 0$, the proof is completed. $\square$

Remark 3.2. From the beginning, we can assume the compensator includes some inhomogeneous local speed function, i.e.,

$$\tilde{N}(dtdy) = N(dtdy) - a(t, S_{t-})\nu(dy)dt$$

for some function $a(t, z)$, then the double exponential tail becomes

$$\phi = \phi(t, x) = \begin{cases} \int_x^\infty (e^z - e^x)a(t, z)\nu(dz) & \text{if } x > 0, \\ \int_{-\infty}^x (e^x - e^z)a(t, z)\nu(dz) & \text{if } x < 0. \end{cases} \tag{3.4}$$

As long as we assume $a(t, z)$ is bounded and continuous, and there exists $x_0 \neq 0$ such that $\phi(0, x_0) > 0$, our proof of Lemma 2.1 and Theorem 3.1 still applies.

Recall (3.1), we rewrite it as

$$\sigma(T,K)^2 = \frac{\tilde{\sigma}(T,K)^2K^2C_{KK}(T,K) + 2\int_0^\infty yC_{KK}(T,y)\phi(\ln(K/y))dy}{K^2C_{KK}(T,K)}, \tag{3.5}$$

and this is the local volatility surface recovered by Dupire's formula from the option prices generated in the jump diffusion model. If we assume Lemma 2.2 also holds for S_t from a jump diffusion model, then by the same proof, choosing the strike $K \neq S_0$ such that $\phi(\ln(K/S_0)) > 0$ and letting $T \to 0$, the numerator in (3.5) converges to some positive number while the denominator converges to zero, so $\sigma(T,K) \to \infty$. Therefore we see the blow up of local volatility surface in the short time maturities when applying Dupire's formula to model with jumps.

Remark 3.3. The exponential decay of the tail of density function of $\ln(S_t/S_0)$ is not necessary. We only need proper decay to pass to the limit.

In summary, we proved that two classes of equations share no common solutions, which indicates models with jumps and models with continuous price dynamics lead to disjoint sets of call price surfaces. It further sheds light on model calibration in the sense that Dupire's formula does not necessarily apply to models with jumps. Furthermore, we show that under appropriate conditions, the recovered local volatility surface from models with jumps explodes in the short time maturities.

References

[1] Y. Achdou and O. Pironneau, *Computational methods for option pricing*, vol. 30, Society for Industrial Mathematics, 2005. 1

[2] L. Andersen and J. Andreasen, *Jump diffusion models: Volatility smile fitting and numerical methods for pricing*, Review of Derivatives Research, 4 (2000), pp. 231–262. 1, 2

[3] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 3 (1973), p. 637654. 1

[4] P. Carr, H. Geman, D. B. Madan, and M. Yor, *From local volatility to local Lévy models*, Quant. Finance, 4 (2004), pp. 581–588. 2

[5] P. Carr and A. Hirsa, *Why be backward? Forward equations for American options*, RISK, (2003). 2

[6] R. Cont, *Model calibration*, in Encyclopedia of Quantitative Finance, Wiley, Chichester, 2010, pp. 1210–1219. 2

[7] R. Cont and A. Bentata, *Forward equations for option prices in semimartingale models*, Finance and Stochastics, to appear (2009). 2

[8] R. Cont and I. Savescu, *Forward equations for portfolio credit derivatives*, Frontiers in Quantitative Finance, (2008), pp. 269–293. 2

[9] R. Cont and E. Voltchkova, *Integro-differential equations for option prices in exponential lévy models*, Finance and Stochastics, 9 (2005), pp. 299–325. 1

[10] B. Dupire, *Pricing with a smile*, Risk, 7 (1994), pp. 18–20. 1

[11] ——, *Pricing and hedging with smiles*, in Mathematics of Derivative Securities, M. Dempster and S. Pliska, eds., Cambridge University Press, 1997, pp. 103–111. 1, 2

[12] ——, *A unified theory of volatility*, tech. report, Working paper, Banque Paribas, 1997. 2

[13] A. Friedman, *Partial differential equations of parabolic type*, vol. 38, Prentice-Hall Englewood Cliffs, NJ, 1964. 4

[14] B. Jourdain, *Stochastic flow approach to dupires formula*, Finance and Stochastics, 11 (2007), pp. 521–535. 2

[15] F. Klebaner, *Option price when the stock is a semimartingale*, Electronic Communications in Probability, 7 (2002), pp. 79–83. 2

[16] R. Merton, *Theory of rational option pricing*, Bell Journal of Economics, 4 (1973), pp. 141–183. 1