Ordinary Differential Equations

Macroeconomic Analysis Recitation 1

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1 Introduction

We will cover some basics of ordinary differential equations (ODE). Within this class, we deal with differential equations, whose variable of interest takes derivative with respect to time t. Denote $\dot{Y}_t = \frac{dY_t}{dt}$, where Y_t can be a scalar or vector. A general explicit form of ODE is

$$\dot{Y}_t = f(Y_t, t) \tag{1}$$

2 First-Order Differential Equations

- Autonomous equation: $\dot{y}_t = f(y_t)$, an equation is autonomous when it depends on time only through the variable itself. Example: $k_t = sk_t^\alpha - \delta k_t$, where s, α and δ are constants.

- Linear equation: $\dot{y}_t = a_t y_t + b_t$, where a_t and b_t are taken as given. Example: $\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}(r_t - \rho)$, where γ and ρ are parameters, while r_t is a given function of t.

- Homogeneous: set the above linear differential equation $b_t = 0$. This terminology also applies to high-order differential equations: e.g. $\ddot{y}_t = g_t \dot{y}_t + h_t y_t$.

Autonomous equation can be solved (illustrated) graphically, while linear equation admits analytical solution.

2.1 Analytical Solution

A homogeneous differential equation

$$\dot{y}_t = a_t y_t \tag{2}$$

Divide both sides by y_t,

$$\frac{\dot{y}_t}{y_t} = a_t \tag{3}$$

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$$\Rightarrow \frac{dlog(y_t)}{dt} = a_t \tag{4}$$

Therefore,

$$y_t = C\exp(\int_0^t a_s ds) \tag{5}$$

where C is determined by boundary condition.

A linear differential equation

$$\dot{y}_t = a_t y_t + b_t \tag{6}$$

Rearrange the above equation as

$$\dot{y}_t - a_t y_t = b_t \tag{7}$$

Multiply both sides by $\exp(-\int_0^t a_s ds)$, we obtain

$$\dot{y}_t \exp(-\int_0^t a_s ds) - a_t y_t \exp(-\int_0^t a_s ds) = b_t \exp(-\int_0^t a_s ds) \tag{8}$$

That is

$$\frac{d[y_t \exp(-\int_0^t a_s ds)]}{dt} = b_t \exp(-\int_0^t a_s ds) \tag{9}$$

$$\Rightarrow y_t = \exp(\int_0^t a_s ds)(\int_0^t b_u \exp(-\int_0^u a_s ds)du + C) \tag{10}$$

where C is pinned down by boundary condition.

Example

$$\dot{k}_t = s k_t^\alpha - \delta k_t \tag{11}$$

Define $z_t = k_t^{1-\alpha}$, then

$$\dot{z}_t = (1-\alpha)k_t^{-\alpha}\dot{k}_t \tag{12}$$

Substitute $\dot{k}_t, k_t$ by $\dot{z}_t, z_t$, we arrive at

$$\dot{z}_t = s(1-\alpha) - \delta(1-\alpha)z_t \tag{13}$$

Let $a_t = -\delta(1-\alpha)$ and $b_t = s(1-\alpha)$, and use the solution we already get in the linear differential equation.

$$z_t = \frac{s}{\delta} + (z_0 - \frac{s}{\delta})\exp[-\delta(1-\alpha)t] \tag{14}$$

$$\Rightarrow k_t = \{\frac{s}{\delta} + (k_0^{1-\alpha} - \frac{s}{\delta})\exp[-\delta(1-\alpha)t]\}^{\frac{1}{1-\alpha}} \tag{15}$$

2.2 Graphical Solution

An autonomous equation

$$\dot{y}_t = f(y_t) \tag{16}$$

Solution Steps:

- Plot $f(y)$ in the space of $(y, \dot{y})$. Put y on the x-axis and $\dot{y}$ on the y-axis.
- Draw rightward arrows when $f > 0$, and leftward arrows when $f < 0$.
- Given an initial point, follow the arrows to track the dynamics of y.

Equilibrium and Stability: When $\dot{y}_t = 0$, we have $f(y) = 0$. The solutions to $f(y_t) = 0$ are equilibrium points. An equilibrium point y^* is said to be stable if $f'(y^*) < 0$ (or equivelently, write it as $\frac{\partial \dot{y}}{\partial y}|_{y^*} < 0$). Intuitively, the arrows around the equilibrium point direct to the stable equilibrium point. Or roughly speaking, after a small perturbation, y_t will finally go back to the equilibrium point.

Examples:

- $\dot{y}_t = ay_t + b$, where $a < 0$. The equilibrium point is $y^* = -\frac{b}{a}$. Since $\frac{\partial \dot{y}}{\partial y}|_{y^*} = a < 0$, this equilibrium point is stable. See Figure 1. (Figures are on the last two pages.)
- $\dot{y}_t = ay_t + b$, where $a > 0$. The equilibrium point is $y^* = -\frac{b}{a}$. Since $\frac{\partial \dot{y}}{\partial y}|_{y^*} = a > 0$, this equilibrium point is unstable. See Figure 2.
- Go back to our old friend, $\dot{k}_t = sk_t^\alpha - \delta k_t$, with $0 < \alpha < 1$. We have two equilibrium points: $k^* = (\frac{s}{\delta})^{\frac{1}{1-\alpha}}$, $k^{**} = 0$. Then $\frac{\partial \dot{k}}{\partial k}|_{k^*} = \delta(\alpha - 1) < 0$ and $\frac{\partial \dot{k}}{\partial k}|_{k^{**}+} = +\infty$. Therefore, k^* is stable and k^{**} is unstable. See Figure 3.

2.3 Linearization

Suppose we are interested in the dynamics around the equilibrium y^*.

$$\dot{y}_t = f(y_t) \approx f(y^*) + f'(y^*)(y_t - y^*) = f'(y^*)(y_t - y^*) \tag{17}$$

Applying it to the above $\dot{k}_t = sk_t^\alpha - \delta k_t$, we obtain

$$\dot{k}_t = (\alpha s k^{*\alpha-1} - \delta)(k_t - k^*) \tag{18}$$

3 Systems of Differential Equations

3.1 Analytical Solution

Consider the linear system of differential equation $\dot{Y}_t = A_t Y_t + B_t$ with $A_t = A, B_t = 0$. A is a $n \times n$ matrix with constant elements.

A simple case is when A has n linearly independent eigenvectors $v_1, v_2....v_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. It is equivalent to say A is diagonalizable $D = P^{-1}AP$, where the diagonal elements of D are $\lambda_1, \lambda_2, ..., \lambda_n$ and the columns of P are $v_1, v_2....v_n$.

$$\dot{Y_t} = AY_t = PDP^{-1}Y_t \tag{19}$$

$$\Rightarrow P^{-1}\dot{Y_t} = P^{-1}AY_t = DP^{-1}Y_t \tag{20}$$

Denote $Z_t = P^{-1}Y_t$, we have

$$\dot{Z_t} = DZ_t \tag{21}$$

$$\Rightarrow \dot{z_{it}} = \lambda_i z_{it}, i = 1, 2, ..., n \tag{22}$$

$$\Rightarrow z_{it} = c_i \exp(\lambda_i t), i = 1, 2, ..., n \tag{23}$$

Given Z_t, we immediately get $Y_t = PZ_t$.

One can follow the same steps above to show that $\dot{Y_t} = AY_t + B_t$ has analytical solution as well.

Remark 1. If A is not diagnonalizable (for example, in a two-dimension case, we may only have one linearly independent eigenvector), we can use another decomposition $T = U^{-1}AU$, where T is an upper triangular matrix. Denote $W_t = U^{-1}Y_t$, we have $\dot{W_t} = TW_t + U^{-1}B_t$, then solve w_{it} by the order of $w_{nt}, w_{n-1,t}, ..., w_{1t}$, where w_{it} is the ith element of W_t.

Remark 2. It is possible that eigenvalues are complex numbers, and eigenvectors are complex vectors thus we obtain complex solutions. However, we want real solutions instead of complex solutions. Apply the following observation: if $p_t + iq_t$ is a solution to $\dot{Y_t} = AY_t$ ($p_t - iq_t$ should also be a solution), where p_t and q_t are real vectors, then p_t and q_t are also the solutions. This is because $\dot{Y_t} = \dot{p_t} + i\dot{q_t} = A(p_t + iq_t) = Ap_t + iAq_t$, then we arrive at $\dot{p_t} = Ap_t$ and $\dot{q_t} = Aq_t$.

3.2 Graphical Solution (Phase Diagram)

Here we focus on two dimensions of system of differential equations. That is we have two variables y_1 and y_2 of interest.

$$\dot{y_{1t}} = f(y_{1t}, y_{2t}) \tag{24}$$

$$\dot{y_{2t}} = g(y_{1t}, y_{2t}) \tag{25}$$

Solution Steps:

- In the space (y_1, y_2), draw the lines of $\dot{y_{1t}} = 0$ and $\dot{y_{2t}} = 0$ respectively. Or equivalently to say, draw both $f(y_1, y_2) = 0$ and $g(y_1, y_2) = 0$
- Draw rightward arrows when $\dot{y_{1t}} > 0$, and leftward arrows when $\dot{y_{1t}} < 0$
- Draw upward arrows when $\dot{y_{2t}} > 0$, and downward arrows when $\dot{y_{2t}} < 0$
- The intersections of $\dot{y_{1t}} = 0$ and $\dot{y_{2t}} = 0$ are equilibrium points. These points can be stable, unstable or saddle path stable.

- Given an initial point, follow the direction of arrows to track the dynamics of (y_{1t}, y_{2t})

Remark. In macroeconomics,we usually have initial values of state varibles (instead of all the varibles) and a transversality condition to uniquely determine the initial point and thus the whole dynamics. For example, given y_{10} (y_{1t} is a state variable), we will pick up y_{20} (y_{2t} is a control variable). After picking up y_{20}, we follow arrows and we need to ensure that we will reach an equilibrium point which satisfies the transversality condition.

Example 1 Consider the following system of differential equations of c_t and k_t.

$$\dot{k_t} = k_t^\alpha - \delta k_t - c_t \tag{26}$$

$$\dot{c_t} = \frac{1}{\gamma}(\alpha k_t^{\alpha-1} - \delta - \rho)c_t \tag{27}$$

Boundary condition: initial state k_0 and transversality condition $lim_{t\to+\infty}\lambda_t k_t = 0$. You will see what transversality condition is in class, and it basically guarantees that the economy will not explode or converge to a non-sense point (no-ponzi scheme). Note we will have a unique saddle path. See Figure 4.

Example 2 Now we turn to a two dimension linear case $\dot{Y_t} = AY_t$, where $Y_t = (y_{1t}, y_{2t})'$ and A is a 2 by 2 matrix with constant elements. We first write out the analytical solutions (see the above Section 3.1 for how to solve it.). Suppose the eigenvalues of matrix A are λ_1 and λ_2.

- If $\lambda_1 \neq \lambda_2$ and the two linearly independent eigenvectors are v_1, v_2, the solution is of the form:
$$Y_t = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t} \tag{28}$$

- If $\lambda_1 = \lambda_2$ (must be a real number) and we have two linearly independent eigenvectors v_1, v_2 , the solution is of the form:
$$Y_t = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_1 t} \tag{29}$$

- If $\lambda_1 = \lambda_2$ (must be a real number) and we only have one linearly independent eigenvector v_1, the solution is of the form:
$$Y_t = C_1 v_1 e^{\lambda_1 t} + C_2(v_1 t e^{\lambda_1 t} + v_2 e^{\lambda_1 t}) \tag{30}$$
where v_2 is the solution of $(A - \lambda_1 I)v_2 = v_1$

Constants C_1 and C_2 are determined by boundary conditions.

Eigenvalues and Stability (Example 2)

- If the eigenvalues are both positive real numbers, the equilibrium is unstable. See Figure 5.

- If the eigenvalues are both negative real numbers, the equilibrium is stable. See Figure 6.

- If the real parts of the eigenvalues are of opposite sign (it also implies that there is no complex part), the equilibrium is saddle path stable. Note this case is of special interest in this class. See Figure 7.
- If the eigenvalues are both complex numbers and the real parts are both positive, the system is unstable and oscillating. See Figure 8.
- If the eigenvalues are both complex numbers and the real parts are both negative, the system converges to the steady state in an oscillating manner. See Figure 9.

3.3 Linearization

Suppose we are interested in the dynamics around equilibrium point. We proceed with a two-dimension case.

$$\dot{y_{1t}} = f(y_{1t}, y_{2t}) \approx f(y_1^*, y_2^*) + f_1(y_1^*, y_2^*)(y_{1t} - y_1^*) + f_2(y_1^*, y_2^*)(y_{2t} - y_2^*) \tag{31}$$

$$\dot{y_{2t}} = g(y_{1t}, y_{2t}) \approx g(y_1^*, y_2^*) + g_1(y_1^*, y_2^*)(y_{1t} - y_1^*) + g_2(y_1^*, y_2^*)(y_{2t} - y_2^*) \tag{32}$$

Since we linearize around the equilibrium point, $f(y_1^*, y_2^*) = 0$ and $g(y_1^*, y_2^*) = 0$. That implies

$$\dot{Y}_t = A(Y_t - Y^*) \tag{33}$$

where $Y_t = (y_{1t}, y_{2t})'$ or

$$\dot{Z}_t = AZ_t \tag{34}$$

where $Z_t = Y_t - Y^*$

Then the above formula goes back to the aformentioned homogeneous system of differential equation with constant coefficient matrix.

Example

$$\dot{k_t} = k_t^\alpha - \delta k_t - c_t \approx (\alpha k^{*\alpha-1} - \delta)(k_t - k^*) - (c_t - c^*) \tag{35}$$

$$\dot{c_t} = \frac{1}{\gamma}(\alpha k_t^{\alpha-1} - \delta - \rho)c_t \approx \frac{\alpha(\alpha-1)}{\gamma} c^* k^{*\alpha-2}(k_t - k^*) \tag{36}$$

Pick up the equilibrium point with positive (k^*, c^*) and write the above in a more compact form:

$$\dot{k_t} = \beta(k_t - k^*) - (c_t - c^*) \tag{37}$$

$$\dot{c_t} = -\tau(k_t - k^*) \tag{38}$$

with $\beta > 0$ and $\tau > 0$. The coefficient matrix is (assume parameter $0 < \alpha < 1$)

$$A = \begin{pmatrix} \beta & -1 \\ -\tau & 0 \end{pmatrix}$$

One can show the eigenvalues of the above have opposite sign. Therefore, the equilibrium is saddle path stable.

3.4 Time Elimination Method

An illustrative example

$$\dot{k}_t = k_t^\alpha - \delta k_t - c_t \tag{39}$$

$$\dot{c}_t = \frac{1}{\gamma}(\alpha k_t^{\alpha-1} - \delta - \rho)c_t \tag{40}$$

If we have policy function $c(k)$, we can solve k_t from

$$\dot{k} = k^\alpha - \delta k - c(k) \tag{41}$$

by using standard numerical method. In order to get $c(k)$, we do the following

$$\frac{dc}{dk} = \frac{\dot{c}_t}{\dot{k}_t} = \frac{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)c(k)}{k^\alpha - \delta k - c(k)} \tag{42}$$

Initial condition is given by steady state (k^*, c^*)

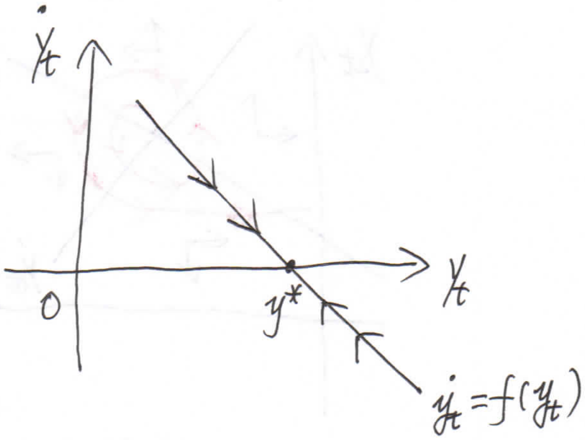

Figure 1.

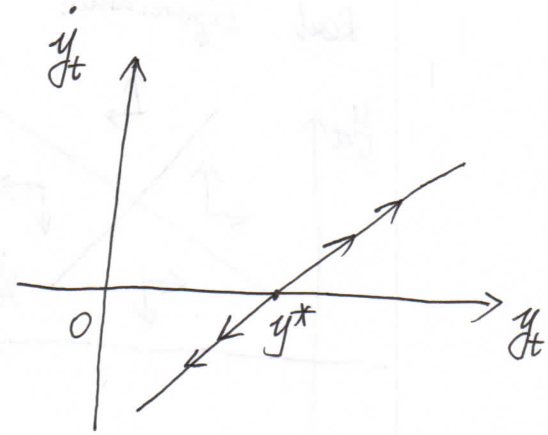

Figure 2.

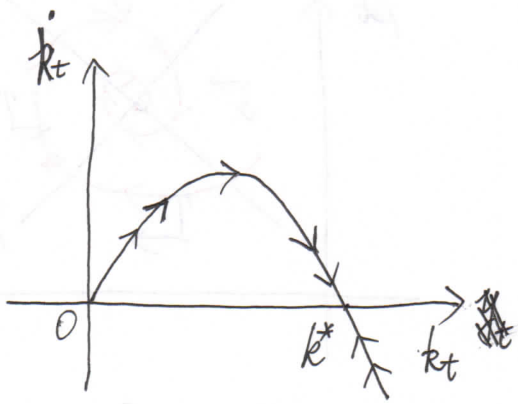

Figure 3.

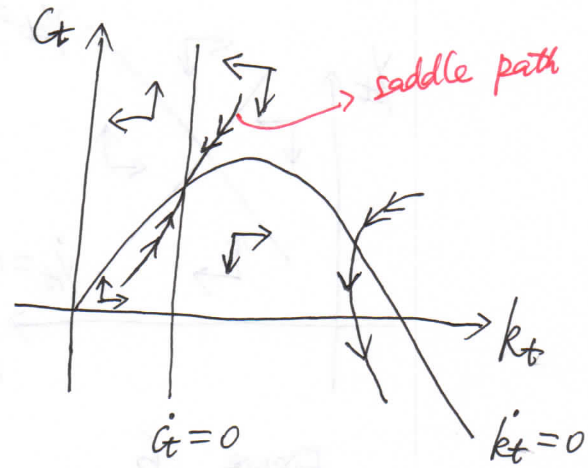

Figure 4

	Real Eigenvalues	Complex Eigenvalues
Real parts are positive	Figure 5	Figure 8
Real parts are negative	Figure 6	Figure 9
Real parts have different signs.	Figure 7	No this case.

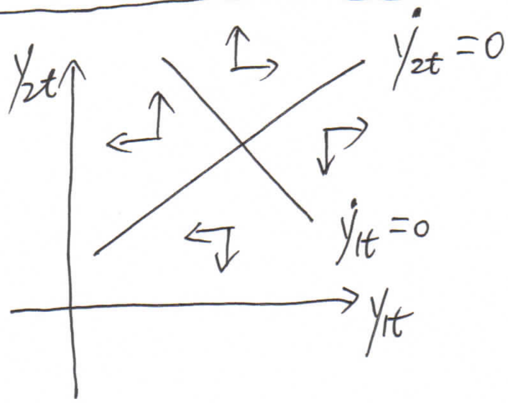

Figure 5

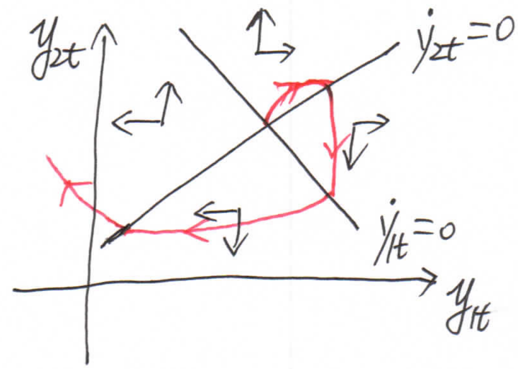

Figure 8

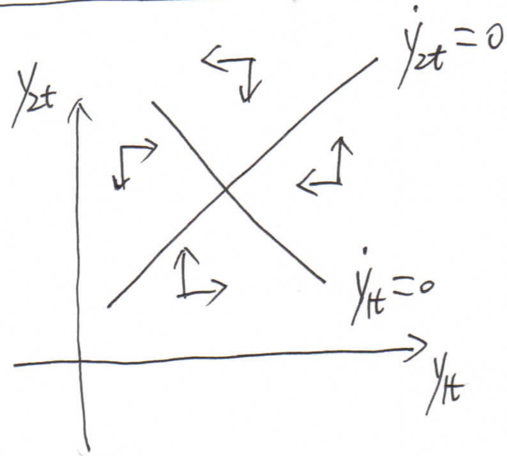

Figure 6

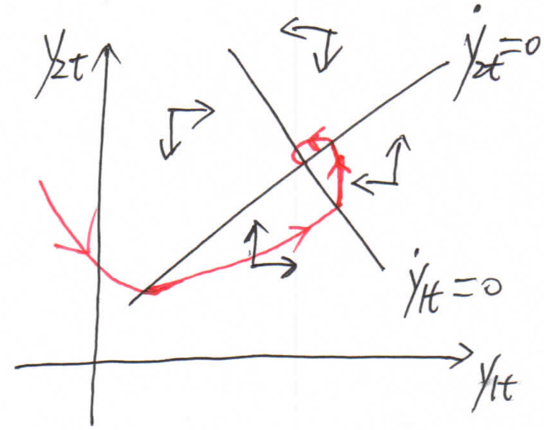

Figure 9

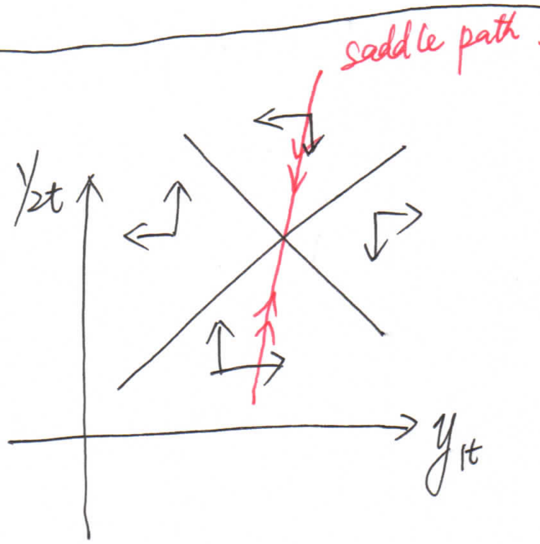

Figure 7

Investment Theory

Macroeconomic Analysis Recitation 2

Yang Jiao*

In this note, our focus is 1). Linearization of the internal adjustment cost model. 2). External adjustment cost model. I type the setup of internal adjustment cost as well in case you can refer to it. We will come back to the comparison of the two investment models next time.

1 Internal Adjustment Cost Model

1.1 Model Setup

Internal adjustment cost means in order to invest I_t, firms have to themselves forego additional resources $I_t\varphi(\frac{I_t}{K_t})$. Assume $\varphi(0) = 0$, $\varphi'(\cdot) > 0$ and $2\varphi' + \frac{I}{K}\varphi'' > 0$. For simplicity, assume depreciation rate $\delta = 0$.

$$V_0 = \max_{K_t,L_t,I_t} \int_0^{+\infty} e^{-rt}[AF(K_t, L_t) - w_tL_t - I_t(1 + \varphi(\frac{I_t}{K_t}))]dt \tag{1}$$

s.t.

$$\dot{K}_t = I_t \tag{2}$$

$$\text{Non-ponzi scheme: } \lim_{t\to\infty} e^{-rt}K_t \geq 0 \tag{3}$$

$$K_0 \text{ is given.} \tag{4}$$

1.2 Solve the Model

We first set up the Hamiltonian:

$$\mathscr{H} = e^{-rt}[AF(K_t, L_t) - w_tL_t - I_t(1 + \varphi(\frac{I_t}{K_t}))] + \lambda_tI_t \tag{5}$$

First order conditions are:

$$AF_L = w_t \tag{6}$$

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$$e^{-rt}\{-[1+\varphi(\frac{I_t}{K_t})+\frac{I_t}{K_t}\varphi'(\frac{I_t}{K_t})]\}+\lambda_t=0 \tag{7}$$

$$e^{-rt}[AF_K+(\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})]=-\dot{\lambda}_t \tag{8}$$

$$TVC:\lim_{t\to+\infty}\lambda_t K_t=0 \tag{9}$$

Denote current shadow price $q_t=e^{rt}\lambda_t$, then F.O.C.s change to

$$AF_L=w_t \tag{10}$$

$$q_t=1+\varphi(\frac{I_t}{K_t})+\frac{I_t}{K_t}\varphi'(\frac{I_t}{K_t}) \tag{11}$$

$$\dot{q}_t=rq_t-[AF_K+(\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})] \tag{12}$$

$$TVC:\lim_{t\to+\infty}e^{-rt}q_tK_t=0 \tag{13}$$

Equation (11) establishes the relationship between q_t and $\frac{I_t}{K_t}$. It is a one-to-one mapping (by $2\varphi'+\frac{I}{K}\varphi''>0$, we know q_t is an increasing function of $\frac{I_t}{K_t}$).

$$\frac{I_t}{K_t}=h(q_t) \tag{14}$$

From equation (11), when $\frac{I}{K}=0$, $q=1$, thus $h(1)=0$ in equation (14).

To solve the model, capital accumulation equation has to be used as well (don't foreget it, since it is not listed in the first order conditions.).

$$\dot{K}_t=I_t=h(q_t)K_t \tag{15}$$

From the first order conditions, we also have

$$\dot{q}_t=rq_t-[AF_K+h^2(q_t)\varphi'(h(q_t)] \tag{16}$$

Now we have got a system of differential equations for q_t, K_t with initial condition K_0 given and TVC $\lim_{t\to+\infty}e^{-rt}q_tK_t=0$.

Remark. In fact, the problem is complicated because labor input L_t will depend on K_t, see equation (10). And L_t shows up in equation (16) $F_K(K_t,L_t)$ as well. Denote the functional relationship derived from equation (10) as $L_t=g(K_t)$. Then in equation (16), one needs to substitute $F_K(K,L)$ by $F_K(K,g(K))$. However, to simplify the analysis, we assume labor demand L_t keeps fixed at L^* for the moment.

1.3 Graphical Solution (Phase Diagram)

- We first need to find the steady state. We will focus on the steady state with positive (k^*,q^*).

$$h(q^*)=0 \tag{17}$$

$$rq^* = [AF_K(K^*, L^*) + h^2(q^*)\varphi'(h(q^*)] \tag{18}$$

Since $h(1) = 0$ and h is monotonic, we immediately have $q^* = 1$. And then $K^* = F_K^{-1}(r/A)$.

- It is easy to draw $\dot{K}_t = 0$, that is $h(q) = 0$ thus $q = 1$, a horizontal line in the space of (K, q).

  Since we will concentrate on the dynamics around the steady state, we would like to know the slope of $\dot{q}_t = 0$ in a small neighborhood of (K^*, q^*). Denote the line $\dot{q}_t = 0$ as $m(K, q) = 0$,

  $$m(K, q) = rq - [AF_K(K, L^*) + h^2(q)\varphi'(h(q))] \tag{19}$$

  For the line $m(K, q) = 0$, at point (K^*, q^*), by the implicit function theorem, the slope is

  $$\frac{dq}{dK}|_{(K^*,q^*)} = -\frac{\partial m/\partial K}{\partial m/\partial q} = \frac{AF_{KK}(K^*, L^*)}{r - 2h(q^*)\varphi'(h(q^*)) - h^2(q^*)\varphi''(h(q^*))h'(q^*)} = \frac{AF_{KK}(K^*, L^*)}{r} < 0 \tag{20}$$

  By continuity, in a small neighborhood of (K^*, q^*), the slope is also negative.

  Remark 1. If we forget about the assumption that L^* is fixed, what we need for $\frac{dq}{dK}|_{(K^*,q^*)} < 0$ to hold is $F_{KK}(K^*, g(K^*)) + F_{K,L}(K^*, g(K^*))g'(K^*) < 0$. In fact, to justify the fixed L^*, we may consider a general equilibrium model with fixed labor supply L^*, so wages will adjust to guarantee that labor demand equals labor supply $L_t = L^*$.

  Remark 2. When we have capital depreciation rate $\delta > 0$ in this internal adjustment cost model, the steady state q^* will not be 1. Please check how depreciation rate will affect steady state level of q^*.

See Figure 1 for the phase diagram. When $q_t > 1$, firms invest, thus capital stock increases, while when $q_t < 1$, firms disinvest thus capital stock decreases. That is to say investment depends on q_t.

1.4 Linearization

Around the steady state (K^*, q^*):

$$\dot{K}_t \approx h(q^*)(K_t - K^*) + K^* h'(q^*)(q_t - q^*) \tag{21}$$

$$\dot{q}_t \approx -AF_{KK}(K^*, L^*)(K_t - K^*) + [r - 2h(q^*)\varphi'(h(q^*)) - h^2(q^*)\varphi''(h(q^*))h'(q^*)](q_t - q^*) \tag{22}$$

Substituting the steady state property that $h(q^*) = h(1) = 0$, we obtain

$$\dot{K}_t \approx K^* h'(q^*)(q_t - q^*) \tag{23}$$

$$\dot{q}_t \approx -AF_{KK}(K^*, L^*)(K_t - K^*) + r(q_t - q^*) \tag{24}$$

Write the above in a more compact form

$$\dot{Z}_t = GZ_t \tag{25}$$

where $Z_t = (K_t - K^*, q_t - q^*)'$ and the coefficient matrix is

$$G = \begin{pmatrix} 0 & K^* h'(q^*) \\ -AF_{KK}(K^*, L^*) & r \end{pmatrix}$$

The eigenvalues λ_1 and λ_2 satisty that

$$\lambda_1 * \lambda_2 = det(G) = K^* h'(q^*) * AF_{KK}(K^*, L^*) < 0 \tag{26}$$

We conclude that eigenvalues must have opposite signs and they are real numbers thus this system is saddle path stable.

Solving the eigenvalues explicitly, we have

$$\lambda = \frac{r \pm \sqrt{r^2 - 4AF_{KK} h'(q^*) K^*}}{2} \tag{27}$$

Let λ_1 be the eigenvalue smaller than 0, and λ_2 larger than 0. The solution to the above system of differential equation is

$$Z_{1t} = K_t - K^* = \Psi_{11} e^{\lambda_1 t} + \Psi_{12} e^{\lambda_2 t} \tag{28}$$

$$Z_{2t} = q_t - q^* = \Psi_{21} e^{\lambda_1 t} + \Psi_{22} e^{\lambda_2 t} \tag{29}$$

(Recall the result from recitation 1, a two-dimension linear system of differential equation with two different eigenvalues should have the solution form $Z_t = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$, where C_1 and C_2 are constants, v_1 and v_2 are eigenvectors, and λ_1 and λ_2 are eigenvalues. Therefore, $\Psi_{11} = C_1 v_{11}$, $\Psi_{21} = C_1 v_{12}$, $\Psi_{12} = C_2 v_{21}$, and $\Psi_{22} = C_2 v_{22}$)

To let the system converge to (K^*, q^*), we must have $\Psi_{12} = \Psi_{22} = 0$ (just set C_2=0). Otherwise, the solution would have a term $e^{\lambda_2 t}$ going to infinity and TVC will be violated. Now we are on the saddle path.

To determine Ψ_{11} and Ψ_{21}, we need two conditions. First, the initial K_0 is given

$$K_0 - K^* = \Psi_{11} \tag{30}$$

Second, $(\Psi_{11}, \Psi_{21})'$ is the eigenvector of λ_1(v_1 is an eigenvalue, then $C_1 v_1$ is also an eigenvector when $C_1 \neq 0$),

$$\Psi_{21} = \frac{\lambda_1 \Psi_{11}}{K^* h'(q^*)} = \frac{\lambda_1 (K_0 - K^*)}{K^* h'(q^*)} \tag{31}$$

In sum, we have determined all the dynamics of this linearized system analytically.

Additionally, initial q is given by

$$q_0 = q^* + \Psi_{21} = q^* + \frac{\lambda_1 (K_0 - K^*)}{K^* h'(q^*)} \tag{32}$$

2 External Adjustment Cost

In the above internal adjustment cost model, capital is owned by firms and firms bear the adjustment cost. Now consider an alternative setting in which final goods producers purchase capital from capital goods producers. Final goods producers bear no adjustment cost while capital good producers incur adjustment cost (modeled as a convex cost of production, i.e. decreasing return to scale production technology). One can think that installation of capital is done by capital good producers instead of final good producers.

2.1 Model Setup

2.1.1 Capital Goods Firms

To produce I_t capital goods, capital goods firms have to incur cost $C(I_t)$. We assume the cost function satisfies $C(0) = 0$, $C'(I) > 0$ for $I > 0$, $C''(I) > 0$ for $I > 0$ and $\lim_{I \to +\infty} C'(I) = +\infty$. In short, cost function is convex and marginal cost goes to infinity when producing infinite capital.

These firms are price takers, and their problem is

$$\max_{I_t} \int_0^{+\infty} e^{-rt}[P_{It}I_t - C(I_t)]dt \tag{33}$$

This is a static problem. It is equivalent to maximize $P_{It}I_t - C(I_t)$ at each moment. The first order condition is

$$P_{I_t} = C'(I_t) \tag{34}$$

Since $C'' > 0$, P_I is a strictly increasing function of I and the reverse is true as well:

$$I = h(P_I) \tag{35}$$

with $h'(\cdot) > 0$. It is a supply function of capital goods.

2.1.2 Final Goods Producers

We assume final goods producers directly buy capital from capital good producers and they don't pay adjustment cost.

$$V_0 = \max_{K_t, L_t, I_t} \int_0^{+\infty} e^{-rt}[P_t AF(K_t, L_t) - w_t L_t - P_{It}I_t]dt \tag{36}$$

s.t.

$$\dot{K}_t = I_t - \delta K_t \tag{37}$$

$$\text{Non-ponzi scheme: } \lim_{t \to \infty} e^{-rt} K_t \geq 0 \tag{38}$$

$$K_0 \text{ is given.} \tag{39}$$

2.2 Solve the Model

We set up the Hamiltonian of final goods producers:

$$\mathscr{H} = e^{-rt}[P_t AF(K_t, L_t) - w_t L_t - P_{It}I_t] + \lambda_t(I_t - \delta K_t) \tag{40}$$

First order conditions are:

$$AF_L = w/P \tag{41}$$

$$-e^{-rt}P_{It} + \lambda_t = 0 \tag{42}$$

$$e^{-rt}PAF_K - \lambda_t\delta = -\dot{\lambda}_t \tag{43}$$

$$TVC: \lim_{t\to+\infty} \lambda_t K_t = 0 \tag{44}$$

Denote current shadow price $q_t = e^{rt}\lambda_t$, then F.O.C.s change to

$$AF_L = w_t/P_t \tag{45}$$

$$P_{It} = q_t \tag{46}$$

$$\dot{P}_{It} = (r+\delta)P_I - PAF_K \tag{47}$$

$$TVC: \lim_{t\to+\infty} e^{-rt}q_t K_t = 0 \tag{48}$$

Remember we also have the capital accumulation equation, and plugging in the solution of capital goods producers yields

$$\dot{K}_t = h(P_{It}) - \delta K_t \tag{49}$$

Combining

$$\dot{P}_{It} = (r+\delta)P_{It} - PAF_K, \tag{50}$$

initial condition K_0 and TVC $\lim_{t\to+\infty} e^{-rt}P_{It}K_t = 0$, we are ready to solve a two-dimension differential equation system.

2.3 Housing Market Interpretation

- One can think of the capital goods producers as firms in construction sector who build new houses. Final goods producers are real estate agents who will hire labor to provide housing services.

- Capital stock is housing stock and real estate agents buy newly built houses from the construction sector. Real estate agents are the owners of housing stock.

- We can view the newly produced capital goods I_t as residential investment. $PAF(K, L)$ are the rent income of real estate agents. P_I is the price of a unit of newly built house. We take final goods price P as given, but one can have a household side utility function to derive a demand function for final goods, then P can be endogenized.

- Some of you asked why capital goods production has convex cost. Here is an interpretation: since land supply is limited, building one house on the top of a skyscraper will be more difficult than building a house on the ground.

- Interest rate

$$r = \frac{\dot{P}_I + PAF_K - \delta P_I}{P_I} \tag{51}$$

given by equation (50). It is a no-arbitrage condition: return from investing in a unit of house is equal to the interest rate.

- Rearrange equation (50):

$$(r+\delta)P_{It} - \dot{P}_{It} = PAF_K \tag{52}$$

Multiply both sides by $e^{-(r+\delta)t}$, and take integral from 0 to $+\infty$ on both sides,

$$P_{I0} = \int_0^{+\infty} e^{-(r+\delta)t} PAF_K(K_t, L^*)dt \tag{53}$$

The price of a house is equal to the present value of all future rent income from the house. Notice we have an addtional depreciation rate δ in the equation, because in the setup we assume $\delta \neq 0$.

Remark. You can use similar steps to get q_0 in the **internal adjustment cost** model:

$$q_0 = \int_0^{+\infty} e^{-rt}[AF_K + (\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})]dt \tag{54}$$

The term $(\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})$ captures the learning by doing benefit from installing capital today.

2.4 Graphical Solution (Phase Diagram)

Setting $\dot{K}_t = 0$ and $\dot{P}_{It} = 0$ leads to

$$h(P_I) = \delta K \tag{55}$$

$$P_I = \frac{PAF_K(K, L^*)}{r+\delta} \tag{56}$$

One is a positive relationship between K and P_I, and the other is a negative relationship. The steady state is given by the intersection of the above two lines. See Figure 2 for the phase diagram.

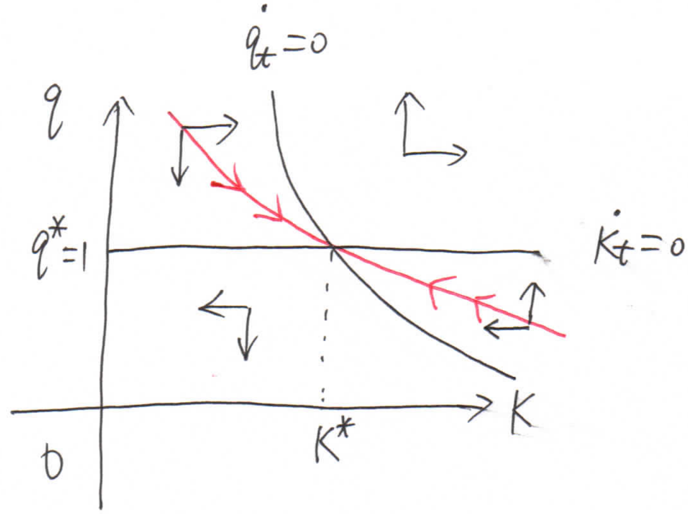

Figure 1.

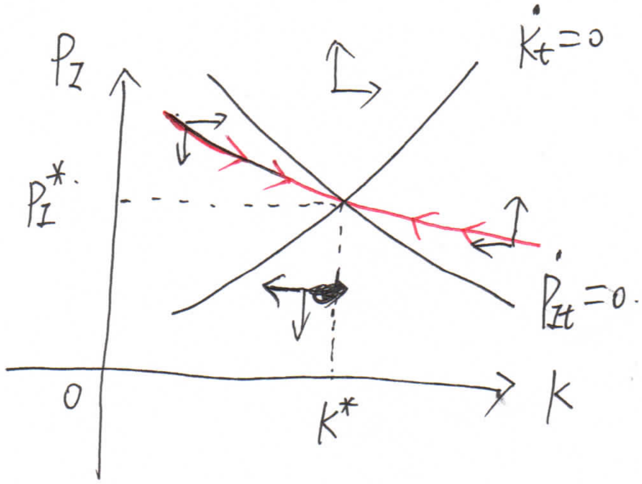

Figure 2

Investment Theory (continued)- Neoclassical Production Functions

Macroeconomic Analysis Recitation 3

Yang Jiao*

1 Comparison of Internal and External Adjustment Cost Models

Consider a permanent positive productivity shock to both models. See Figure 1 (internal cost model) and Figure 2 (external cost model). In the external adjustment cost model, long run $q^* = P_I^*$ will increase, while in the internal adjustment cost model, steady state q^* is fixed at 1.

One may wonder are these two models really different? Adjustment cost is just adjustment cost after all. Note the difference we introduced in these two models: in the external adjustment cost model, we add depreciation rate of capital, and assume the cost of producing new capital is $C(I)$ instead of $I(1+\varphi(\frac{I}{K}))$ as in the internal adjustment cost model. Now the question is whether these two differences in modeling generate the different predictions as shown in Figure 1 and Figure 2.

In the external adjustment cost model, in steady state, when K^* changes, $I^* = \delta K^*$ will change as well since we have depreciation rate $\delta > 0$. Then $q^* = P_I^* = C'(I^*)$ will also change. In order to make $q^* = P_I^*$ not change with K^* (i.e. a flat $\dot{K}_t = 0$ line), we need to revise the cost function in the external adjustment cost model so that the marginal cost of capital goods will not differ when we have a different K^*. Here are two ways: 1) let the marginal cost of new capital goods be a function of $I - \delta K$ (in steady state $I^* - \delta K^*$ will always be 0). 2) let the marginal cost of new capital goods be a function of $\frac{I}{K}$ (in steady state $\frac{I^*}{K^*}$ will always be δ).

1.1 Capital Depreciation

Assume in the external adjustment cost model, cost function takes the form $\bar{C}(I) = C(I - \delta K)$, therefore, replacing depreciated capital costs nothing: $\bar{C}(I^*) = C(I^* - \delta K^*) = C(0) = 0$. Resolving the model shows:

$$P_{It} = \bar{C}'(I) = C'(I_t - \delta K_t) \tag{1}$$

*Please email me if you find errors or typos to yj2279@columbia.edu. Your help will be greatly appreciated. All comments and suggestions are especially welcome.

Capital accumulation function becomes

$$\dot{K}_t = I_t - \delta K_t = h(P_{It}) \tag{2}$$

In this way, you have a flat $\dot{K}_t = 0$ schedule: $P_I = h^{-1}(0)$.

1.2 $\frac{I}{K}$ and I

In the internal adjustment cost model, the cost function is related to capital stock to capture the learning by doing effect, while in the external adjustment cost, we ignore it. Now suppose in the external adjustment cost model, we also take into account the learning by doing effect and assume the margial cost of producing capital goods is a function of $\frac{I}{K}$: $g(\frac{I}{K})$. For example, if $C(I, K) = I(1+\varphi(I/K))$, we have $g(I/K) = \frac{\partial C(I,K)}{\partial I} = 1 + \varphi(I/K) + \frac{I}{K}\varphi'(I/K)$. Resolving the external adjustment cost model shows:

$$P_{It} = g(\frac{I_t}{K_t}) \tag{3}$$

The above establishes a relation

$$\frac{I_t}{K_t} = h(P_{It}) \tag{4}$$

Capital accumulation function becomes

$$\dot{K}_t = I_t - \delta K_t = [h(P_{It}) - \delta]K_t \tag{5}$$

We reach a flat $\dot{K}_t = 0$ as well: $P_I = h^{-1}(\delta)$.

Conclusion: these two investment models are essentially the same if we revise the cost function in the external adjustment cost model to eliminate the differences caused by different modeling strategies in the two models.

Remark. We have discussed the line of $\dot{K}_t = 0$ for the two models, but what about the line of $\dot{q}_t = 0$? In fact, there is externality that is not internalized. In the external adjustment cost model, final goods producers don't take into account that their purchasing of newly built capital goods can decrease the cost of capital goods producers in the future. That's why when you set $\delta = 0$ and $C(I, K) = I(1 + \varphi(I/K))$ in the external adjustment cost model, you will get a different line for $\dot{q}_t = 0$ when comparing to the internal adjustment cost model (please check what I am saying is correct).

2 Neoclassical Production Functions $Y = F(K, L, A)$

2.1 Basic Properties

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}$ is twice continuously differentiable to its three arguments. A production function is called a neoclassical production function if the following properties are satistied.

- **Constant returns to scale:**

$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A) \tag{6}$$

That is $F(K, L, A)$ is homogeneous of degree one in K and L. Note A is non-rivalry, so the replication principle doesn't apply to A.

- **Positive and diminishing returns to K and L:**

$$\frac{\partial F}{\partial K} > 0, \qquad \frac{\partial F}{\partial L} > 0 \tag{7}$$

$$\frac{\partial^2 F}{\partial K^2} < 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0 \tag{8}$$

If we increase the amount of one input, output will increase, but marginal product will decrease as the input increases.

- **Inada Conditions**

$$\lim_{K \to 0} \frac{\partial F}{\partial K} = \lim_{L \to 0} \frac{\partial F}{\partial L} = \infty \tag{9}$$

$$\lim_{K \to +\infty} \frac{\partial F}{\partial K} = \lim_{L \to +\infty} \frac{\partial F}{\partial L} = 0 \tag{10}$$

Inada conditions can help us nail down interior solutions.

- **Essentiality**

$$F(K, 0, A) = F(0, L, A) = 0 \tag{11}$$

Therefore, to produce a positive amount of output, a positive amount of each input is required.

The first three properties imply the last essentiality property, so we don't need to write down the last essentiality property. See the following proof of this argument:
Proof. Recall **L' Hôpital's Rule:**
If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ around I which is a small neighborhood of c, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$
Apply it to the following:

$$\lim_{K \to +\infty} \frac{Y}{K} = \lim_{K \to \infty} \frac{\frac{\partial Y}{\partial K}}{\frac{\partial K}{\partial K}} = \lim_{K \to \infty} \frac{\partial Y}{\partial K} = 0 \tag{12}$$

The last equality comes from Inada conditions.
Note that by CRS (constant returns to scale)

$$\lim_{K \to +\infty} \frac{Y}{K} = \lim_{K \to +\infty} F(1, \frac{L}{K}, A) = F(1, 0, A) \tag{13}$$

which leads to

$$F(1, 0, A) = 0 \tag{14}$$

And $F(L,0,A) = L * F(1,0,A) = 0$ follows immediately. Similarly, one can prove that $F(0,K,A) = 0$ as well. ■

Examples A constant elasticity of substitution (CES) production function

$$Y = F(K,L) = (aK^{\frac{\sigma-1}{\sigma}} + bL^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \tag{15}$$

where $a \geq 0$, $b \geq$ and $\sigma \geq 0$ are constants. $\sigma = -\frac{d\ln(L/K)}{d\ln(F_L/F_K)}$, where F_K and F_L are partial derivatives.

- When $\sigma \to +\infty$, $Y = aK + bL$, perfect substitutability
- When $\sigma \to 0$, $Y = \min\{K, L\}$, perfect complementarity, Leontief. When $Y = F(K,L) = [(aK)^{\frac{\sigma-1}{\sigma}} + (bL)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$, and $\sigma \to 0$, we have $Y = \min\{aK, bL\}$. Note the slight difference.
- When $\sigma = 1$, $Y = K^\alpha L^{1-\alpha}$ with $\alpha = \frac{a}{a+b}$, Cobb-Douglas.

We can use L' Hôpital's Rule to show when $\sigma \to 0$ and $\sigma \to 1$, we are approaching Leontief production function and Cobb-Douglas production function respectively.

2.2 Constant Returns to Scale and Zero Profit

Euler Theorem Suppose $f : \mathbb{R}^M \to \mathbb{R}$ is continuously differentiable and homogeneous of degree α, i.e.

$$\forall x \in \mathbb{R}^M, f(\lambda x) = \lambda^\alpha f(x) \tag{16}$$

then

$$\sum_{i=1}^{M} \frac{\partial f(x)}{\partial x_i} x_i = \alpha f(x) \tag{17}$$

Proof. Differentiate both sides of equation (11) with respective to λ and set $\lambda = 1$. ■

For a constant returns to scale production function $F(K,L,A)$, we know $\alpha = 1$. Apply Euler Theorem,

$$F_K K + F_L L = F \tag{18}$$

A profit maximization firm takes input price R and W and output price P as given and first order conditions are $PF_K = R$ and $PF_L = W$. Therefore, $RK + WL = PF$, or firms' profit $PF - RK - WL = 0$. Note the implicit assumption is that firms are price takers of input and output.

Some may wonder that in investment theory, firms are price takers but they do have profits. The reason is that in investment theory we study, firms are capital owners, and they start with a capital stock $K_0 > 0$.

2.3 Technological Progress

2.3.1 Three Production Function Forms

Three forms of production functions:

- Hicks neutral: $Y = A_t F(K, L)$;
- Harrod neutral: $Y = F(K, A_t L)$;
- Solow neutral: $Y = F(A_t K, L)$.

Cobb-Douglas production function can be written as all of the above three forms.

2.3.2 Kaldor Facts

Kaldor facts about growth:

- Per capita output $\frac{Y}{L}$ grows over time, and its growth rate doesn't tend to diminish.
- Physical capital per worker $\frac{K}{L}$ grows over time
- Return of capital R keeps nearly constant
- Ratio of physical capital to output $\frac{K}{Y}$ keeps nearly constant
- Labor share $\frac{WL}{Y}$ and capital share $\frac{RK}{Y}$ are nearly constant
- The growth rate of output per worker differs substantially across countries

2.3.3 What Form of Technological Progress?

Suppose there is a production function:

$$Y_t = F(K_t, L_t, A_t) \tag{19}$$

, if

- F exhibits constant returns to scale in K and L
- Resource constraint: $\dot{K}_t = Y_t - C_t - \delta K_t$ and saving rate is constant s
- Labor grows at a constant rate $\frac{\dot{L}_t}{L_t} = n$
- Capital stock grows at a constant rate $\frac{\dot{K}_t}{K_t} = \gamma_K$

then the production function must be labor augmenting, i.e. Harrod neutral
Proof.

From the resource constraint

$$\frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} - \delta \tag{20}$$

Since the left hand side is a constant γ_K, we immediately conclude that $\frac{Y}{K}$ is a constant (consistent with one of the Kaldor facts).

$$Y_t = F(B_t K_t, A_t L_t) \tag{21}$$

Note when B_t and A_t grow at the same rate, we will go to Hicks neutral case by CRS, therefore, this form includes all three possible production function forms.

We additionally assume that technology will grow at a constant rate. Assume that $\frac{\dot{B}_t}{B_t} = z$ and $\frac{\dot{A}_t}{A_t} = x$. Without loss of generality, we let $B_0 = 1$ and $A_0 = 1$ so that $B_t = e^{zt}$ and $A_t = e^{xt}$. Then

$$\frac{Y_t}{K_t} = \frac{F(B_tK_t, A_tL_t)}{K_t} = F(B_t, A_t\frac{L_t}{K_t}) = B_tF(1, \frac{A_t}{B_t}\frac{L_t}{K_t}) = e^{zt}F(1, e^{(x-z)t}\frac{L_t}{K_t}) \tag{22}$$

Since labor grows at a constant rate n and capital grows at a constant rate of γ_K (again, for simplicity assume $L_0 = 1$ and $K_0 = 1$), we get

$$\frac{Y_t}{K_t} = e^{zt}F(1, e^{(x-z+n-\gamma_K)t}) \tag{23}$$

Define $\varphi(\cdot) = F(1, \cdot)$ to obtain

$$\frac{Y_t}{K_t} = e^{zt}\varphi(e^{(x-z+n-\gamma_K)t}) \tag{24}$$

We have proved that $\frac{Y_t}{K_t}$ is a constant, and now we discuss two scinarios:

1). If $x - z + n - \gamma_K = 0$ thus $x = \gamma_K - n$, we need to have $z = 0$, which means A_t grows at a constant rate and B_t is a constant . So the production function is labor augmenting.

2). If $x - z + n - \gamma_K \neq 0$, we still need to have $\frac{\partial[e^{zt}\varphi(e^{(x-z+n-\gamma_K)t})]}{\partial t} = 0$ which implies

$$\frac{\varphi'(\chi)\chi}{\varphi(\chi)} = \frac{-z}{n + x - z - \gamma_K} \tag{25}$$

where $\chi = e^{(x-z+n-\gamma_K)t}$. Notice that we need to require $n + x - z - \gamma_K$ is non-zero here.

Solve the above differential equation to reach

$$\varphi(\chi) = constant \cdot \chi^{1-\alpha} \tag{26}$$

where α is a constant. Substitute back to the production function to finally write out

$$F(B_tK_t, A_tL_t) = B_tK_t \cdot F(1, \frac{A_tL_t}{B_tK_t}) = B_tK_t(\frac{A_tL_t}{B_tK_t})^{1-\alpha} = constant \cdot K_t^\alpha(L_te^{\nu t})^{1-\alpha} \tag{27}$$

where $\nu = \frac{z\alpha + x(1-\alpha)}{1-\alpha}$. This Cobb-Douglas production function also belongs to labor augmenting production function.

We then conclude that production function takes the labor augmenting form. ■

Remark. We can use other ways to prove the labor augmenting production function. First, since Y/K is constant and K grows at a constant rate of γ_K, Y will also grow at a constant rate γ_Y and $\gamma_Y = \gamma_K$. At time 0 (pick up an arbitrary time T should work as well), the production function is

$$Y_0 = F(K_0, L_0, A_0) \tag{28}$$

Multiply both sides by $e^{\gamma_K t}$, we obtain

$$e^{\gamma_K t} Y_0 = F(e^{\gamma_K t} K_0, e^{\gamma_K t} L_0, A_0) \tag{29}$$

$$Y_t = F(K_t, e^{(g_K - n)t} L_t, A_0) \tag{30}$$

Denote $\bar{A}_t = e^{(g_K - n)t}$ and re-write

$$Y_t = F(K_t, e^{(\gamma_K - n)t} L_t, A_0) = F(K_t, \bar{A}_t L_t, A_0) = \bar{F}(K_t, \bar{A}_t L_t) \tag{31}$$

where $\bar{A}_t = e^{(\gamma_K - n)t}$. The above $\bar{F}(K_t, \bar{A}_t L_t)$ is already a labor augmenting production function.

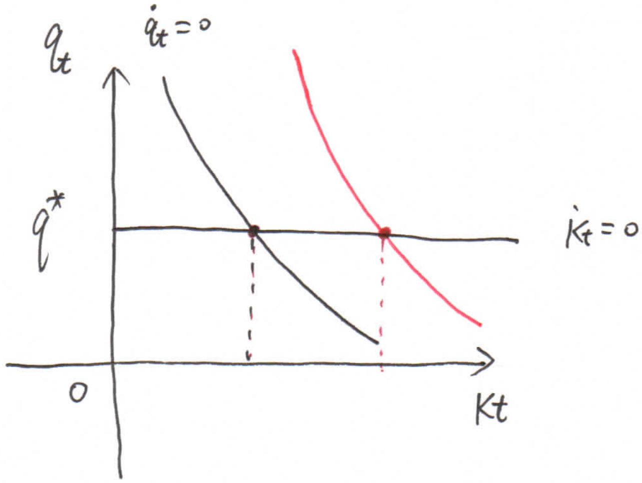

Figure 1

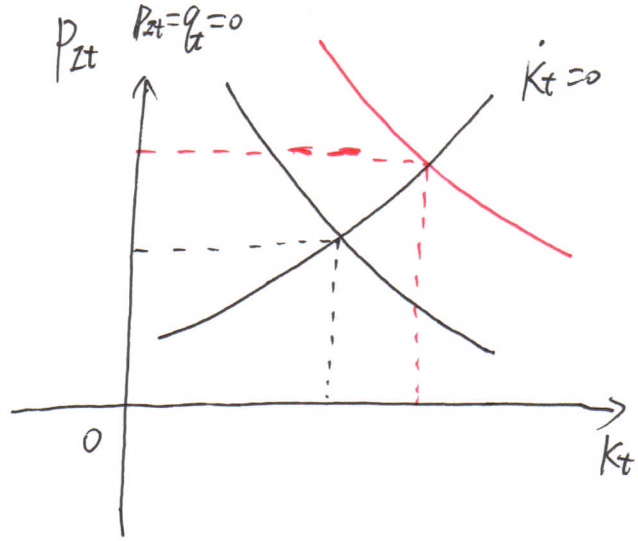

Figure 2.

after a permanent technology shock,

$\dot{q}_t = 0$ will shift

Growth Model with Exogenous Saving Rate

Macroeconomic Analysis Recitation 4&5

Yang Jiao*

1 Extensions of Solow-Swan Model

1.1 Solow-Swan Model with Technolgy Progress

Now we introduce technology progress as well. Assume we have a neoclassical production function $Y = F(K_t, A_t L_t)$, where technology A_t grows at a constant rate γ_A and labor L_t grows at n. All else equal as in the Solow-Swan model we studied in class.

$$\frac{\dot{A}_t}{A_t} = \gamma_A \tag{1}$$

$$\frac{\dot{L}_t}{L_t} = n \tag{2}$$

Define lowercase letter variable $x_t = \frac{X_t}{A_t L_t}$ where $X = Y, K$.

$$y_t = \frac{Y_t}{A_t L_t} = F(k_t, 1) = f(k_t) \tag{3}$$

The last equality is just a definition of $f(\cdot)$.

Capital accumulation equation

$$\dot{K}_t = I_t - \delta K_t = sY_t - \delta K_t \tag{4}$$

Divide both sides by K_t to obtain

$$\frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} + \delta \tag{5}$$

First,

$$\frac{Y_t}{K_t} = \frac{y_t}{k_t} = \frac{f(k_t)}{k_t} \tag{6}$$

Second,

$$k_t = \frac{K_t}{A_t L_t} \to \frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} \tag{7}$$

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$$\rightarrow \frac{\dot{K}_t}{K_t} = \frac{\dot{k}_t}{k_t} + \gamma_A + n \tag{8}$$

Substitute equation (6) and (8) to equation (5) to get

$$\frac{\dot{k}_t}{k_t} = s\frac{f(k_t)}{k_t} - (\delta + n + \gamma_A) \tag{9}$$

Note the interpretation of k_t or y_t is no longer per capita variables. $k_t A_t$ and $y_t A_t$ are per capita terms instead.

1.2 A Model with Poverty Trap

Suppose the economy has access to two possible technologies,

$$Y_A = AK^\alpha L^{1-\alpha} \tag{10}$$

$$Y_B = BK^\alpha L^{1-\alpha} - bL \tag{11}$$

where $B > A$, and $b > 0$.

In per capita terms, the production functions become

$$y_A = Ak^\alpha \tag{12}$$

$$y_B = Bk^\alpha - b \tag{13}$$

The economy will compare and decide which technology to use and it will depend on the level of k. There is a level of capital $\tilde{k} = (\frac{b}{B-A})^{1/\alpha}$ such that when $k > \tilde{k}$, the economy chooses technology B to produce, otherwise when $k \leq \tilde{k}$, the economy chooses technology A .

From the Solow-Swan model, we have

$$\frac{\dot{k}}{k} = sf(k)/k - (n + \delta) \tag{14}$$

Here $f(k) = Ak^\alpha$ when $k \leq \tilde{k}$, and $f(k) = Bk^\alpha - b$ when $k > \tilde{k}$.

It is easy to draw $sf(k)/k$ when $k \leq \tilde{k}$, since it is strictly decreasing. However, when $k > \tilde{k}$, function $h(k) := sf(k)/k = Bk^{\alpha-1} - \frac{b}{k}$ may not be a monotonic function. There is a cutoff $\bar{k} = (\frac{b}{B(1-\alpha)})^{\frac{1}{\alpha}}$ such that when $k < \bar{k}$, $h(k)$ is decreasing in k, and when $k \geq \bar{k}$, $h(k)$ is increasing in k.

If $\bar{k} > \tilde{k}$, $sf(k)/k$ will first decrease then increase and finally decrease. To determine the steady state, we also need to plot a horizontal line $n + \delta$. One particularly interesting case is shown in Figure 1. There are three steady states. And two of them are stable. It means the initial level k_0 matters for the long run steady state. A country with very little k_0 will end up with a steady state of lower output per capita (poverty trap).

If $\bar{k} \leq \tilde{k}$, $sf(k)/k$ will always decrease with k. There is only one stable steady state. See Figure 2.

1.3 A Model with Physical Capital K and Human Capital H

Now we introduce both physical capital and human capital into the Slow-Swan model. Let's assume a Cobb-Douglas production function:

$$Y_t = K_t^\alpha H_t^\eta (A_t L_t)^{1-\alpha-\eta}$$

where A_t is technology which grows at rate γ_A and L_t is labor force which grows at rate n. Parameters satisty $0 < \alpha + \eta < 1$.

Divide both sides by $A_t L_t$:

$$y_t = k_t^\alpha h_t^\eta$$

There are two possible ways to introduce exogenous saving rates. 1) exogenous saving rate s for the sum of physical capital and human capital, and the economy will decide how to allocate between physical capital and human capital 2) exogenous saving rates s_k and s_h for physical capital and human capital respectively.

For the first case, the law of motion of the sum of physical and human capital is

$$\dot{k} + \dot{h} = sk^\alpha h^\eta - (\delta + n + \gamma_A)(k + h)$$

Additionally, households allocate between human capital and physical capital so that the marginal returns are equalized:

$$MPK - \delta = MPH - \delta$$

That is $\alpha h = \eta k$. Substitute back to the law of motion equation to eliminate h, we can get

$$\dot{k} = s(\frac{\eta^\eta \alpha^{1-\eta}}{\alpha + \eta})k^{\alpha+\eta} - (\delta + n + \gamma_A)k$$

For the second case, the law of motions of physical and human capital are

$$\dot{k} = s_k k^\alpha h^\eta - (\delta + n + \gamma_A)k$$

$$\dot{h} = s_h k^\alpha h^\eta - (\delta + n + \gamma_A)h$$

Notice that in the second case, we cannot impose the condition that $MPK - \delta = MPH - \delta$. This is because, once we have exogenous saving rates for both physical and human capital, then given k_0, h_0, the path for k_t and h_t are pinned down by the two law of motions. There is no freedom to adjust k_t or h_t to equalize marginal returns of physical and human capital.

2 Golden Rule

In the Solow-Swan model, consumption is

$$c_t = (1-s)f(k_t) \tag{15}$$

In the steady state k^* is a function of saving rate s and higher s leads to higher k^* thus higher output per capita $f(k^*)$. However, higher saving rate also means smaller fraction of output per capita will go to consumption. So there is a trade-off here if one wants to achieve higher consumption. The question is what value of saving rate s can deliver the highest steady state consumption c^*.

We want to maximize the following by choosing s

$$c^* = (1-s)f[k^*(s)] \tag{16}$$

In steady state, $sf(k^*) = (n+\delta)k^*$, therefore, our objective is to maximize

$$c^* = (1-s)f[k^*(s)] = f[k^*(s)] - (n+\delta)k^*(s) \tag{17}$$

First order condition:

$$\{f'[k^*(s)] - (n+\delta)\}\frac{dk^*(s)}{ds} = 0 \tag{18}$$

As $\frac{dk^*(s)}{ds} > 0$, we have

$$f'[k^*(s)] = (n+\delta) \tag{19}$$

Denote the corresponding k^* as k_{gold}, saving rate s as s_{gold}.

$$f'(k_{gold}) = n+\delta \tag{20}$$

Once we have k_{gold}, we can use the relationship $k^*(s)$ to infer s_{gold}.

Remark. With Cobb-Douglas production function (capital share α), the above condition becomes

$$\alpha k_{gold}^{\alpha-1} = n+\delta \tag{21}$$

and we also know that in the Solow-Swan model, in steady state

$$sf(k_{gold}) = (n+\delta)k_{gold} \tag{22}$$

i.e.

$$sk_{gold}^{\alpha} = (n+\delta)k_{gold} \tag{23}$$

Comparing equation (21) and (22) we conclude that $s_{gold} = \alpha$ for this special case.

3 Absolute and Conditional Convergence

3.1 Absolute Convergence

In the Solow-Swan model,

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta+n) \tag{24}$$

So the growth rate $\frac{\dot{k}}{k}$ depends on the level of k.

$$\frac{d(\frac{\dot{k}}{k})}{dk} = s\frac{f'(k) - kf(k)}{k^2} < 0 \tag{25}$$

This implies higher the level of capital per capita, the slower the growth rate of capital per capita: poor countries should grow faster than rich countries. Note the underlying assumption is that poor countries and rich countries share the similar characteristics (parameters). The hypothesis that poor countries tend to grow faster than rich ones without conditioning on any other characteristics of economies is called absolute convergence (i.e. a negative relationship between the level of output per capita and the growth rate of output per capita).

In the data, when we have a broad set of countries (countries from both poor countries and rich countries), the hypothesis in fact fails. While if we look at more homogeneous groups, e.g. only OECD countries or states within U.S., evidence accepts the hypothesis instead.

3.2 Conditional Convergence

Next we drop the assumption that countries share similar characteristics. We will proceed with a simple example. Suppose two countries only differ in their saving rates. So they will also differ in their steady state k^*. The country with higher saving rate will have higher k^*. Again the growth rate of capital per capita is

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n) \tag{26}$$

and in steady state

$$s\frac{f(k^*)}{k^*} = (\delta + n) \tag{27}$$

Substitute equation (27) to equation (26) to eliminate s.

$$\frac{\dot{k}}{k} = (\delta + n)[\frac{f(k)/k}{f(k^*)/k^*} - 1] \tag{28}$$

Assume the production function is in Cobb-Douglas form, then

$$\frac{\dot{k}}{k} = (\delta + n)[(\frac{k}{k^*})^{\alpha-1} - 1] \tag{29}$$

Then it is clear that the growth rate depends on the distance of k to steady state k^* ($\frac{k}{k^*}$ matters). It becomes possible that a richer country can grow faster than a poorer country as they have different steady states.

Therefore, in order to account for the difference in these two countries' growth rate, one has to look at both the current level of k_t and the steady state level k^* which depends on country characteristics, i.e. country characteristics matter. After controling variables that proxy for differences in steady state positions, we can get a significantly negative relationship between per capita growth rate and the log of initial real per capita GDP. In short, data supports the conditional convergence hypothesis.

4 The Speed of Convergence

We have discussed roughly about the speed of convergence. Now we formally define this concept speed of convergence as

$$\beta = -\frac{\partial(\dot{k}/k)}{\partial logk} \tag{30}$$

We assume that countries are near to their steady state (in order to use approximations). We will also adopt Cobb-Douglas production function.

$$\frac{\dot{k}}{k} = sAk^{\alpha-1} - (\delta + n + \gamma_A) = sAe^{(\alpha-1)log(k)} - (\delta + n + \gamma_A) \tag{31}$$

Around the steady state

$$\beta^* = -\frac{\partial(\dot{k}/k)}{\partial logk}|_{k^*} = -\frac{\partial(sAe^{(\alpha-1)logk} - (\delta + n + \gamma_A))}{\partial logk}|_{k^*} = (1-\alpha)sAk^{*\alpha-1} \tag{32}$$

Recall that with technology progress in steady state $sAk^{*\alpha-1} = \delta + n + \gamma_A$.

$$\beta^* = (1-\alpha)(\delta + n + \gamma) \tag{33}$$

The above procedures are equivalent (the same thing) to first log-linearize the right hand side of equation (31) and take the coefficient directly

$$\frac{\dot{k}}{k} = sAk^{\alpha-1} - (\delta+n) \approx -\frac{\partial(sAe^{(\alpha-1)logk} - (\delta + n + \gamma_A))}{\partial logk}|_{k^*}(logk - logk^*) = -\beta^*(logk - logk^*) \tag{34}$$

Remark. A more general form of log-linearization is the following. Assume we want to log-linearize a multivariate function $y = f(x_1, ..., x_n)$ around x^*.

$$y = f(x_1, ..., x_n) = f(e^{log(x_1)}, ..., e^{log(x_n)}) \tag{35}$$

Define $z_i = log(x_i)$, then the first order approximation is

$$y = f(x_1, ..., x_n) = f(e^{z_1}, ..., e^{z_n}) \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}|_{x^*} \cdot e^{z_i^*} \cdot (z_i - z_i^*) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}|_{x^*} \cdot x_i^* \cdot [log(x_i) - log(x_i^*)] \tag{36}$$

Notice for the left hand side, we didn't do any thing. In the future (the second half of this semester), we will do log-linearization on both sides. We are interested in the percentage deviation of variables from its steady state. You can ignore the following now.

$$y = f(x_1, ..., x_n) \tag{37}$$

Then

$$log(y) - log(y^*) = \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* \cdot [log(x_i) - log(x_i^*)] \tag{38}$$

We will come back to this later in the second half of the semester.

5 Some Other Empirical Issues

5.1 Capital Stock Measurement

Perpetual Inventory Method (PIM) is so far perhaps the most popular way to compute gross capital stock. In discrete time, capital accumulation is

$$K_{t+1} = (1-\delta)K_t + I_t \tag{39}$$

$$\to K_t = \sum_{i=0}^{+\infty} (1-\delta)^i I_{t-(i+1)} \tag{40}$$

But unfortunately in the data, we don't have an infinite series of investment.

$$K_t = (1-\delta)^{t-1} K_0 + \sum_{i=0}^{t-1} (1-\delta)^i I_{t-(i+1)} \tag{41}$$

How to get the initial capital stock K_0?
In neoclassical growth model, under balanced growth path (constant growth rate)

$$g_{GDP} = g_K = \frac{K_t - K_{t-1}}{K_{t-1}} = \frac{I_t}{K_{t-1}} - \delta \tag{42}$$

Therefore,

$$K_0 = \frac{I_1}{g_{GDP} + \delta} \tag{43}$$

But the economy may not be on a balanced growth path, we can use g_I instead of g_{GDP}, it is still an approximation anyway.

$$K_0 = \frac{I_1}{g_I + \delta} \tag{44}$$

To obtain g_I, we can choose a three-year average or more years' average.

5.2 Growth Accounting

Assume the production function is

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{45}$$

Define

$$SR_t = \frac{Y_t}{K_t^{\alpha} L_t^{1-\alpha}} \tag{46}$$

Then we get the data of SR.

$$\frac{\dot{SR}_t}{SR_t} = \frac{\dot{Y}_t}{Y_t} - \alpha\frac{\dot{K}_t}{K_t} - (1-\alpha)\frac{\dot{L}_t}{L_t} \tag{47}$$

$sr_t = \frac{\dot{SR}_t}{SR_t}$ is the Solow residual: the growth of output that cannot be accounted by the growth of capital and labor.

$$\frac{\dot{Y}_t}{Y_t} = sr_t + \alpha\frac{\dot{K}_t}{K_t} + (1-\alpha)\frac{\dot{L}_t}{L_t} \tag{48}$$

or write it in discrete time

$$\frac{Y_t - Y_{t-1}}{Y_{t-1}} = \frac{SR_t - SR_{t-1}}{SR_{t-1}} + \alpha\frac{K_t - K_{t-1}}{K_{t-1}} + (1-\alpha)\frac{L_t - L_{t-1}}{L_{t-1}} \tag{49}$$

This equation can be used to find out the sources of a particular economy's economic growth. Alwyn Young (1995, QJE) uses this method (not exactly the same, e.g. they use a more complicated production function and take human capital into account as well etc., but the main idea is similar) to show that the East Asian growth miracles are largely due to capital accumulation $\frac{\dot{K}_t}{K_t}$ and increasing labor force participation $\frac{\dot{L}_t}{L_t}$.

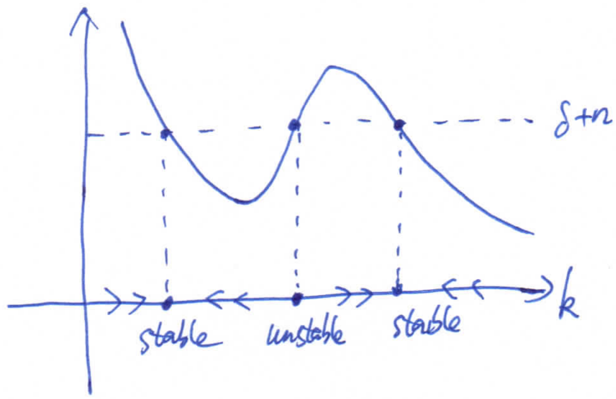

Figure 1.

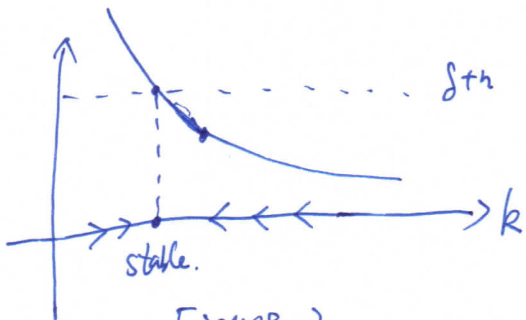

Figure 2

Endogenous Growth Models

Macroeconomic Analysis Recitation 6

Yang Jiao*

1 A Congestion Model

In class, we learned a growth model with government spending. In that model, production function is $Y_i = AL_i^{1-\alpha}K_i^{\alpha}G^{1-\alpha}$, where G is government spending. In this setup, government spending is a public good. However, in reality, many government spending is not, such as highways. Now we introduce a congestion model with a continuum of firms, where the production function for firm $i \in [0,1]$ is

$$Y_i = AK_i^{\alpha}L_i^{1-\alpha}f(G/K) \tag{1}$$

where G is government spending and K is aggregate capital: $K = \int_0^1 K_i di$. We assume that f satisfies $f' > 0$ and $f'' < 0$.

1.1 Decentralization

Firms' problem is static as before

$$\max_{K_{it},L_{it}} AK_{it}^{\alpha}L_{it}^{1-\alpha}f(G/K) - w_tL_{it} - R_tK_{it} \tag{2}$$

F.O.C.s are

$$\alpha AK_{it}^{\alpha-1}L_{it}^{1-\alpha}f(G/K) = R_t \tag{3}$$

$$(1-\alpha)AK_{it}^{\alpha}L_{it}^{-\alpha}f(G/K) = w_t \tag{4}$$

In equilibrium $L = \int_0^1 L_i di$ as well, and all firms are symmetric, so $L_{it} = L$, $K_{it} = K$ for $i \in [0,1]$.

In per capita terms:

$$\alpha Ak_t^{\alpha-1}f(g_t/k_t) = R_t \tag{5}$$

$$(1-\alpha)Ak_t^{\alpha}f(g_t/k_t) = w_t \tag{6}$$

Households' problem is quite standard as in Ramsey model,

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$$\max_{a_t, c_t} \int_0^{+\infty} e^{-(\rho-n)t} \frac{c_t^{1-\theta} - 1}{1-\theta} dt \tag{7}$$

s.t.

$$\dot{a}_t = r_t a_t + w_t - c_t - n a_t - \tau_t \tag{8}$$

$$\lim_{t \to +\infty} e^{-rt} a_t \geq 0 \tag{9}$$

with $a_0 > 0$ given. τ_t is a lump sum tax in per capita term $\tau_t = \frac{T_t}{L_t}$.

Set up the Hamiltonian

$$\mathscr{H} = e^{-(\rho-n)t} \frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t (r_t a_t + w_t - c_t - n a_t - \tau_t) \tag{10}$$

F.O.C.s are

$$e^{-(\rho-n)t} c_t^{-\theta} - \lambda_t = 0 \tag{11}$$

$$-\dot{\lambda}_t = \lambda_t (r_t - n) \tag{12}$$

$$\lim_{t \to +\infty} \lambda_t a_t = 0 \tag{13}$$

Therefore,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) \tag{14}$$

The rate of return from investing in risk free bond is r_t and the rate of return from investing in physical capital is $R_t - \delta$. By **no arbitrage condition**, they are equal

$$r_t = R_t - \delta \tag{15}$$

Market clearing condition gives

$$a_t = k_t + b_t \tag{16}$$

and $b_t = 0$.

Balanced government budget:

$$G_t = T_t \tag{17}$$

or in per capita term

$$g_t = \tau_t \tag{18}$$

We conclude that equation (14) and (15) gives the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\alpha A k_t^{\alpha-1} f(g_t/k_t) - \delta - \rho) \tag{19}$$

The law of motion of capital comes from equation (3)(4)(8)(15)

$$\dot{k}_t = A k_t^{\alpha-1} f(g_t/k_t) - c_t - (n+\delta) k_t - g_t \tag{20}$$

Notice we haven't chosen the sequence of g_t yet to maximize household utility. That's going to be a very complicated problem. Perhaps what we can do is to let the government set a constant g and compare steady states to choose an optimal fiscal policy g.

We are interested in whether the above decentralization problem is socially optimal.

1.2 Social Planner

Social planner maximizes household utility subject to resource constraint.

$$\max_{a_t, c_t} \int_0^{+\infty} e^{-(\rho-n)t} \frac{c_t^{1-\theta} - 1}{1-\theta} dt \tag{21}$$

s.t.

$$\dot{k}_t = A k_t^{\alpha-1} f(g_t/k_t) - c_t - (n+\delta)k_t - g_t \tag{22}$$

From F.O.C.s, we derive Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\alpha A k_t^{\alpha-1}[f(g_t/k_t) - \frac{g_t}{k_t} f'(\frac{g_t}{k_t})] - \delta - \rho) \tag{23}$$

What we find is that the decentralized case doesn't have the same formula of the Euler equation as that of the social planner's problem. The externality comes from the fact that each individual firm doesn't consider their capital choice's congestion effect on other firms, so they over employ capital: a traditional Tragedy of Commons problem.

To have a rough idea why we think decentralized case will over accumulate capital, we can assume a common steady state policy g for these two cases. Then we compare the steady state k, which comes from the Euler equation. For decentralized case, in steady state

$$\alpha A k^{\alpha-1} f(g/k) = \delta + \rho$$

For social planner, in steady state

$$\alpha A k^{\alpha-1} f(g/k) = \frac{g}{k} f'(g/k) + \delta + \rho > \delta + \rho$$

Comparing the above two equations, it is easy to verify that decentralized case have higher steady state k.

Remark. We deliberately choose a lump sum tax in order not to impose additional externality so that we can isolate the aformentioned externality which firms don't internize.

2 A Rising Cost of R&D model

In class, we learned a expanding variety model of growth with constant $R\&D$ cost. Now let's turn to a setting with increasing cost of $R\&D$: $\eta'(N) > 0$. Specifically, let's assume that

$$\eta(N) = \phi N^\sigma \tag{24}$$

Free entry condition says

$$V_t = \eta(N_t) = \phi N_t^\sigma \tag{25}$$

Remark. Free entry condition above holds only when there is positive entry. In general, it should be a complementary slackness condition $\dot{N}_t[\eta(N_t) - V_t] = 0$, $\dot{N}_t \geq 0$ and $\eta(N_t) \geq V_t$. So if $\dot{N}_t$ is strictly greater than 0, we will have $\eta(N_t) = V_t$.

Interest rate

$$r_t = \frac{\pi + \dot{V}_t}{V_t} = \frac{\pi}{\phi N_t^\sigma} + \sigma \frac{\dot{N}_t}{N_t} \tag{26}$$

Households' problem is

$$\int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\theta} - 1}{1 - \theta} dt \tag{27}$$

s.t.

$$\dot{a}_t = w_t L + r_t a_t - C_t \tag{28}$$

The Euler equation from the above households' problem is

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}(\frac{\pi}{\phi N_t^\sigma} + \sigma \frac{\dot{N}_t}{N_t} - \rho) \tag{29}$$

When $\sigma = 0$, we go back to the case where innovation cost is constant, and this equation alone can tell us the growth rate of C_t. However, now we have to deal with the dynamics of N_t as well.

Note in this model, the asset held by households is the market value of all firms $a = \eta \cdot N$. Cobb-Douglas final good production function means $wL = (1 - \alpha)Y$. Plug these two equations and also equation (26) into equation (28) to generate

$$\eta(N_t)\dot{N}_t = N_t \pi + (1 - \alpha)Y_t - C_t \tag{30}$$

Since $Y = AL^{1-\alpha} \sum_{j=1}^N X_j^\alpha$, $X_j = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L$ and $\pi = LA^{\frac{1}{1-\alpha}} \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} = LA^{\frac{1}{1-\alpha}}(\alpha - \alpha^2)\alpha^{\frac{2\alpha}{1-\alpha}}$(One can check that by some manipulation, we can express the above as $\eta \dot{N}_t = Y_t - C_t - X_t$ which is in fact the resource constraint), we obtain

$$\eta(N_t)\dot{N}_t = N_t \pi + (1 - \alpha)Y_t - C_t = (1 - \alpha^2)A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L N_t - C_t \tag{31}$$

That is

$$\frac{\dot{N}_t}{N_t} = \frac{\psi}{\phi} N_t^{-\sigma} - \frac{C_t}{\phi} N_t^{-(1+\sigma)} \tag{32}$$

with $\psi = (1 - \alpha^2)A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L$

Substitute equation (32) to equation (29) to yield

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}(\frac{\pi}{\phi} N_t^{-\sigma} + \sigma \frac{\psi}{\phi} N_t^{-\sigma} - \sigma \frac{C_t}{\phi} N_t^{-(1+\sigma)} - \rho) \tag{33}$$

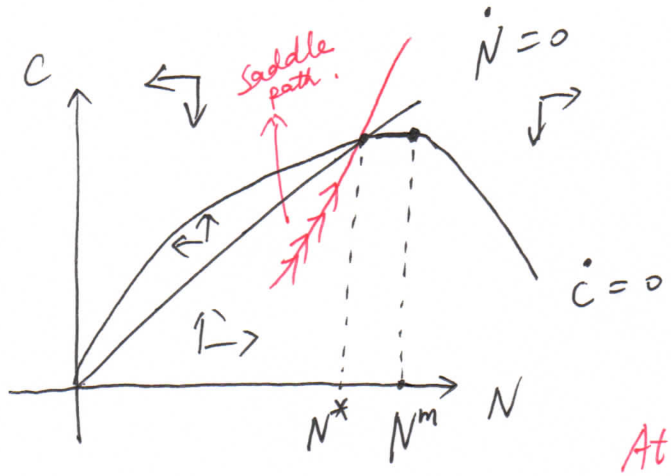
Equation (32) and (33) define a differential equation system.

To sum up, the main procedure is first we start from Euler equation and households' budget constraint, and then we try to substitute variables except C_t and N_t.

We then display the phase diagram. See Figure 1. $N^m = (\frac{\pi + \sigma\psi}{\rho\phi(1+\sigma)})^{1/\sigma}$ and $N^* = (\frac{\pi}{\rho\phi})^{1/\sigma}$.

Notice $N^m > N^*$ because of $\psi > \pi$.

In this rising cost of $R\&D$ setting, there is no long run growth. While with constant $R\&D$ cost, we do have long run growth.



At N^m, $\dot{c}=0$ schedule achieves the maximum.

Figure 1

Consumption & Hours Worked

Macroeconomic Analysis Recitation 7

Yang Jiao*

1 Natural Debt Limit and No Ponzi Scheme

In class, we learned that we can derive the natural debt limit from intertemporal budget constraint and no ponzi scheme. In fact, we can prove the equivalence between the natural debt limit and the no ponzi scheme, given the budget constraint. We first review how to go from no ponzi scheme to natural debt limit.

Assume there is no uncertainty and interest rate is fixed at r.

"$\Longleftarrow$" For arbitrary $j \leq T$, the intertemporal budget constraint is

$$\sum_{t=j}^{T}(1+r)^{j-t}c_t + \frac{a_{T+1}}{(1+r)^{T-j}} = (1+r)a_j + \sum_{t=j}^{T}(1+r)^{j-t}w_t$$

Since $c_t \geq 0$, then the above says

$$(1+r)a_j + \sum_{t=j}^{T}(1+r)^{j-t}w_t \geq \frac{a_{T+1}}{(1+r)^{T-j}}$$

Let T goes to infinity and we know **no ponzi scheme** requires

$$\lim_{T\to+\infty}[\frac{a_{T+1}}{(1+r)^{T+1}}] \geq 0$$

Therefore,

$$a_j \geq \sum_{t=j}^{+\infty}(1+r)^{j-t-1}w_t$$

When w_t is a constant,

$$a_j \geq -\frac{w}{r}$$

which is the **natural debt limit**.

"$\Longrightarrow$" The natural debt limit is

$$a_t \geq -\frac{w}{r}$$

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Then

$$\frac{a_t}{(1+r)^t} \geq -\frac{w}{r(1+r)^t}$$

Take limit of $t \to +\infty$, we have the right hand side goes to 0, thus

$$\lim_{t\to+\infty} \frac{a_t}{(1+r)^t} \geq 0$$

2 Marginal Propensity to Consume

2.1 Continuous Time

$$\max_{c_t} \int_0^{+\infty} e^{-(\rho-n)t} \frac{c_t^{1-\gamma}-1}{1-\gamma} dt$$

s.t.

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t$$

$$\lim_{T\to+\infty} a_T e^{-\int_0^T (r_s-n)ds} \geq 0$$

The Euler equation from the above equation is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}(r_t - \rho)$$

and the budget constraint is

$$\dot{a}_t - (r_t - n)a_t = w_t - c_t$$

Multiply both sides by $e^{-\int_0^t (r_s-n)ds}$ of the budget constraint and integrate from 0 to $+\infty$ to yield

$$\int_0^{+\infty} [\dot{a}_t - (r_t-n)a_t] e^{-\int_0^t (r_s-n)ds} dt = \int_0^{+\infty} w_t e^{-\int_0^t (r_s-n)ds} dt - \int_0^{+\infty} c_t e^{-\int_0^t (r_s-n)ds} dt$$

Euler equation implies

$$c_t = c_0 e^{\int_0^t \frac{1}{\gamma}(r_s-\rho)ds}$$

So we obtain

$$\int_0^{+\infty} [\dot{a}_t - (r_t-n)a_t] e^{-\int_0^t (r_s-n)ds} dt = \int_0^{+\infty} w_t e^{-\int_0^t (r_s-n)ds} dt - \int_0^{+\infty} c_0 e^{\int_0^t \frac{1}{\gamma}(r_s-\rho)ds} e^{-\int_0^t (r_s-n)ds} dt$$

$$0 - a_0 = \int_0^{+\infty} w_t e^{-\int_0^t (r_s-n)ds} dt - c_0 \int_0^{+\infty} e^{\int_0^t \frac{1}{\gamma}(r_s-\rho)ds - \int_0^t (r_s-n)ds} dt$$

$$c_0 \int_0^{+\infty} e^{\int_0^t \frac{1}{\gamma}(r_s-\rho)ds - \int_0^t (r_s-n)ds} dt = (a_0 + \int_0^{+\infty} w_t e^{-\int_0^t (r_s-n)ds} dt)$$

Finally, write

$$c_0 = MPC_0(a_0 + PV_0)$$

where $MPC_0^{-1} = \int_0^{+\infty} e^{\int_0^t \frac{1}{\gamma}(r_s-\rho)ds - \int_0^t (r_s-n)ds} dt$ and $PV_0 = \int_0^{+\infty} w_t e^{-\int_0^t (r_s-n)ds} dt$.

Intuitively, consumption is a fraction of the present value of his wealth.

2.2 Discrete Time

For simplicity, assume $r_t = r$.

The Bellman equation is

$$V(a) = \max_{a' \geq -\frac{w}{r}} \{u((1+r)a + w - a') + \beta V(a')\}$$

F.O.C. for a' is

$$-u'(c) + \beta V'(a') = 0$$

Envelope theorem:

$$V'(a) = (1+r)u'(c)$$

Combine the above two to get Euler equation

$$u'(c) = \beta(1+r)u'(c')$$

For CRRA utility (the same utility as in the continuous time case above), Euler equation becomes

$$c' = [\beta(1+r)]^{\frac{1}{\gamma}} c$$

which leads to

$$c_t = [\beta(1+r)]^{\frac{t}{\gamma}} c_0$$

Recall that the lifetime budget constraint is

$$\sum_{t=0}^{+\infty} (1+r)^{-t} c_t = (1+r)a_0 + \sum_{t=0}^{+\infty} (1+r)^{-t} w_t$$

Substitute c_t,

$$\sum_{t=0}^{+\infty} (1+r)^{-t} [\beta(1+r)]^{\frac{t}{\gamma}} c_0 = (1+r)a_0 + \sum_{t=0}^{+\infty} (1+r)^{-t} w_t$$

Therefore,

$$c_0 = MPC_0[(1+r)a_0 + PV_0]$$

where $MPC_0^{-1} = \sum_{t=0}^{+\infty} (1+r)^{-t} [\beta(1+r)]^{\frac{t}{\gamma}}$ and $PV_0 = \sum_{t=0}^{+\infty} (1+r)^{-t} w_t$.

When $\gamma = 1$, i.e. log-utility, we have $MPC_0 = 1 - \beta$.

3 Log-Linearization

The idea of log-linearization is to look at the percentage change (log-change) of a variable in response to other variables in terms of percentage change as well around a small neighborhood of some state (usually steady state).

$$y = f(x_1, ..., x_n) \tag{1}$$

Then first order expansion is

$$log(y) - log(y^*) = \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* \cdot [log(x_i) - log(x_i^*)] \tag{2}$$

Denote log-change as $\hat{x} = log(x) - log(x^*)$, the above becomes

$$\hat{y} = \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* \cdot \hat{x}_i$$

Useful formulas to remember (derived based on the equation above)

$$\widehat{ax} = \widehat{x}, \text{where } a \text{ is a constant.}$$

$$\widehat{x+y} = \frac{x^*}{x^*+y^*}\hat{x} + \frac{y^*}{x^*+y^*}\hat{y}$$

$$\widehat{xy} = \hat{x} + \hat{y}$$

$$\widehat{a} = 0, \text{where } a \text{ is a constant.}$$

$$\widehat{ax} = \widehat{x}, \text{where } a \text{ is a constant.}$$

$$\widehat{x^\alpha} = \alpha\hat{x}, \text{where } \alpha \text{ is a constant.}$$

Apply to the hours worked model covered in your lecture note 2.

$$h_t = [(\frac{1-\alpha}{\gamma})\frac{1}{c_t/y_t}(\frac{1-\tau_t}{(1+x_t)(1+\mu_t)})]^{\frac{\epsilon}{1+\epsilon}}$$

Assume x_t and μ_t are fixed $\rightarrow$

$$\hat{h_t} = \frac{\epsilon}{1+\epsilon}[-\widehat{(\frac{c_t}{y_t})} + \widehat{(1-\tau_t)}]$$

For example, in page 18 of lecture note 2, substitute France data. $-\widehat{(\frac{c_t}{y_t})} = 0.11$ and $\widehat{(1-\tau_t)} = -0.22$. When $\epsilon = 0$, $\hat{h_t} = 0$. When $\epsilon = 1$, $\hat{h_t} = -0.165$. When $\epsilon = 4$, $\hat{h_t} = -0.264$. When $\epsilon \rightarrow +\infty$, $\hat{h_t} = -0.33$.

4 Balanced Growth Preferences on Labor

Slide 22 of lecture note 3:

$$\frac{u_h(t)}{u_c(t)} = w_t$$

Here is the proof. Since w_t and c_t both grow at the same rate, so $w_t = ac_t$, where a is a constant.

$$u_h = ac_t u_c \tag{3}$$

Take derivative w.r.t. c_t on both sides:

$$u_{ch} = au_c + c_t a u_{cc} \tag{4}$$

Divide (3) by (4) and rearrange to arrive at

$$\frac{u_{ch} c_t}{u_h} = 1 + \frac{u_{cc} c_t}{u_c} = 1 - \gamma$$

Consumption & Hours Worked (continued)-Dynamic Programming

Macroeconomic Analysis Recitation 8

Yang Jiao*

1 Consumption and Hours Worked with GHH Preference

With Balanced Growth Path Preference (abbreviated as BGP preference) we studied in class,

$$U_t = u(c_t, h_t) = \log(c_t) + v(-h_t) = log(c_t) - (1 + x_t)(\frac{\gamma\epsilon}{1+\epsilon})h_t^{1+\frac{1}{\epsilon}},$$

we have derived the labor supply is (slide 20 of lecture note 2)

$$\gamma^\epsilon h_t = [\frac{(1-\tau_{ht})w_t}{1+x_t}]^\epsilon \lambda_t^\epsilon$$

Frisch elasticity is ϵ since this concept is defined as $\frac{\partial log(h_t)}{\partial log(w_t)}|_\lambda$ (i.e. keeping λ_t fixed).

Hicks elasticity is going to be more complicated. It is defined as $\frac{\partial log(h_t)}{\partial log(w_t)}|_U$. We first need to write hours worked h_t as a function of w_t and U_t instead of w_t and λ_t.

Since $\frac{1}{c_t} = \lambda_t(1 + \tau_{ct})$ (slide 11 of lecture note 2), and $U_t = log(c_t) - (1 + x_t)(\frac{\gamma\epsilon}{1+\epsilon})h_t^{1+\frac{1}{\epsilon}}$, the labor supply function can be re-written as

$$\frac{1-\tau_{ht}}{1+\tau_{ct}}\frac{w_t}{\gamma(1+x_t)} = e^{U_t}e^{(1+x_t)\frac{\gamma\epsilon}{1+\epsilon}h_t^{1+\frac{1}{\epsilon}}}h_t^{\frac{1}{\epsilon}}$$

The above implicit function gives h_t as a function of w_t and U_t. Now assume $x_t = 0$, $\tau_t = \tau$ are constant and we are near steady state h^*:

$$log(w_t) = U_t + \frac{\gamma\epsilon}{1+\epsilon}exp[\frac{1+\epsilon}{\epsilon}logh_t] + \frac{1}{\epsilon}log(h_t) + constant$$

Applying implicit function theorem, we obtain

$$\frac{\partial log(h_t)}{\partial log(w_t)}|_U = -\frac{1}{-\gamma(h^*)^{1+\frac{1}{\epsilon}} - \frac{1}{\epsilon}} = \frac{\epsilon}{\gamma\epsilon(h^*)^{1+\frac{1}{\epsilon}} + 1}$$

*Please email me if you find errors or typos to yj2279@columbia.edu. Your help will be greatly appreciated. All comments and suggestions are especially welcome.

- **Remark.** If you use Lagrange multiplier method to solve the model instead of Bellman equation, you will find that λ_t above coincides with the Lagrange multiplier of the budget constraint because there we will also have $\frac{1}{c_t} = \lambda_t(1+\tau_{ct})$.

Now we introduce another utility function form called Greenwood-Hercowitz-Huffman preferences:

$$u(c_t, h_t) = U(c_t - v(h_t))$$

Specifically, let's assume

$$u(c_t, h_t) = log(c_t - (1+x_t)(\frac{\gamma\epsilon}{1+\epsilon})h_t^{1+\frac{1}{\epsilon}})$$

Now the dynamic problem of an agent is

$$\max_{c_t,h_t} \sum_{t=0}^{+\infty}[log(c_t - (1+x_t)(\frac{\gamma\epsilon}{1+\epsilon})h_t^{1+\frac{1}{\epsilon}})]$$

s.t. budget constraint:

$$(1+\tau_{ct})c_t + a_{t+1} = (1-\tau_{at})(1+r_t)a_t + (1-\tau_{ht})w_t h_t$$

with no ponzi scheme and a_0 given. τ_c, τ_a and τ_h are tax rates.

Write the Bellman equation

$$V(a) = \max_{c,h}\{log(c - (1+x)(\frac{\gamma\epsilon}{1+\epsilon})h^{1+\frac{1}{\epsilon}}) + \beta V((1-\tau_a)(1+r)a + (1-\tau_h)wh - (1+\tau_c)c)\}$$

where we use $'$ to represent next period's variable (but V' means derivative)

Two F.O.C.s for c and h are

$$\frac{1}{c - (1+x)(\frac{\gamma\epsilon}{1+\epsilon})h^{1+\frac{1}{\epsilon}}} = (1+\tau_c)\beta V'(a') \tag{1}$$

$$\frac{(1+x)\gamma h^{1/\epsilon}}{c - (1+x)(\frac{\gamma\epsilon}{1+\epsilon})h^{1+\frac{1}{\epsilon}}} = (1-\tau_h)w\beta V'(a') \tag{2}$$

and Envelope theorem implies

$$V'(a) = \beta(1-\tau_a)(1+r)V'(a')$$

Define

$$\lambda_t = \frac{V'(a)}{(1-\tau_{at})(1+r_t)}$$

as well.

Divide (2) by (1), we have

$$(1+x_t)\gamma h_t^{1/\epsilon} = \frac{1-\tau_{ht}}{1+\tau_{ct}}w_t$$

In fact, one can directly get this equation by using the fact marginal rate of substitution between consumption and hours workerd should be equal to the relative price (you can also try it for the previous balanced growth path preference).

We can see that hours worked is a function of w_t and not related to λ_t. Therefore, Frisch elasticity is $\frac{\partial log(h_t)}{\partial log(w_t)}|_\lambda = \epsilon$. Hicks elasticity is $\frac{\partial log(h_t)}{\partial log(w_t)}|_U = \epsilon$ as well.

Remark 1. Notice the difference of labor supply between BGP and GHH preference. Under BGP, labor supply h_t is not only related to w_t but also a function of λ_t. While under GHH, labor supply h_t is not related to λ_t.

The relative price of leisure in terms of consumption good $\frac{1-\tau_{ht}}{1+\tau_{ct}} w_t$ captures the **substitution effect** (if w_t is smaller, enjoy more leisure and thus work less) while λ_t captures the **income effect** (if λ_t is high, i.e. consumption level is low, like in a recession, **income** is smaller, you will want to enjoy less leisure "good" as well, which means you want to work more).

To sum up, in a recession, with BGP, substitution effect means work less but income effect means work more. While with GHH, there is no income effect, so unambiguously, workers will work less. Typically, during a recession, total hours drop significantly, so income effect is very small. That makes it popular to use GHH preference to study business cycles (short run fluctuations).

Remark 2. (minor remark) Jaimovich and Rebelo (2009, AER) have a more general form of utility function.

$$u(c_t, h_t) = \frac{(c_t - \psi h_t^\theta X_t)^{1-\sigma} - 1}{1-\sigma}$$

with $X_t = c_t^\gamma X_{t-1}^{1-\gamma}$. When $\gamma = 0$, we have GHH. When $\gamma = 1$, we have BGP.

2 Dixit-Stiglitz Preference

The cost minimization problem for Dixit-Stiglitz preference is

$$\min_{C_{it}} \int_0^1 P_{it} C_{it} di$$

s.t.

$$C_t = (\int_0^1 C_{it}^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}}$$

This is in fact a static problem, so one can omit all the time subscript here. Denote ψ as the Lagrange multiplier of the constraint.

First order condition is

$$P_i = \psi(\int_0^1 C_i^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}-1} C_i^{-\frac{1}{\theta}}$$

which will hold for any $i \in [0, 1]$.

$$\rightarrow \frac{C_i}{C_j} = (\frac{P_i}{P_j})^{-\theta}$$

Rearrange the above equation to get

$$C_i^{\frac{\theta-1}{\theta}} = C_j^{\frac{\theta-1}{\theta}} P_j^{\theta-1} P_i^{1-\theta}$$

$$\to P_j^{1-\theta} C_i^{\frac{\theta-1}{\theta}} = C_j^{\frac{\theta-1}{\theta}} P_i^{1-\theta}$$

$$\to \int_0^1 P_j^{1-\theta} dj \cdot C_i^{\frac{\theta-1}{\theta}} = \int_0^1 C_j^{\frac{\theta-1}{\theta}} dj \cdot P_i^{1-\theta}$$

$$\to P^{1-\theta} \cdot C_i^{\frac{\theta-1}{\theta}} = C^{\frac{\theta-1}{\theta}} \cdot P_i^{1-\theta}$$

where price index P is defined as $P = [\int_0^1 P_j^{1-\theta} dj]^{\frac{1}{1-\theta}}$.

$$\to C_i = C(\frac{P_i}{P})^{-\theta}$$

$$\to P_i C_i = P_i^{1-\theta} P^{\theta} C$$

Integrate both sides from 0 to 1 to finally have the total expenditure

$$\int_0^1 P_i C_i di = PC$$

This final equation makes it clear why we call P as price index.

3 Dynamic Programming

A standard **sequential problem** (SP) is

$$\max_{\{x_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$$

s.t.

$$x_{t+1} \in \Gamma(x_t)$$

and x_0 given. Γ is a correspondence.

For example, we have studied the following consumption decision problem

$$\max_{\{c_t\},\{a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c_t + a_{t+1} = (1+r)a_t + w$$

$$c_t \geq 0$$

natural debt limit

$$a_{t+1} \geq -NDL$$

with a_0 given.

Write the above in a standard sequential problem form

$$\max_{\{a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(a_{t+1} - (1+r)a_t - w)$$

s.t.

$$a_{t+1} - (1+r)a_t - w \geq 0$$

$$a_{t+1} \geq -NDL$$

with a_0 given.

A **Bellman equation** (BE) has the following form

$$J(x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta J(x')$$

Therefore, the consumption decision problem can be written as

$$J(a) = \max_{a' \geq -NDL, a' \geq (1+r)a - w} F(a, a') + \beta J(a')$$

(See Stockey, Lucas and Prescott's book for details, within this semester's macro class, I don't think you will be required to do these proofs.) SP can imply BE but in order to let the reverse hold, one has to add one more condition. After we have established the equivalence between SP and BE, we can just focus on BE. In order to let BE solution exist and be unique, we usually apply the fixed point theorem. Finally, to apply fixed point theorem, we need some conditions, for example Blackwell sufficient conditions.

Now let's introduce two simple examples of Bellman equation. And we can also make ourselves more familiar with the solving steps of a Bellman equation.

Example 1. Consider the following growth model,

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{+\infty} \beta^t log(c_t)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha}$$

$$c_t \geq 0$$

$$k_{t+1} \geq 0$$

with k_0 given.

The Bellman equation is

$$V(k) = \max_{k'} \{log(k^{\alpha} + (1-\delta)k - k') + \beta V(k')\}$$

F.O.C.:

$$\frac{-1}{k^{\alpha} + (1-\delta)k - k'} + \beta V'(k') = 0$$

Envelope theorem:

$$V'(k) = \frac{\alpha k^{\alpha-1} + (1-\delta)}{k^{\alpha} + (1-\delta)k - k'} \tag{3}$$

TVC:

$$\lim_{t\to\infty} \beta^t V'(k_t) k_t \leq 0$$

Then from equation (3):

$$V'(k') = \frac{\alpha k'^{\alpha-1} + (1-\delta)}{k'^{\alpha} + (1-\delta)k' - k''}$$

Substitute it to the F.O.C.:

$$\frac{1}{c} = \beta \frac{\alpha k'^{\alpha-1} + (1-\delta)}{c'}$$

When $\delta = 1$, this problem admits analytical solution.

We will use "guess and verify" strategy. Guess $V(k) = A + B\log(k)$ (this is value function "guess and verify", but we can alternatively use policy function "guess and verify"). Substitute $V(k') = A + B\log(k')$ to the Bellman equation

$$V(k) = \max_{k'}\{\log(k^{\alpha} - k') + \beta[A + B\log k]\}$$

F.O.C.:

$$k' = \frac{\beta B}{1+\beta B} k^{\alpha} \tag{4}$$

Substitute the FOC back to the value function to get

$$V(k) = \alpha(1+\beta B)\log k - (1+\beta B)\log(1+\beta B) + \beta A + \beta B\log(\beta B)$$

It is indeed a linear function of $\log k$, i.e. we verify our guess is correct here.

Compare coefficients to have

$$A = -(1+\beta B)\log(1+\beta B) + \beta A + \beta B\log(\beta B)$$

$$B = \alpha(1+\beta B)$$

The unique solution is

$$A = \frac{1}{1-\beta}[\frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) + \log(1-\alpha\beta)]$$

$$B = \frac{\alpha}{1-\alpha\beta}$$

Finally, by equation (4) and resource constraint $c + k' = k^{\alpha}$, policy functions satisfy

$$k' = \alpha\beta k^{\alpha}$$

$$c = (1 - \alpha\beta)k^{\alpha}$$

Notice this model generates that consumption is a constant fraction of total output k^{α}.

Example 2. Consider the following investment problem under uncertainty

$$\max_{\{c_t, A_{t+1}\}} E \sum_{t=0}^{+\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma}$$

s.t.

$$A_{t+1} = R_{t+1}(A_t - C_t)$$

When R_{t+1} is i.i.d., this model will admit analytical solution as well (guess policy function is $c_t = \lambda A_t$, where λ is a constant). Assume $E[R_{t+1}^{1-\sigma}] < \frac{1}{\beta}$.

Costs of Fluctuations-Consumption Theory

Macroeconomic Analysis Recitation 9

Yang Jiao*

1 Costs of Fluctuations

Mathematical preparation:

1).If X follows log-normal, i.e. $log(X) \sim N(\mu, \sigma^2)$, we have $E[X] = E(e^{\log X}) = e^{\mu+\frac{1}{2}\sigma^2}$.

2).$\log(1+a) \approx a$, $e^a \approx 1+a$, when $|a|$ is small.

In the data, variables usually not only have a growth trend but also fluctuate around the trend. Our objective is to measure the cost of fluctuations. Our measure is how much to compensate consumers in a world with fluctuations so that he is as well-off as in a world without fluctuations.

$$\sum_{t=0}^{+\infty} e^{-\rho t} u(E_0(C_t)) = E_0\{\sum_{t=0}^{+\infty} e^{-\rho t} u(C_t(1+\lambda))\} \tag{1}$$

This is a representative agent framework and **let's assume $u(\cdot)$ is CRRA:** $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$. Then equation (1) becomes

$$\sum_{t=0}^{+\infty} e^{-\rho t} [E_0(C_t)]^{1-\gamma} = E_0\{\sum_{t=0}^{+\infty} e^{-\rho t} [C_t(1+\lambda)]^{1-\gamma}\}$$

$\rightarrow$

$$\sum_{t=0}^{+\infty} e^{-\rho t} [E_0(C_t)]^{1-\gamma} = (1+\lambda)^{1-\gamma} E_0\{\sum_{t=0}^{+\infty} e^{-\rho t} [C_t]^{1-\gamma}\} \tag{2}$$

We stand at time 0 and **assume:** $\log(C_t)$ **follows normal distribution and** $E_0(C_t) = C_0 e^{gt}$**.**

Denote the variance of $\log(C_t)$ as $Var_0(\log(C_t))$ and mean as $E_0(\log(C_t))$. Since

$$E_0(C_t) = E(e^{log(C_t)}) = e^{E_0(\log(C_t)) + \frac{1}{2} Var_0(\log(C_t))}$$

and we already assume $E_0(C_t) = C_0 e^{gt}$, therefore,

$$E_0(\log(C_t)) + \frac{1}{2} Var_0(\log(C_t)) = \log(C_0) + gt$$

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which means

$$E_0(\log(C_t)) = \log(C_0) + gt - \frac{1}{2}Var_0(\log(C_t))$$

That is

$$\log(C_t) \sim N(\log(C_0) + gt - \frac{1}{2}Var_0(\log(C_t)), Var_0(\log(C_t)))$$

Since $(1-\gamma)\log(C_t)$ is also a normal distribution, then

$$\begin{aligned} E_0(C_t^{1-\gamma}) = E_0(e^{(1-\gamma)\log(C_t)}) &= e^{(1-\gamma)[\log(C_0)+gt-\frac{1}{2}Var_0(\log(C_t))]+\frac{1}{2}(1-\gamma)^2 Var_0^2(\log(C_t))} \\ &= (C_0 e^{gt})^{1-\gamma} e^{\frac{1}{2}\gamma(\gamma-1)Var_0(\log(C_t))} \end{aligned}$$

Remember we also have Euler equation $r = \rho + \gamma g$. The left hand side of equation (2) is

$$LHS = \sum_{t=0}^{+\infty} C_0^{1-\gamma} e^{-[\rho-g(1-\gamma)]t} = C_0^{1-\gamma}\frac{1}{1-e^{-(\rho-g(1-\gamma))}} = C_0^{1-\gamma}\frac{1}{1-e^{-(r-g)}}$$

The right hand side of equation (2) is

$$RHS = (1+\lambda)^{1-\gamma}\sum_{t=0}^{+\infty} e^{-\rho t}(C_0 e^{gt})^{1-\gamma} e^{\frac{1}{2}\gamma(\gamma-1)Var_0(\log(C_t))} = (1+\lambda)^{1-\gamma} C_0^{1-\gamma}\sum_{t=0}^{+\infty} e^{-(r-g)t} e^{\frac{1}{2}\gamma(\gamma-1)Var_0(\log(C_t))}$$

Finally, $LHS = RHS$ gives

$$(1+\lambda)^{\gamma-1} = [1-e^{-(r-g)}]\sum_{t=0}^{+\infty} e^{-(r-g)t} e^{\frac{1}{2}\gamma(\gamma-1)Var_0(\log(C_t))}$$

Case 1. If $\log(C_t) = \log(C_0) + gt - \frac{1}{2}\sigma^2 + \epsilon_t$, with ϵ_t i.i.d. normal with variance σ^2 (notice that we need to guarantee that $E(C_t) = C_0 e^{gt}$), then $Var_0(\log(C_t))) = \sigma^2$.

$$(1+\lambda)^{\gamma-1} = e^{\frac{1}{2}\gamma(\gamma-1)\sigma^2}$$

$$\log(1+\lambda) \approx \lambda = \frac{\gamma\sigma^2}{2}$$

Case 2. If $\log(C_t) = g - \frac{1}{2}\sigma^2 + \log(C_{t-1}) + \epsilon_t$, with ϵ_t i.i.d. normal with variance σ^2, then $Var_0(\log(C_t))) = \sigma^2 t$.

$$(1+\lambda)^{\gamma-1} = \frac{1-e^{-(r-g)}}{1-e^{-(r-g+\frac{1}{2}\gamma(1-\gamma)\sigma^2)}}$$

$$(1+\lambda)^{1-\gamma} = \frac{1-e^{-(r-g+\frac{1}{2}\gamma(1-\gamma)\sigma^2)}}{1-e^{-(r-g)}} = \frac{r-g+\frac{1}{2}\gamma(1-\gamma)\sigma^2}{r-g}$$

$$(1-\gamma)log(1+\lambda) = log(\frac{r-g+\frac{1}{2}\gamma(1-\gamma)\sigma^2}{r-g}) \approx \frac{\frac{1}{2}\gamma(1-\gamma)\sigma^2}{r-g}$$

$$log(1+\lambda) \approx \lambda = \frac{\gamma\sigma^2}{2}\frac{1}{r-g}$$

Case 3. If $m_t = \theta m_{t-1} + \epsilon_t$ with $m_0 = 0$ and $log(C_t) = m_t + gt + ln(C_0) - \frac{1}{2}\sigma^2\frac{1-\theta^{2t}}{1-\theta^2}$ (again, expost we add the mean term to ensure that $E_0(C_t) = C_0 e^{gt}$), then $Var_0(\log(C_t))) = var_0(\sum_{i=1}^{t}\theta^{i-1}\epsilon_i) = \sigma^2\frac{1-\theta^{2t}}{1-\theta^2}$.

$$(1+\lambda)^{\gamma-1} = [1-e^{-(r-g)}]\sum_{t=0}^{+\infty} e^{-(r-g)t}e^{\frac{1}{2}\gamma(\gamma-1)\sigma^2\frac{1-\theta^{2t}}{1-\theta^2}} \approx [1-e^{-(r-g)}]\sum_{t=0}^{+\infty} e^{-(r-g)t}[1+\frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)}(1-\theta^{2t})]$$

$$= [1-e^{-(r-g)}]\sum_{t=0}^{+\infty} e^{-(r-g)t}[1+\frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)} - \frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)}\theta^{2t}] = 1 + \frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)} + \frac{-(1-e^{-(r-g)})\frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)}}{1-e^{-(r-g)}\theta^2}$$

$\rightarrow$

$$(\gamma-1)log(1+\lambda) = \frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)} + \frac{-(1-e^{-(r-g)})\frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)}}{1-e^{-(r-g)}\theta^2}$$

$$= \frac{\gamma(\gamma-1)\sigma^2}{2(1-\theta^2)}\frac{e^{-(r-g)}(1-\theta^2)}{1-e^{-(r-g)}\theta^2} = \frac{\gamma(\gamma-1)\sigma^2}{2}\frac{e^{-(r-g)}}{1-e^{-(r-g)}\theta^2} = \frac{\gamma(\gamma-1)\sigma^2}{2}\frac{1}{e^{r-g}-\theta^2}$$

$\rightarrow$

$$\lambda \approx log(1+\lambda) \approx \frac{\gamma\sigma^2}{2}\frac{1}{1-\theta^2+r-g}$$

Notice that when $\theta = 0$ and $\theta = 1$, we can go back to the first two cases. Intuitively, when θ is larger, given σ^2, $var_0(\log(C_t))$ will be larger, leading to higher uncertainty thus higher costs of fluctuations.

2 Consumption Theory

2.1 CRRA Utility

Euler equation:

$$u'(c_t) = \beta(1+r)E_t(u'(c_{t+1}))$$

With CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, we know that $\frac{u''(c)c}{u'(c)} = -\gamma$, for any $c > 0$. Then do the following approximation

$$\log u'(c_{t+1}) = \log u'(e^{\log(c_{t+1})}) \approx \log u'(c_t) + \frac{u''(c_t)c_t}{u'(c_t)}(\log(c_{t+1}) - \log(c_t))$$

The above equation is a first order approximation of function $f(x) = \log u'(e^x)$ around $x^* = \log(c_t)$, where $x = \log(c_{t+1})$.

$\rightarrow$

$$\log u'(e^{\log(c_{t+1})}) \approx \log u'(c_t) - \gamma(\log(c_{t+1}) - \log(c_t))$$

$\rightarrow$

$$u'(c_{t+1}) = u'(c_t)e^{-\gamma(\log(c_{t+1})-\log(c_t))}$$

Assume $\log(c_{t+1})$ follows normal distribution with variance σ^2, and then plug the above into the Euler equation to get

$$1 = \beta(1+r)e^{-\gamma E_t(\log(c_{t+1})-\log(c_t))+\frac{1}{2}\gamma^2\sigma^2}$$

$\rightarrow$

$$E_t(\log(c_{t+1}) - \log(c_t)) = \frac{1}{\gamma}\log(\beta(1+r)) + \frac{1}{2}\gamma\sigma^2$$

When $\sigma^2 \neq 0$, even if $\beta(1+r) = 1$, in expectation, consumption will have a positive growth rate and the growth rate is increasing with variance σ^2, in contrast with quadratic utility function where consumption growth rate is 0 and volatility doesn't matter.

2.2 CARA Utility

(We will finish up this problem next time.)

Setup a consumption problem with CARA utility (i.e. $u(c) = -\frac{e^{-\alpha c}}{\alpha}$, which satisfies $-\frac{u''}{u'}$ is a constant):

$$V(w) = \max_c \frac{-e^{-\alpha c}}{\alpha} + \beta E[V(R(w-c)+y')]$$

where y is i.i.d. normal $N(0,\sigma^2)$. Assume $\beta R = 1$. We also relax the constraint that c_t has to be non-negative so that we don't need to worry about natural debt limit here.

F.O.C.:

$$e^{-\alpha c} = E(V'(w'))$$

$$V'(w) = E(V'(w'))$$

Euler equation

$$e^{-\alpha c} = E[e^{-\alpha c'}]$$

Guess policy function is $c = Aw + B$.

Then

$$E(e^{-\alpha c'}) = E(e^{-\alpha(Aw'+B)}) = E(e^{-\alpha[A(R(w-c)+y')+B]})$$
$$= E(e^{-\alpha[A(R(w-c))+B]}e^{-\alpha Ay'}) = e^{-\alpha AR(1-A)w-\alpha B(1-AR)}e^{\alpha^2A^2\sigma^2/2} = e^{-\alpha c}$$

which implies

$$c = AR(1-A)w + B(1-AR) - \alpha A^2\sigma^2/2$$

It verifies our guess. Compare coefficients

$$A = AR(1-A)$$

$$B = B(1-AR) - \alpha A^2\sigma^2/2$$

Solution

$$A = \frac{R-1}{R}$$

$$B = -\alpha \frac{(R-1)\sigma^2}{2R^2}$$

or

$$A = 0$$

and B can be any number.

But the latter solution violates the TVC

$$\lim_{t\to+\infty} E[\beta^t V'(w_t) w_t] = 0$$

This is because if it is the solution then $V'(w_t) = e^{-\alpha c_t} = e^{-\alpha B}$ and

$$w_{t+1} = R(w_t - B) + y_{t+1} = R^{t+1}w_0 - RB - R^2B - ... - R^{t+1}B + y_{t+1} + Ry_t + ... + R^t y_1$$

which implies

$$\lim_{t\to+\infty} E[\beta^{t+1} V'(w_{t+1}) w_{t+1}] = e^{-\alpha B}[w_0 - \frac{B}{1-\beta}]$$

It is not valid to argue that if $w_0 = \frac{B}{1-\beta}$, that solution satisfies TVC, since TVC has to be satisfied at all time, i.e. for w_1, w_2...etc. As we have shocks, $\{w_t\}$ will fluctuate. Therefore, TVC is violated.

So our solution should be

$$A = \frac{R-1}{R}$$

$$B = -\alpha \frac{(R-1)\sigma^2}{2R^2}$$

$\rightarrow$

$$w_{t+1} = R(w_t - c_t) + y_{t+1} = R(1-A)w_t - RB + y_{t+1} = w_t + \alpha \frac{(R-1)\sigma^2}{2R} + y_{t+1}$$

$\rightarrow$

$$E_t[C_{t+1} - C_t] = E_t[A(w_{t+1} - w_t)] = E_t[\frac{R-1}{R} y_{t+1}] + \frac{1}{2}\frac{(R-1)^2}{R^2}\alpha\sigma^2 = \frac{1}{2}\frac{(R-1)^2}{R^2}\alpha\sigma^2 > 0$$

This example also shows that consumption growth is positive, and the higher the variance of income shock, the higher the difference between expected tomorrow's consumption and today's consumption, in contrast with quadratic utility function where consumption growth is 0 and volatility doesn't matter.

Consumption Theory (continued)

Macroeconomic Analysis Recitation 10

Yang Jiao*

1 Consumption Theory

1.1 CARA Utility

See Recitation note 9.

1.2 Borrowing Constraint

We have known that with quadratic utility and no borrowing constraint, we will have consumption is a random walk. There is no precautionary saving. However, we have seen from Recitation note 9 that with CRRA utility or CARA utility, but without borrowing constraint (in CRRA case, since Inada condition is satisfied, even if we impose natural debt limit, we don't need to worry because it never binds) that we will have consumption growth, and more uncertainty leads to more savings: i.e. precautionary savings.

Suppose now we introduce borrowing constraint but preserve quadratic utility function, we will see that we will also have precautionary saving.

Consider the following problem:

$$\max_{c_t, b_{t+1}} \sum_{t=0}^{+\infty} \beta^t E_t[u(c_t)]$$

s.t. budget constraint

$$b_{t+1} = (1+r)(b_t + y_t - c_t)$$

and borrowing constraint

$$b_{t+1} \geq 0$$

Write down the Lagrangian problem:

$$\max_{c_t, b_{t+1}} \sum_{t=0}^{+\infty} \beta^t E_t[\{u(c_t) + \lambda_t[(1+r)(b_t + y_t - c_t) - b_{t+1}] + \mu_{t+1} b_{t+1}\}]$$

*Please email me if you find errors or typos to yj2279@columbia.edu. Your help will be greatly appreciated. All comments and suggestions are especially welcome.

F.O.C.:

$$u'(c_t) - (1+r)\lambda_t = 0$$

$$\mu_{t+1} - \lambda_t + \beta(1+r)E_t[\lambda_{t+1}] = 0$$

Complementary slackness condition:

$$\mu_{t+1}b_{t+1} = 0, \mu_{t+1} \geq 0, b_{t+1} \geq 0$$

Therefore,

$$u'(c_t) = \beta(1+r)E_t[u'(c_{t+1})] + \mu_{t+1}(1+r) \geq \beta(1+r)E_t[u'(c_{t+1})]$$

If $\mu_{t+1} > 0$, we must have $b_{t+1} = 0$ thus $c_t = y_t + b_t = w_t$, which means $u'(c_t) = u'(w_t) > \beta(1+r)E_t u'(c_{t+1})$.

If $\mu_{t+1} = 0$, we directly have $u'(c_t) = \beta(1+r)E_t u'(c_{t+1})$.

In sum,

$$u'(c_t) = \max\{u'(w_t), \beta(1+r)u'(c_{t+1})\}$$

Check the above is what you have on slide 7 of lecture note 6: generalized Euler equation.

Remark. Suppose $\beta(1+r) = 1$.

When $u''' = 0$, but with borrowing constraint so that sometimes we will have $\mu_{t+1} > 0$, which implies $c_t < E_t(c_{t+1})$.

When u' is strictly convex, i.e. $u''' > 0$, by Jensen's inequality, $u'(c_t) > E_t[u'(c_{t+1})] > u'(E_t(c_{t+1}))$. We also assume u is concave, so $c_t < E_t(c_{t+1})$.

■

Next by Benveniste-Scheinkman formula,

$$u'(c) = V'(w)$$

Therefore,

$$V'(w_t) \geq \beta(1+r)V'(w_{t+1})$$

Remark. See Benveniste-Scheinkman (1979, Econometrica), it basically says the following Envelope theorem: a Bellman equation

$$V(x) = \max_{x' \in \Gamma(x)} u(x, x') + \beta V(x'),$$

implies

$$V'(x) = \frac{\partial u(x, x'^*)}{\partial x}.$$

where x'^* is the policy function as a function of x.

■

Notice that we haven't introduced restrictions on the income process or utility function yet. Therefore, we have reviewed what we studied in class here.

Now let's use quadratic utility: $u(c) = -\frac{1}{2}(c - \bar{c})^2$ and assume $\beta(1+r) = 1$.

$$c_t = \min\{y_t + b_t, E_t[c_{t+1}]\} = \min\{y_t + b_t, E_t[\min\{y_{t+1} + b_{t+1}, E_t[c_{t+2}]\}]\}$$

First, c_t is no longer a random walk and $c_t \leq E_t[c_{t+1}]$. Second, with larger uncertainty in y_{t+1}, c_t will be smaller since we have a larger chance of getting smaller realization of y_{t+1}. (but of course, this is not a rigorous proof, since variables b_{t+1}, c_{t+2} are also endogenous).

To conclude, **either** we have quadratic utility function $u''' = 0$ but with borrowing constriant **or** we have $u''' > 0$ (check that it is true for CRRA and CARA), but regardless of whether you have borrowing constraint, precautionary saving will show up.

1.3 Role of Market Incompleteness

One caution is that here we have assumed that financial market is incomplete: only risk free bond is traded in the economy. Note that borrowing constraint itself means financial market is not complete. But $u''' > 0$ is nothing related to the financial market completeness or not.

Let's consider a case without borrowing constraint, and with $u''' > 0$. Instead of assuming only a risk free bond is traded, we assume market is complete. idiosyncratic income process is i.i.d. with distribution $N(\mu, \sigma^2)$. Assume there is a continuum of agents who start with the same wealth $y_0 + b_0$, but there is no aggregate risk for the economy, i.e. under an aggregate state s_t, some guys get good shocks, and some get bad shocks. Denote the c.d.f. of aggregate state $p(s_t)$. For agent i:

$$\max \sum_{t=0}^{+\infty} \beta^t \int u(c^i(s_t))p(s_t)ds_t$$

s.t.

$$c^i(s_t) + \int q(s_{t+1})b^i(s_{t+1})ds_{t+1} \leq y^i(s_t) + b^i(s_t)$$

Denote $\lambda^i(s_t)$ the Lagrange multiplier of the budget constraint. F.O.C.:

$$u'(c^i(s_t)) = \lambda^i(s_t)$$

$$\lambda^i(s_t)q(s_{t+1})p(s_t) = \beta\lambda^i(s_{t+1})p(s_{t+1})$$

Therefore,

$$u'(c^i(s_t))q(s_{t+1}) = \beta\frac{p(s_{t+1})}{p(s_t)}u'(c^i(s_{t+1}))$$

In period 0, by symmetry, we should have all agents choose the same consumption c_0. Then the above equation says, in future days, no matter what the aggregate state is, all agents also have the same consumption. Since the aggregate income is μ, we will have that each guy consumes μ no matter what idiosyncratic shock he has. Note idiosyncratic income process's volatility σ doesn't matter in this economy.

One can use a two-agent, two state, complete market setting to illustrate the above as well. Assume income process is i.i.d. binary with probability 1/2 the outcome is $\mu + \sigma$ and with probability 1/2 the outcome is $\mu - \sigma$.

There is no precautionary saving in this economy since they can insure against each other (compare with Aiyagari model).

2 Some General Results

We have already derived

$$u'(c_t) \geq \beta(1+r)E_t[u'(c_{t+1})]$$

Assume Inada condition: $\lim_{c \to 0} u'(c) = +\infty$

Case 1. If $\beta(1+r) > 1$,

$$u'(c_t) \geq \beta(1+r)E_t[u'(c_{t+1})]$$

Denote $h_t = [\beta(1+r)]^t u'(c_t)$, then we have $h_t \geq E_t h_{t+1}$. By super martingale convergence theorem, $h_t \to x$ a.s., where random variable x satisfies that $E[x] < M$. Therefore, $u'(c_t) = [\beta(1+r)]^{-t} h_t \to 0$ a.s., which means $c_t \to +\infty$ a.s.

Case 2. If $\beta(1+r) = 1$,

$$u'(c_t) \geq E_t[u'(c_{t+1})]$$

See Chamberlain and Wilson (2000, RED), we also get consumption grows without bound with probability 1.

Case 3. If $\beta(1+r) < 1$,

Suppose additionally we assume

$$\lim_{c \to +\infty} \frac{u_{cc}}{u_c} = 0$$

then asset w_t has an upper bound w_m.

Intuition: with the decreasing absolute risk aversion, agents care less about uncertainty when they get richer, so they choose to consume instead of continue to accumulate, dragging down asset.

3 Log-Linearization (continued)

In recitation 7, we have considered how to implement log-linearization. However, there, the euqation is with contemporaneous variables. Now let's consider a more general case with both time t variables $\{x_{1t}, x_{2t}, ...x_{nt}\}$ and time $t+1$ variables $\{z_{1,t+1}, z_{2,t+1}...z_{m,t+1})\}$. Some x and z can coincide.

$$y_t = E_t[f(x_{1t}, x_{2t}, ..., x_{nt}, z_{1,t+1}, z_{2,t+1}, ..., z_{m,t+1})]$$

$$\begin{aligned}
y_t &= E_t[e^{\log f(e^{\log x_{1t}}, e^{\log x_{2t}}, ... e^{\log x_{nt}}, e^{\log z_{1,t+1}}, e^{\log z_{2,t+1}}, ..., e^{\log z_{m,t+1}})}] \\
&\approx E_t[e^{\log f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*) + \sum_{i=0}^n \frac{\frac{\partial f}{\partial x_i} \cdot x_i^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)}[\log x_{it} - \log x_i^*] + \sum_{j=0}^m \frac{\frac{\partial f}{\partial z_j} \cdot z_j^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)}[\log z_{j,t+1} - \log z_j^*]}] \\
&= e^{\log f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*) + \sum_{i=0}^n \frac{\frac{\partial f}{\partial x_i} \cdot x_i^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)}[\log x_{it} - \log x_i^*]} \\
&\quad \cdot e^{\sum_{j=0}^m \frac{\frac{\partial f}{\partial z_j} \cdot z_j^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)}[E_t \log z_{j,t+1} - \log z_j^*] + \frac{1}{2} k' \Sigma k}
\end{aligned}$$

where the last equality comes from the moment generating function of a multivariate normal distribution. Here $k = (\frac{\frac{\partial f}{\partial z_1} \cdot z_1^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)}, \frac{\frac{\partial f}{\partial z_2} \cdot z_2^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)}, ..., \frac{\frac{\partial f}{\partial z_m} \cdot z_m^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)})'$. Note we assume that $(\log z_{1,t+1} - \log z_1^*, \log z_{2,t+1} - \log z_2^*, ..., \log z_{n,t+1} - \log z_n^*)'$ follows multivariate normal distribution with variance matrix Σ.

Finally,

$$\hat{y}_t = \log y_t - \log y^* = \sum_{i=0}^{n} \frac{\frac{\partial f}{\partial x_i} \cdot x_i^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)} \hat{x}_{it} + \sum_{j=0}^{m} \frac{\frac{\partial f}{\partial z_j} \cdot z_j^*}{f(x_1^*, x_2^*, ..., x_n^*, z_1^*, z_2^*, ..., z_m^*)} E_t \hat{z}_{j,t+1} + \frac{1}{2} k' \Sigma k$$

where definition $\hat{x} = \log x - \log x^*$ is for any variable x.

We additionally have a variance term $\frac{1}{2}k'\Sigma k$ compared to the result in recitation 7 without uncertainty. But we usually neglect that variance term. For example, Euler equation with log-utility:

$$E_t(\frac{\beta(1+r_t)C_t}{C_{t+1}}) = 1$$

$\rightarrow$

$$\widehat{1+r_t} + \hat{C}_t - E_t \widehat{C_{t+1}} = 0$$

Approximately, $\widehat{1+r_t} = \log(1+r_t) - \log(1+r^*) \approx r_t - r^*$

Price Level Determination

Macroeconomic Analysis Recitation 11

Yang Jiao*

1 Aiyagari Economy (Quick Review)

Aiyagari economy is a general equilibrium model with each individual subject to borrowing constraint. By definition, stationary equilibrium implies aggregate variables keep constant, e.g. interest rate r, aggregate capital K etc., but since individuals are subject to idiosyncratic shocks, every agent's variables behave stochastically.

Supply Side of Capital

At the individual level, agent i solves a standard precautionary saving problem.

$$\max_{\{c_t^i\}} E[\sum_{t=0}^{+\infty} \beta^t \frac{(c_t^i)^{1-\gamma}-1}{1-\gamma}]$$

s.t. budget constraint

$$c_t^i + a_{t+1}^i = (1+r)a_t^i + y_t^i$$

borroing constraint:

$$a_{t+1}^i \geq -\phi$$

where labor income $y_t^i = s_t^i w$, with s_t^i a Markov process as the source of uncertainty.

First, we must have $\beta(1+r) < 1$, otherwise, individuals accumulate infinite asset. That is, $1+r$ is bounded by $\frac{1}{\beta}$. Or alternatively speaking, when $1+r$ approaches $1/\beta$ from below, individuals accumulate more and more asset. When r is larger, individuals accumulate more assets a_t^i. Therefore, capital stock in the economy $K = \sum_i a_t^i$ will also be larger.

That's why we have an upward sloping capital supply curve and capital goes to infinity when $1+r$ approaches $1/\beta$.

Demand Side of Capital

Demand side is simple: representative firms rent capital and hire labor from households:

$$\max_{K_t, N_t} K_t^{\alpha} N_t^{1-\alpha} - RK_t + (1-\delta)K_t - wN_t$$

where R is the gross interest rate.

*Please email me if you find errors or typos to yj2279@columbia.edu. Your help will be greatly appreciated. All comments and suggestions are especially welcome.

F.O.C.s give the demand for capital

$$R = 1 - \delta + \alpha K_t^{\alpha-1} N_t^{1-\alpha}$$

Labor market clearing: $N_t = 1$. In stationary equilibrium, aggregate capital K_t is a constant K.

Asset market no arbitrage condition: $R = 1 + r$.

Therefore, $1 + r = 1 - \delta + \alpha K^{\alpha-1}$, which is the demand function for capital (a downward sloping curve).

Demand curve and supply curve determine the equilibrium capital and interest rate.

Remark 1. When borrowing constraint is looser, i.e. an increase in ϕ, what will happen to the supply curve of capital? First, given an interest rate, agents will not accumulate so much wealth as before, i.e. supply curve shift left. Second, at the point $1 + r = 1$, since individuals' constraints become

$$c_t^i + (a_{t+1}^i + \phi) = (a_t^i + \phi) + y_t$$

$$a_{t+1}^i + \phi \geq 0,$$

an increase in ϕ means optimal choice of a_t^i will decrease the same amount as ϕ increases. That is to say, aggregate capital also decrease the same amount as ϕ at point $1 + r = 1$.

Remark 2. Decrease in the volatility of idiosyncratic shock to labor income will also shift the supply side of capital because the precautionary saving motive is smaller.

2 Inflation in a Cashless Economy

See your Lecture note 7.

Although the economy is cashless, we have nominal account. One can think every transaction is done by revising the balance sheet of the nominal account.

$$\max E[\sum_{t=0}^{+\infty} \beta^t \log(C_t)]$$

s.t.

$$P_t C_t + B_{t+1} + P_t K_{t+1} = P_t Y_t + B_t(1 + i_{t-1}) + P_t K_t(1 + r_{t-1})$$

where K_{t+1} is real bonds and B_{t+1} is nominal bonds. i_{t-1} and r_{t-1} are risk-free nominal interest rate and real interest rate, which are determined at time $t - 1$. Notice this is an endowment economy, so Y_t is an exogenous process.

Euler equation for **real bonds**

$$1 = \beta E_t[\frac{(1 + r_t)C_t}{C_{t+1}}]$$

In steady state $(1 + r^*) = \beta^{-1}$ thus $r^* \approx \log(1 + r^*) = -\log \beta$. Log-linearize the above Euler equation

$$E_t(\widehat{C_{t+1}} - \widehat{C}_t) = \hat{r}_t$$

where $\widehat{C}_t = \log C_t - \log C^*$, $\hat{r}_t = \log(1+r_t) - \log(1+r^*) = r_t - r^*$. Define $c_t = \log C_t$, then re-write the above as

$$E_t(c_{t+1} - c_t) = \hat{r}_t$$

Euler equation for **nominal bonds**

$$1 = \beta E_t[\frac{(1+i_t)P_tC_t}{P_{t+1}C_{t+1}}]$$

Notice we have steady sate for inflation but not necessarily price level (only when steady state inflation is 0, price level has steady state constant value). So let's do it carefully instead of directly apply the log-linearization rule as in recitation note 7. Define $p_t = \log P_t$ (instead of $\log P_t - \log P^*$ since P^* may not exist), thus $\pi_{t+1} \approx p_{t+1} - p_t = \Delta p_{t+1}$. Let $\rho = -\log \beta$. The Euler equation becomes

$$1 = E_t\{\exp(i_t - \Delta c_{t+1} - \pi_{t+1} - \rho)\}$$

We used the approximation $exp(i_t) \approx 1 + i_t$.

In steady state $1 = \exp(i - 0 - \pi - \rho)$, i.e. $i = \pi + \rho$.

First order Taylor expansion of the Euler equation above around steady state

$$1 = E_t\{1 + i_t - i - (\Delta c_{t+1} - 0) - (\pi_{t+1} - \pi)\}$$

$\rightarrow$

$$E_t(c_{t+1} - c_t) = i_t - E_t(\pi_{t+1}) - \rho$$

Combine the two log-linearized equations to obtain

$$r_t - r^* = i_t - E_t(\Delta p_{t+1}) - \rho$$

As $r^* = \rho = -\log \beta$, we get Fisher equation

$$i_t - r_t = E_t(\Delta p_{t+1})$$

Remark. In fact, you can use the simiar steps of the derivation following the Euler equation for nominal bonds, to get $r_t - \rho = E(c_{t+1} - c_t)$ from the Euler equation for real bonds. That is, forget about log-linearization rules, and do it in the following way.

$$1 = \beta E_t\{\frac{(1+r_t)C_t}{C_{t+1}}\}$$

$\rightarrow$

$$1 = E_t\{\exp(r_t - \rho - \Delta c_{t+1})\}$$

In steady state: $r^* = \rho$.

First order approximation:

$$1 \approx E_t\{1 + r_t - \rho - \Delta c_{t+1}\}$$

i.e.

$$r_t - \rho = E(c_{t+1} - c_t)$$

Classical Dichotomy

Suppose now there is no government intervention. Assets market clearing condition

$$K_t = 0$$

$$B_t = 0$$

Goods market clearing

$$C_t = Y_t$$

It gives the following equilibrium solution for real variables:

$$K_{t+1} = 0$$

$$C_t = Y_t$$

Euler equation for **real bonds**

$$(1 + r_t)^{-1} = E_t(\frac{\beta C_t}{C_{t+1}})$$

We can see that since Y_t is an exogenous process, C_t, K_{t+1}, r_t will be determined only by the process of real variable Y_t. There is no role for nominal stuff to affect the outcome of real variables.

Price Level Indeterminacy

We need to pin down P_t, i_t but we only have the following one equation: Euler equation for **nominal bonds**

$$(1 + i_t)^{-1} = E_t(\frac{\beta C_t}{C_{t+1}} \frac{P_t}{P_{t+1}}) = E_t(\frac{\beta Y_t}{Y_{t+1}} \frac{P_t}{P_{t+1}})$$

One can choose i_t to move correspondingly with P_t, but we cannot pin down P_t, there is no other equation for us to determine P_t.

Therefore, we need additional conditions (where the number of conditions must exceed the number of newly introduced variables by 1). We will introduce government/ central bank to resolve the price indeterminacy problem.

2.1 Nominal Reserves 1

Fiscal Policy: $T_t + D_t = G_t$, where T_t is tax, G_t is government spending. G_t is exogenous. D_t is called fiscal deficit. T_t is chosen by policy as well. Fiscal policy brings one new endogenous variable D_t, and one new equation.

Central Bank: There is fiscal deficit. Central bank issues liability to fill this gap. Specifically, the central bank promises to pay $P_t(1 + x_{t-1})$ dollars back at time t if he borrows 1 dollar at time $t - 1$. Notice this is neither a real bond nor a nominal bond as in the consumer problem. It links the payment schedule to **price level**. The law of motion of libability (called reserves)

$$V_{t+1}^R = P_t(1 + x_{t-1})V_t^R + P_t D_t$$

Note x_t is a policy variable. That is we have one new endogenous variable V_t^R and one new equation here.

Remark. Alternatively, you can take V_t^R as exogenous policy, but tax T_t as endogenous variable. ■

We still lack one equation.The central bank's libability is an asset traded in the market as well. We should have an Euler equation for this asset.

Recall Euler equation for nominal bonds:

$$\beta E_t[\frac{P_t C_t (1 + i_t)}{P_{t+1} C_{t+1}}] = 1$$

Similarly, Euler equation for the central bank's liability is

$$\beta E_t[\frac{P_t C_t P_{t+1}(1 + x_t)}{P_{t+1} C_{t+1}}] = 1$$

i.e.

$$1 = (1 + x_t) P_t E_t[\frac{\beta C_t}{C_{t+1}}]$$

Substitute the Euler equation for real bonds to arrive at:

$$P_t = \frac{1 + r_t}{1 + x_t}$$

where $(1 + r_t)^{-1} = E_t[\frac{\beta C_t}{C_{t+1}}]$ and $C_t = Y_t - G_t$. Therefore, here we are able to pin down price level.

2.2 Nominal Reserves 2

Fiscal Policy no change.

Central Bank directly issues a nominal bonds with nominal interest rate i_t^v, which is set by the central bank (thus it is exogenous). Nominal interest rate must be the same as the policy choice as a result of no arbitrage condition

$$i_t = i_t^v$$

which is the one additional equation we are looking for.

Fisher equation

$$r_t = i_t - E_t(\Delta p_{t+1}) = i_t^v - E_t(\Delta p_{t+1})$$

$\rightarrow$

$$p_t = E_t p_{t+1} + r_t - i_t^v,$$

then

$$p_t = \sum_{s=0}^{+\infty}(r_{t+s} - i^v_{t+s}) + \lim_{s\to+\infty} E_t p_{t+s+1}$$

But we cannot do this, since in steady state, we cannot do the summation for the first term because it doesn't converge and we don't know $\lim_{s\to+\infty} E_t p_{t+s+1}$, either.

Come back to

$$p_t = E_t p_{t+1} + r_t - i^v_t,$$

If we expect tomorrow's price level will jump up, today's price will immediately jump up, and vice versa (expectation driven).

One may have found that the problem lies in the fact that the coefficient before p_t is not larger than 1. Let's link the nominal interest rate policy to price level or inflation to resolve this problem.

Wicksellian Rules (Price Targeting)

Let $i_t = i^v_t = \phi(p_t - p^*)$, where p^* is the constant (not steady state value) price target, taken as given, and parameter $\phi > \mathbf{0}$.

Fisher equation becomes

$$(\phi + 1)p_t = E_t p_{t+1} + r_t + \phi p^*$$

i.e.

$$p_t = \frac{1}{1+\phi} E_t p_{t+1} + \frac{1}{1+\phi}(r_t + \phi p^*)$$

Iterate forward to get

$$p_t = \sum_{s=0}^{+\infty}(1+\phi)^{-s-1}E_t[r_{t+s} + \phi p^*] + \lim_{s\to+\infty}(1+\phi)^{-s-1}E_t p_{t+s} = \sum_{s=0}^{+\infty}(1+\phi)^{-s-1}E_t[r_{t+s} + \phi p^*]$$

Taylor Rules (Inflation Targeting)

Let $i_t = \phi(\Delta p_t - \Delta p^*)$, where $\Delta p^* = \pi^*$ is the constant inflation target, taken as given and parameter $\phi > \mathbf{1}$.

Fisher equation becomes

$$\phi \Delta p_t = E_t \Delta p_{t+1} + r_t + \phi \Delta p^*$$

Similarly, iterate forward to get

$$\Delta p_t = \sum_{s=0}^{+\infty} \phi^{-s-1} E_t[r_{t+s} + \Delta p^*]$$

Notice at time t, either you have inflation Δp_t or price level p_t, you can determine the other, since p_{t-1} is already known.

Remark. Imagine inflation is really really negative, Taylor rule above with linear form means interest rate has to be negative as well, violating ZLB (zero lower bound). We argue that we only consider locally bounded equilibrium around the steady state so $i_t \geq 0$ will be satisfied locally. However, a global interest rule has to consider the ZLB constraint $i_t \geq 0$.

2.3 Global Analysis and the ZLB

For simplicity, assume there is no uncertainty, and $C_t = Y_t$ is constant. Combine the two Euler equations to get

$$\frac{1}{\beta} = (1 + i_t)\Pi_{t+1}$$

where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$. Again one equation is not enough to pin down two variables P_t, i_t.

Suppose central bank takes the following policy rule, taking care of the ZLB

$$1 + i_t = \max\{\frac{\bar{\Pi}}{\beta}(\frac{\Pi_t}{\bar{\Pi}})^{\phi}, 1\}$$

where $\bar{\Pi}$ is the constant inflation target.

By eliminating $1 + i_t$, we conclude

$$1 = \frac{\beta}{\Pi_{t+1}} \max\{\frac{\bar{\Pi}}{\beta}(\frac{\Pi_t}{\bar{\Pi}})^{\phi}, 1\}$$

Draw Π_{t+1} against Π_t, one can find globally we have two steady states. But previous local equilibrium analysis will ignore the possibility of two steady states.

The high inflation steady state is not stable and if you perturbate leftward a little, the economy will run into a deflation spiral, until reaching another steady state at the ZLB.

3 Inflation with Money

See your lecture note 8.

In the previous section, we studied an economy without money (I mean, cash), where one can think transactions happen through the nominal account. Now let's introduce money to the model. Then the question is, why do people hold money, since it is an asset which only pays 0 nominal interest rate. As long as, nominal bonds pay $i_t > 0$, there is no reason to hold money. To let money have a role, we assume that real money balance directly enters into utility function, which, of course, is a shortcut.

Households problem

$$E_0\{\sum_{t=0}^{+\infty} \beta^t u(C_t, M_t/P_t)\}$$

where $u_m > 0, u_{mm} \leq 0, u_{cm} \geq 0$.

s.t.

$$P_tC_t + B_{t+1} + P_tK_{t+1} + M_t \leq P_t(Y_t - T_t) + B_t(1 + i_{t-1}) + P_tK_t(1 + r_{t-1}) + M_{t-1}$$

Notice that money as an asset pays zero interest, so the reason that we hold money comes from that holding money itself brings utility. Since money itself provides utility but nominal bond doesn't. We must require

$$i_t \geq 0$$

Otherwise, one can sell bonds to exchange money without any restriction, leading to infinite utility. Furthermore, one cannot short money (but you can issue bond to others):

$$M_t \geq 0$$

However, when the utility function satisfies Inada condition for real money balance, we never need to worry about this constraint.

First order condition gives

$$\frac{u_m(\cdot)}{u_c(\cdot)} = \frac{i_t}{1+i_t}$$

Assume that the utility function is

$$u(C, M/P) = \log C + \chi \log(M_t/P_t)$$

Remark. One can use more general CRRA utility for both consumption and real money balance, or other more sophisticated formula, and follow similar analysis below. Note the utility function specified in your lecture note is not consistent with the later used Euler equations. That's why I choose to use this simple utility function instead of that one. ■

Then

$$\frac{P_t C_t}{M_t} = \frac{1}{\chi}\frac{i_t}{1+i_t} \tag{1}$$

which is the **quantity theory of money**.

Take *log* of both sides (instead of log-linearization, since price level P_t doesn't have a steady state)

$$p_t + c_t - m_t = -\log\chi + \log(\frac{i_t}{1+i_t}) \tag{2}$$

Here

$$\log(\frac{i_t}{1+i_t}) = \log(1-\frac{1}{1+i_t}) \approx \log(1-\frac{1}{\exp(i_t)}) = \log(1-\exp(-i_t)) \approx \log(1-\exp(-i^*))+\frac{1}{\exp(i^*)-1}(i_t-i^*)$$

The last approximation is a first order Taylor expansion.

Denote $\eta = \frac{1}{\exp(i^*)-1}$ and **ignore** constant terms, we have

$$p_t + c_t - m_t = \eta i_t$$

Fiscal Policy Similar to before.

$$T_t + D_t = G_t$$

where G_t is exogenous.

Monetary Policy There is no nominal reserves. But the central bank can print money to fill the fiscal deficit.

$$M_t^S = M_{t-1}^S + P_t D_t$$

The central bank is the only source who can print money, so aggregate money in the economy is equal to the money central bank supplies: $M_t = M_t^S$.

Equilibrium

Real variables are still not affected by nominal stuff, i.e. classical dichotomy holds.

Market clearing for goods market:

$$C_t = Y_t - G_t$$

Market clearing for real bonds:

$$K_{t+1} = 0$$

Euler equation for real bonds

$$(1 + r_t)^{-1} = E_t(\frac{\beta C_t}{C_{t+1}})$$

Fiscal balance

$$T_t = G_t - D_t$$

In fact, T_t and D_t are both endogenous, where D_t will be determined by monetary policy.

Nominal variables satisfy the following.

Market clearing for nominal bonds:

$$B_{t+1} = 0.$$

Euler equation for nominal bonds

$$(1 + i_t)^{-1} = E_t\{\frac{\beta C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\}$$

Notice we have come across this "one equation, two variables" problem in the cashless economy. However, now we addtionally have the equation of quantity theory of money

$$\frac{P_t C_t}{M_t} = \frac{1}{\chi}\frac{i_t}{1 + i_t}$$

If we know M_t, then we will have two equations for two variables i_t, P_t.

3.1 Money Supply Rules

This rule directly specifies an **exogenous path for money supply** M_t thus m_t. From above, we know we are able to pin down price level and nominal interest rate: two equations for two endogenous variables.

Using log- forms as before. Combine two Euler equations to get Fisher equation

$$i_t - r_t = E_t(\Delta p_{t+1})$$

Substitute the above into quantity theory:

$$m_t - p_t = c_t - \eta i_t = c_t - \eta(r_t + E_t(\Delta p_{t+1})) \tag{3}$$

$\rightarrow$

$$p_t(1+\eta) = \eta E_t p_{t+1} + m_t - c_t + \eta r_t$$

Rearrange and iterate forward :

$$p_t = (\frac{1}{1+\eta})\sum_{s=0}^{+\infty}(\frac{\eta}{1+\eta})^s E_t(m_{t+s} - c_{t+s} + \eta r_{t+s})$$

This **Cagan equation** implies that price level is determined by money supply, not only today's money supply but also expected money supply in the future. Suppose there is a news that in the future money supply will jump up, price level will immediately jump up as well. Inflation is **always and everywhere a monetary phenomenon (by Milton Friedman)** (later, when we change the way to pin down price, we can see inflation is always and everywhere a fiscal phenomenon...).

This theory has the following implications. With a random walk process $m_t = m_{t-1} + \epsilon_t$, we have $p_t = m_t$ (ignore constant terms), i.e. $p_{t+1} - p_t = m_{t+1} - m_t = \epsilon_t$. Putting more realistic process, we find the model predicts inflation is less serially correlated than money growth and inflation is more volatile than money growth. But these predictions are not consistent with facts.

3.2 Fiscal Rules

Fiscal Policy Suppose now T_t **is chosen exogenously**. Then we can pin down $D_t = G_t - T_t$ (government spending still exogenous).

Monetary Policy

Given D_t above from the fiscal policy, we are able to pin down price level now. This D_t is the additional condition we are going to use to pin down price level (compared to the Monetary rules, where the addtional condition is exogenous M_t).

First, equation (3) says

$$m_t = p_t + c_t - \eta(r_t + E_t(\Delta p_{t+1}))$$

secondly,

$$M_t - M_{t-1} = P_t D_t$$

Substitute the first equation (use it twice) to the second equation to see that it is an equation with only price as unknown. In order to solve it formally, one needs to do log-linearization for the second equation (but since M_t doesn't have steady state, one has to divide both sides by $P_t C_t$ first). The important thing is you know you have one equation, and one unknown p_t.

Monetary policy has to accomodate the fiscal policy, following the law of motion

$$M_t = P_t D_t + M_{t-1} \tag{4}$$

with D_t, P_t, already solved above. That is, money supply's dynamics has to obey this equation. Monetary policy becomes passive. If we have a deficit D_t, the central bank has to print money to fill this gap (collects **seignorage**). However, that would cause inflation based on Cagan equation. Now you see fiscal deficit leads to inflation. **Inflation is always and everywhere a fiscal phenonmenon**.

■

Remark 1. Either monetary rules or fiscal rules can pin down price. However, you cannot employ these two rules at the same time, because that would "over-determine" the equilibrium (more equations than unknown variables). You have to give up one rule.

Remark 2. Suppose the government runs a fiscal deficit, can he always fill this gap by printing money? Notice that although printing money can fill some of this gap, we also know that when money supply increases, price level goes up. Go back to your equation (4), we can see that since price goes up, now the nominal deficit gap will be higher as well. Therefore, it is possible that printing money is not able to fill that gap (notice we assume the deficit D_t is in real terms).

As a matter of fact, assuming Y_t is constant, hence there is no uncertainty, we have

$$\frac{S_t}{C_t} = \frac{M_t - M_{t-1}}{P_t C_t} = \frac{M_t}{P_t C_t} - \frac{M_{t-1}}{P_{t-1} C_{t-1}} \frac{P_{t-1}}{P_t}$$

Since

$$\frac{M_t}{P_t C_t} = \exp(m_t - p_t - c_t) = \exp(-\eta i_t) \approx (1 + i_t)^{-\eta}$$

then

$$\frac{S_t}{C_t} = (1 + i_t)^{-\eta} - (1 + i_{t-1})^{-\eta} \Pi_t^{-1}$$

Since $\beta(1 + r) = 1$ and $(1 + i_t) = (1 + r_t)\Pi_{t+1}$, we get

$$\frac{S_t}{C_t} = \beta^\eta (\Pi_{t+1}^{-\eta} - \Pi_t^{-\eta-1})$$

Assume inflation is at its steady state:

$$\frac{S}{C} = \beta^\eta (\Pi^{-\eta} - \Pi^{-\eta-1})$$

The above has a maximum, taking η as a parameter.

Remark. What is η? It depends on steady state nominal interest rate i, which is affected by steady state inflation as well (real interet rate r already known, by Fisher equation). When I directly substitute $\frac{M_t}{P_t C_t}$ from the quantity theory without log. I still get S/C bounded above, but not bounded below. So here we can just take the quantity theory as an empirical result and η is a fixed parameter, instead of a function of i as we have derived.

4 Fiscal Theory in a Cashless Economy

We have analyzed a cashless economy with active monetary policies (lecture 7), and an economy with cash with active monetary policy or active fiscal policy (lecture 8). However, we haven't discussed a cashless economy with active fiscal policy. Now we come to this scenario (lecture 9).

Monetary Policy There is no cash, thus no seignorage. Central bank doesn't exist: $D_t = 0$. That is we have passive monetary policy.

Fiscal Policy Before, we do have nominal bonds and real bonds, but in equilibrium, net supply is 0. Now suppose government issues bonds to the households, then the net supply to the households will not be 0.

Government budget constraint

$$B_{t+1} + P_t K_{t+1} + P_t T_t = (1 + i_{t-1})B_t + (1 + r_{t-1})P_t K_t$$

Assume $Y_t - G_t$ is constant (Y_t, G_t are exogenous as usual), so by Euler equation for real bonds $(1 + r) = \beta^{-1}$. Denote $\tilde{B}_t = (1 + i_{t-1})B_t + (1 + r_{t-1})P_t K_t$.

Government budget constraint can be re-written as

$$\frac{\tilde{B}_t}{P_t} = (T_t - G_t) + \beta \frac{\tilde{B}_{t+1}}{P_{t+1}} = \sum_{j=0}^{+\infty} \beta^j [T_{t+j} - G_{t+j}]$$

When T_t **is exogenous**, we can pin down price P_t since $\tilde{B}_t$ is a state variable already determined at period $t - 1$. This exogenous T_t is again the "additional condition" we use to pin down price.

Remark 1. We understand this model can pin down price, but in fact, we haven't solved it explicitly since $\{\tilde{B}_t\}$ sequence hasn't been solved. However, the choice of sequence $\{\tilde{B}_t\}$ will be determined by households problem, given $\{P_t\}$ sequence (Fixed point argument).

Remark 2. We have both nominal government bonds B_t and real government bonds K_t. But we can only determine the aggregate bonds $\tilde{B}_t$ (instead of K_t and B_t separately), which is in nominal terms. This nominal debt fills the gap of exogenous fiscal deficit (real deficit). Price will adjust to make sure that nominal debt matches real deficit.

Remark 3. In models we studied before, we only have real bonds. That is $B_t = 0$. Government budget constraint becomes

$$K_t = \sum_{j=0}^{+\infty} \beta^j [T_{t+j} - G_{t+j}]$$

As K_t is a given initial condition, T_{t+j} **becomes endogenous**. Furthermore, we have **Ricardian equivalence**. Since the sequence of $\{T_{t+j}\}$ has to satisfy the above budget constraint, a reduction of tax today means an increase of tax tomorrow, i.e. the present value of tax cannot change.

New Keynesian Model

Macroeconomic Analysis Recitation 12

Yang Jiao*

1 Baseline New Keynesian Model

Households Problem

$$\max_{Cit,h_{it}} E\sum_{t=0}^{+\infty}\beta^t[\log C_t - \frac{\gamma\epsilon}{1+\epsilon}\int_0^1 h_{it}^{1+\frac{1}{\epsilon}}di]$$

s.t.

$$C_t = (\int_0^1 C_{it}^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}}$$

$$\int_0^1 P_{it}C_{it}di + B_{t+1} + P_tK_{t+1} + P_tT_t \leq B_t(1+i_t) + K_t(1+r_t)P_t + \int_0^1 W_{it}h_{it}di + X_t$$

where X_t is the total dividends that distributed by firms to households. Notice we assume that for the households, labor supply h_{it} is differentiated across firms as well.

After solving the dyamic problem, we will get the log-linearized equations (lowercase variables mean log of the original uppercase variables):

Euler equation for real bonds

$$E_t(c_{t+1} - c_t) = r_t$$

Demand for each variety of good

$$c_{it} = c_t - \theta(p_{it} - p_t)$$

Price index

$$p_t = \int_0^1 p_{it}di$$

Labor supply to firm i

$$c_t + h_{it}/\epsilon = w_{it} - p_t$$

Fisher equation (derived from Euler equation for real bonds and nominal bonds)

$$r_t = i_t - E_t(\Delta p_{t+1})$$

*Please email me if you find errors or typos to yj2279@columbia.edu. Your help will be greatly appreciated. All comments and suggestions are especially welcome.

Remark. Price index

$$P_t^{1-\theta} = \int_0^1 P_{it}^{1-\theta} di$$

In steady state $P^* = P_i^*$. Apply our rules for log-linearization (refer to the summation case, and for integrals, it is the same rule)

$$(1-\theta)\hat{p}_t = \int_0^1 \frac{P_i^{*1-\theta}}{P^{*1-\theta}}(1-\theta)\hat{p_{it}} di$$

$\rightarrow$

$$\hat{p}_t = \int_0^1 \hat{p_{it}} di$$

$\rightarrow$

$$p_t = \int_0^1 p_{it} di$$

■

Firms Problem

$$\max_{L_{it}, Y_{it}, P_{it}} (1+\tau)P_{it}Y_{it} - W_{it}L_{it}$$

s.t.

production function

$$Y_{it} = A_{it}L_{it}$$

and demand function

$$Y_{it} = Y_t(P_{it}/P_t)^{-\theta}$$

After solving the problem, we will have the log-linearized equations (lowercase variables mean log of the original uppercase variables)

$$y_{it} = a_{it} + l_{it}$$

$$p_{it} = \log \frac{\theta}{(\theta-1)(1+\tau)} + w_{it} - a_{it}$$

Fiscal Policy $P_tT_t = \tau \int_0^1 P_{it}Y_{it}$, a static budget constraint.

Monetary Policy Reduced form nominal income policy: $N_t = P_tY_t$, i.e. $n_t = p_t + y_t$.

In equilibrium All markets clear: $B_{t+1} = 0$, $K_{t+1} = 0$, $l_{it} = h_{it}$, $c_{it} = y_{it}$.

■

Labor supply

$$c_t + h_{it}/\epsilon = w_{it} - p_t$$

$\rightarrow$

$$c_t + l_{it}/\epsilon = w_{it} - p_t$$

$\rightarrow$

$$c_t + (y_{it} - a_{it})/\epsilon = w_{it} - p_t$$

$\rightarrow$

$$c_t + (c_{it} - a_{it})/\epsilon = w_{it} - p_t$$

$\rightarrow$

$$c_t + (c_t - \theta(p_{it} - p_t) - a_{it})/\epsilon = w_{it} - p_t$$

Use

$$p_{it} = \log \frac{\theta}{(\theta - 1)(1 + \tau)} + w_{it} - a_{it},$$

to substitute w_{it} to get

$$p_{it} = p_t + \mu + \alpha(c_t - a_{it})$$

where $\alpha = \frac{\epsilon+1}{\epsilon+\theta}$. We define $\mu = \log \frac{\theta}{(\theta-1)(1+\tau)}/(1 + \frac{\theta}{\epsilon})$ and we can choose tax rate so that $\mu = 0$.

Summary of equilibrium equations

$$r_t = E_t c_{t+1} - c_t$$

$$r_t = i_t - E_t(\Delta p_{t+1})$$

$$p_{it} = p_t + \alpha(c_t - a_{it})$$

$$p_t = \int_0^1 p_{it} di$$

$$n_t = p_t + y_t$$

Take integrals of $p_{it} = p_t + \alpha(c_t - a_{it})$ to have $c_t = \int_0^1 a_{it} di$, thus $p_{it} = p_t$.

We can see that c_t doesn't depend on nominal stuff and thus y_t, r_t doesn't depend on nominal stuff. Therefore, **classical dichotomy** holds in this economy. Nominal income process n_t is not relevant for real variables.

Note in the above economy, firms are free to adjust their prices. However, if there is physical menu cost k and real rigidity α (large k or small α), nominal rigidity (no change in prices) shows up. It is possible that with nominal rigidity, classical dichotomy breaks down.

Strategic Complementarity

Recall

$$p_{it} = p_t + \mu + \alpha(c_t - a_{it})$$

For simplicity, assume $\mu = 0$ and there is no supply side shock $a_{it} = 0$.

$$p_{it} = p_t + \alpha c_t$$

Since $c_t = y_t$ and $n_t = p_t + y_t$,

$$p_{it} = (1-\alpha)p_t + \alpha n_t$$

That is the desired price of a firm (without other frictions, such as nominal rigidity),

$$p_{it}^* = (1-\alpha)p_t + \alpha n_t$$

It basically says an individual firm wants to set his price close to others if α is small (0¡α¡1): strategic complementarity.

Strategic complementarity will "amplify" aggregate price rigidity, since if some adjust and some not, those who adjust want to be close to those non-adjusters. As a result, aggregate price doesn't change much.

Remark. It just happens that the real rigidity parameter α and the strategic complementarity parameter α are the same. Real rigidity and strategic complementarity are two different concepts. Real rigidity refers to the flatness of the profit function.

2 Phillips Curve

In the data, there is a relationship between nominal variables and real economic activities. Phillips curve describes one important aspect of this relationship

$$\pi_t = \pi_t^e + \kappa y_t$$

Inflation (nominal stuff) can affect real output.

However, in the baseline New Keynesian model (lecture 10), classical dichotomy still holds. We will introduce two frictions in addition to the baseline New Keynesian model to link nominal variables and real variables. One is sticky price model (menu cost): the Calvo model. The other is sticky information model: the Mankiw-Reis Model.

3 Sticky Price: the Calvo Model

Each period with probability λ, a firm can adjust its price. The firm understands that this price may stay in the future, so he takes into account today's price's effect on future periods.

Suppose the firm sets price P_{it} at time t. With probability $(1-\lambda)$, his price is still P_{it} at time $t+1$. With probability $(1-\lambda)^2$, his price is still P_{it} at time $t+2$ ...With probability $(1-\lambda)^k$, his price is still P_{it} at time $t+k$...That's where the following summation comes from. The firm chooses P_{it} to maximize the following:

$$\max_{P_{i,t}} \sum_{k=0}^{+\infty} (1-\lambda)^k E_t\{\frac{M_{t,t+k}}{P_{t+k}}[P_{i,t} - MC_{i,t+k}]Y_{i,t+k}\}$$

s.t.

$$Y_{i,t+k} = Y_{t+k}(P_{it}/P_{t+k})^{-\theta}$$

where $[P_{i,t} - MC_{i,t+k}]Y_{i,t+k}$ is nominal profit. $M_{t,t+k}$ is the stochastic discount factor. The constraint is the demand function for firm i product.

Remark. This stochastic discount factor comes from households' problem. Households trade the firm's stock shares. For example, consider the following household problem

$$\max_{C_t, h_t, q_{i,t+1}} E_t \sum_{t=0}^{+\infty} \beta^t [\log C_t - \frac{\gamma\epsilon}{1+\epsilon} \int_0^1 h_{jt}^{1+1/\epsilon} dj]$$

s.t.

$$P_tC_t+B_{t+1}+P_tK_{t+1}+P_tT_t+\int_0^1 q_{i,t+1}(v_{it}-d_{it})di \leq B_t(1+i_t)+K_t(1+r_t)P_t+\int_0^1 W_{jt}h_{jt}dj+\int_0^1 q_{it}v_{it}di$$

where $q_{i,t+1}$ is the share of stocks of firm i households buy. v_{it} is firm i's stock price before paying dividends d_{it}.

Denote Λ_t the Lagrange multiplier of the budget constraint. F.O.C. for C_t

$$\frac{1}{C_t} = P_t \Lambda_t$$

F.O.C. for $q_{i,t+1}$

$$\Lambda_t (v_{it} - d_{it}) = \beta E_t (\Lambda_{t+1} v_{i,t+1})$$

$\rightarrow$

$$v_{it} = d_{it} + \beta E_t (\frac{\Lambda_{t+1}}{\Lambda_t} v_{i,t+1})$$

Iterate forward to get

$$v_{it} = \sum_{k=0}^{+\infty} \beta^k E_t (\frac{\Lambda_{t+k}}{\Lambda_t} d_{i,t+k})$$

Firm stock price is the discounted dividends of the firm. Now substitute Λ_t to have

$$v_{it} = \sum_{k=0}^{+\infty} E_t(\beta^k \frac{P_tC_t}{P_{t+k}C_{t+k}} d_{i,t+k}) = P_t \sum_{k=0}^{+\infty} E_t(\beta^k \frac{C_t}{P_{t+k}C_{t+k}} d_{i,t+k})$$

Define $M_{t,t+k} = \beta^k \frac{C_t}{C_{t+k}}$, then

$$\frac{v_{it}}{P_t} = \sum_{k=0}^{+\infty} E_t (\frac{M_{t,t+k}}{P_{t+k}} d_{i,t+k})$$

Firms should maximize its stock price (firm value for equity holders).

What is $d_{i,t+k}$ for each firm? Its nominal profits in each period. Then check the above is your firm maximiazation objective for the Calvo model (up to the consideration of changing price probability).

In equilibrium, stock shares should be $q_{it} = 1(100\%)$, then we can see that all dividends $\int_0^1 d_{it}di$ go to households.

■

First order condition of the Calvo firm is (substitute the constraint into the objective function):

$$\sum_{k=0}^{+\infty}(1-\lambda)^k E_t\{\frac{M_{t,t+k}}{P_{t+k}}[(1-\theta)+\frac{\theta}{P_{it}}MC_{i,t+k}]Y_{i,t+k}\}=0$$

i.e.

$$\sum_{k=0}^{+\infty}(1-\lambda)^k E_t\{\frac{M_{t,t+k}}{P_{t+k}}Y_{i,t+k}[P_{it}-\frac{\theta}{\theta-1}MC_{i,t+k}]\}=0 \tag{1}$$

$\rightarrow$

$$P_{it}=\frac{\frac{\theta}{\theta-1}\sum_{k=0}^{+\infty}(1-\lambda)^k E_t\{\frac{M_{t,t+k}}{P_{t+k}}Y_{i,t+k}MC_{i,t+k}\}}{\sum_{k=0}^{+\infty}(1-\lambda)^k E_t\{\frac{M_{t,t+k}}{P_{t+k}}Y_{i,t+k}\}}$$

Log-linearize the above equation to get (apply several formulas of log-linearization, see recitation note 7)

$$p_{it}=\sum_{k=0}^{+\infty}\frac{(1-\lambda)^k\beta^k\frac{MC^*}{P^*}Y^*(m_{t,t+k}+y_{i,t+k}+mc_{i,t+k}-p_{t+k})}{\sum_{s=0}^{+\infty}(1-\lambda)^s\beta^s\frac{MC^*}{P^*}Y^*}-\sum_{k=0}^{+\infty}\frac{(1-\lambda)^k\beta^k\frac{Y^*}{P^*}(m_{t,t+k}+y_{i,t+k}-p_{t+k})}{\sum_{s=0}^{+\infty}(1-\lambda)^s\beta^s\frac{Y^*}{P^*}}$$

i.e.

$$p_{it}=[1-\beta(1-\lambda)]\sum_{k=0}^{+\infty}[\beta(1-\lambda)]^k E_t mc_{i,t+k}$$

Notice under flexible price, price $p^*_{i,t+k}=mc_{i,t+k}$. Therefore,

$$p_{it}=[1-\beta(1-\lambda)]\sum_{k=0}^{+\infty}[\beta(1-\lambda)]^k E_t(p^*_{i,t+k})$$

Substitute t by $t-j$, where $j=0,1,2,...$:

$$p_{i,t-j}=[1-\beta(1-\lambda)]\sum_{k=0}^{+\infty}[\beta(1-\lambda)]^k E_{t-j}(p^*_{i,t-j+k})$$

We can ignore index i, since each firm is subject to the same shock,

$$p_t(j)=[1-\beta(1-\lambda)]\sum_{k=0}^{+\infty}[\beta(1-\lambda)]^k E_{t-j}(p^*_{t-j+k}) \tag{2}$$

At time t, we know in the economy, λ firms set price at time t with price $p_t(0)$, $\lambda(1-\lambda)$ firms set price at time t with price $p_t(1)$, $\lambda(1-\lambda)^2$ firms set price at time t with price $p_t(2)$, etc. Hence

$$p_t=\int_0^1 p_{it}di=\lambda\sum_{j=0}^{+\infty}(1-\lambda)^j p_t(j)$$

which implies

$$p_t=\lambda p_t(0)+(1-\lambda)p_{t-1} \tag{3}$$

Equation (2) implies

$$p_t(0) = [1 - \beta(1-\lambda)]p_t^* + \beta(1-\lambda)E_t[E_{t+1}(p_{t+1}(0))]$$

i.e.

$$p_t(0) = [1 - \beta(1-\lambda)]p_t^* + \beta(1-\lambda)E_t(p_{t+1}(0)) \tag{4}$$

Combine equation (3) and (4) to eliminate $p_t(0)$:

$$(1-\lambda)\pi_t + \lambda p_t = \lambda[1 - \beta(1-\lambda)]p_t^* + \beta(1-\lambda)E_t[(1-\lambda)\pi_{t+1} + \lambda p_{t+1}]$$

Substitute flexible price $p_t^* = p_t + \alpha y_t$ to the above equation:

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa y_t$$

where $\kappa = \frac{[1-\beta(1-\lambda)]\alpha\lambda}{1-\lambda}$.

$\beta \approx 1$, so

$$\pi_t = E_t(\pi_{t+1}) + \kappa y_t$$

which is the Phillips curve.

Additionally, we have

$$y_t = n_t - p_t$$

where n_t is an exogenous process. Assume $n_t = n_{t-1} + \epsilon_t$, our objective is to solve p_t as a function of exogenous shocks.

- **Algebra Method**

  Our Phillips curve can be re-written as

  $$E_t(p_{t+1}) - (2+\kappa)p_t + p_{t-1} = -\kappa n_t$$

  which is a stochastic second-order difference equation.

  Denote L as the lag-operator and denote $F = L^{-1}$. It is easy to show (solve a second-order equation)

  $$(F^2 - (2+\kappa)F + 1)LE_t p_t = -\kappa E_t n_t$$

  i.e.

  $$(1-\theta L)(F - \frac{1}{\theta})E_t p_t = -\kappa E_t n_t$$

  where $\theta < 1$ is the smaller root of $(1-\theta)^2/\theta = \kappa$ (because $\theta + \frac{1}{\theta} = 2 + \kappa$), and $1/\theta > 1$ is the other root.

  That is

  $$(1-\theta L)E_t p_t = -\kappa(F - \frac{1}{\theta})^{-1}E_t n_t = \theta\kappa(1-\theta F)^{-1}E_t(n_t)$$

  $\rightarrow$ (substitute κ by $(1-\theta)^2/\theta$)

  $$(1-\theta L)E_t p_t = (1-\theta)^2(1-\theta F)^{-1}E_t(n_t) = (1-\theta)^2[1 + \theta F + \theta^2 F^2 + ...]E_t n_t$$

$\rightarrow$

$$p_t - \theta p_{t-1} = (1-\theta)^2 \sum_{k=0}^{+\infty} \theta^k E_t(n_{t+k})$$

As n_t is random walk, then $E_t(n_{t+k}) = n_t$, which means

$$p_t = \theta p_{t-1} + (1-\theta) n_t.$$

- **Undetermined Coefficients Method**

  By Wold decomposition, we know

$$p_t = \sum_{j=0}^{+\infty} \theta_j \epsilon_{t-j}$$

By Wold decomposition, $p_t = \sum_{j=0}^{+\infty} \theta_j \varepsilon_{t-j}$, where θ_j are coefficients to be determined. Replace it into the Phillips curve to get

$$E_t \sum_{j=0}^{+\infty} \theta_j \varepsilon_{t+1-j} - (2+\kappa) \sum_{j=0}^{+\infty} \theta_j \varepsilon_{t-j} + \sum_{j=0}^{+\infty} \theta_j \varepsilon_{t-1-j} = -\kappa \sum_{j=0}^{+\infty} \varepsilon_{t-j}$$

i.e.

$$\sum_{j=0}^{+\infty} \theta_{j+1} \varepsilon_{t-j} - (2+\kappa) \sum_{j=0}^{+\infty} \theta_j \varepsilon_{t-j} + \sum_{j=1}^{+\infty} \theta_{j-1} \varepsilon_{t-j} = -\kappa \sum_{j=0}^{+\infty} \varepsilon_{t-j}$$

Compare coefficients to get

$$\theta_{j+1} - (2+\kappa)\theta_j + \theta_{j-1} = -\kappa$$

when $j \geq 1$.

$$\theta_1 - (2+\kappa)\theta_0 = -\kappa$$

when $j = 0$.

$\theta_j \equiv 1$ is a special solution. θ and $1/\theta$ are two roots of $x^2 - (2+\kappa)x + 1 = 0$. Therefore, the general solution is

$$\theta_j = c_1 \theta^j + c_2 (\frac{1}{\theta})^j + 1$$

$c_2 = 0$ to guarantee stationarity. Solve $c_1 = -\theta$ to guarantee that $\theta_1 - (2+\kappa)\theta_0 = -\kappa$. Therefore,

$$\theta_j = 1 - \theta^{j+1}$$

Check that it is the same as in the **algebra method**.

- **Guess and Verify Method**

  Please check by yourself.

  After solving p_t, it is straightforward to get y_t from $y_t = n_t - p_t$.

4 Sticky Information: the Mankiw-Reis Model

Each period, λ firms obtain up-to-date information. Therefore, at period t, λ firms have period t information set, $(1-\lambda)\lambda$ have period $t-1$ information set, $(1-\lambda)^2\lambda$ firms have period $t-2$ information set etc.

$$p_t = \int_0^1 p_{it} di = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-j}[p_t + \alpha y_t] \tag{5}$$

Expand the above with the first term before the summation of other terms

$$p_t = \lambda E_t[p_t + \alpha y_t] + \lambda \sum_{j=1}^{+\infty} (1-\lambda)^j E_{t-j}[p_t + \alpha y_t] \tag{6}$$

$$= \lambda E_t[p_t + \alpha y_t] + (1-\lambda)\lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-1-j}[p_t + \alpha y_t] \tag{7}$$

$\rightarrow$

$$p_t = \frac{\lambda\alpha}{1-\lambda} y_t + \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-1-j}[p_t + \alpha y_t] \tag{8}$$

Additionally, write equation (5) for time $t-1$ to get

$$p_{t-1} = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-1-j}[p_{t-1} + \alpha y_{t-1}] \tag{9}$$

Substract equation (8) by equation (9) to obtain the Phillips curve for sticky information model.

$$\pi_t = \frac{\lambda\alpha}{1-\lambda} y_t + \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-1-j}[\pi_t + \alpha \Delta y_t]$$

Assume exogenous $n_t = \sum_{k=0}^{+\infty} \gamma_k \epsilon_{t-k}$, we would like to derive the solution for p_t. Note when $\gamma_k = 1$, n_t is a random walk.

- Undetermined Coefficients Method

  By Wold decomposition,

$$p_t = \sum_{k=0}^{+\infty} \theta_k \epsilon_{t-k}$$

  where θ_k are coefficients to be determined.

  Go back to equation (5).

$$p_t = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-j}[p_t + \alpha y_t] = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-j}[\alpha n_t + (1-\alpha) p_t]$$

  $\rightarrow$

$$\sum_{k=0}^{+\infty} \theta_k \epsilon_{t-k} = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-j}[\alpha \sum_{k=0}^{+\infty} \gamma_k \epsilon_{t-k} + (1-\alpha) \sum_{k=0}^{+\infty} \theta_k \epsilon_{t-k}]$$

  $\rightarrow$

$$\sum_{k=0}^{+\infty} \theta_k \epsilon_{t-k} = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j \sum_{k=j}^{+\infty} [\alpha \gamma_k \epsilon_{t-k} + (1-\alpha) \theta_k \epsilon_{t-k}] = \sum_{k=0}^{+\infty} \sum_{j=0}^{k} \lambda (1-\lambda)^j [\alpha \gamma_k + (1-\alpha) \theta_k]$$

  Compare coefficients to have

$$\theta_k = \lambda \sum_{j=0}^{k} (1-\lambda)^j [\alpha \gamma_k + (1-\alpha) \theta_k]$$

  $\rightarrow$

$$\theta_k = \frac{\alpha[1-(1-\lambda)^{k+1}]}{\alpha + (1-\alpha)(1-\lambda)^{k+1}} \gamma_k$$

  After solving p_t, it is straightforward to get y_t from $y_t = n_t - p_t$.

  When $\gamma_k = 1$, i.e. n_t is a random walk,

$$\theta_k = \frac{\alpha[1-(1-\lambda)^{k+1}]}{\alpha + (1-\alpha)(1-\lambda)^{k+1}}$$

  Inflation

$$\pi_t = p_t - p_{t-1} = \theta_0 \epsilon_t + \sum_{k=1}^{+\infty} (\theta_k - \theta_{k-1}) \epsilon_{t-k}$$

5 Comparison between Sticky Price and Sticky Information

They both generate **inflation persistence and less volatile inflation** compared to money growth. When exogenous shocks arrive, only some firms adjust, so the volatility of inflation is smaller than the volatility of money growth. Inflation persistence is related to the fact that some firms adjust today, some firms adjust tomorrow, and we have strategic complementarity, firms price near to other firms' prices.

Additionally, if there is a **negative shock** to n_t, price will not fall that much, $y_t = n_t - p_t$ predicts both models produce a **recession**, which is consistent with facts.

However, the Calvo model produces some undesirable results.

- Natural Rate property

  In steady state:

$$\pi = \beta\pi + \kappa y \rightarrow y = \frac{(1-\beta)\pi}{\kappa}$$

  It is a **non-vertical long run phillip curve**, which means you can use long run high inflation to achieve high real output. In the long run, we would like monetary policy to be neutral.

  Even if some may argue that β is near 1, so that $y = 0$. One can use an alternative monetary policy $\pi_{t+1} = (1-g)\pi_t$ to get $\pi_t = (\kappa/g)y_t$. If g is very high, y_t will also be very high.

- Accelerationist Problem

  A simple example to illustrate why **change in inflation and real output are negatively correlated by the Calvo model**: suppose that the government announces a disinflation $\pi_{t+1} < \pi_t$(caused by contractionary monetary shocks), recall that the Phillips curve is

$$\pi_t = E_t\pi_{t+1} + \kappa y_t$$

  $\rightarrow$

$$\pi_t - \pi_{t+1} = \kappa y_t > 0$$

  That is the Calvo model predicts that the correlation between change in inflation and output is negative.

  This is at odds with data.

  Remark. Acceleration phenomenon means the positive correlation between real output and change in inflation in the data, i.e. when real output is high, inflation tends to rise.

- Non Hump-Shape

  Iterate forward the Phillips Curve:

$$\pi_t = \kappa \sum_{k=0}^{+\infty} \beta^k E_t(y_{t+k})$$

  When the economy gets hit by a negative shock to n_t, and the shock dies out gradually. Real output also dies out gradually. Then the above Phillips curve would predict a **non-**

hump shaped path for inflation. This result is not in line with empirical evidence, where the response of inflation to monetary shocks tends to be hump-shaped.

However, the above three problems of the Calvo model are not problems for the Mankiw-Reis model. Let's investigate the above three points one by one.

- From equation (8), in steady state, we will have $y = 0$. So in the steady state, money is neutral.

- Suppose the government announces a disinflation in the future (caused by contractionary monetary shocks), firms don't change their pricing before that shock happens. The reason is as follows:

$$p_t = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-j}[p_t + \alpha y_t] = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j E_{t-j}[\alpha n_t + (1-\alpha)p_t]$$

  p_t is not related to future n_{t+j} or expectation about n_{t+j}, $j \geq 1$.

  However, when the disinflation comes at time $t+1$, firms start to decrease price (still, not too much, since only some firms know it), $y_t = n_t - p_t < 0$, which is a recession. This experiment means we have a positive correlation between real output and change in inflation.

  Remark. For comparison, let's consider the intuition for the Calvo result: in the Calvo model, when government announces disinflation in the future, if firms have the opportunity to adjust prices, they will immediately respond. Because they understand in the future they may not have opportunity to decrease their price to accomodate with the contractionary monetary shock, they immediately decrease their price. Now their n_t is still 0 as in previous periods, so when they get the news (announcement by the government), $y_t = n_t - p_t > 0$ means the economy will have a boom. This is a negative relationship between real output and change in inflation.

- Hump shaped response of inflation can be achieved in the Mankiw-Reis model. Suppose there is a positive monetary shock.

  In the sticky information model, those firms who get the information will increase price initially, but not so much. They understand that now not so many firms know it yet. They will "wait for others" and increase their price further when more firms get informed (due to strategic complementarity). Therefore, inflation response is hump-shaped. Note sticky information doesn't mean a flat price plan.

  While for the sticky price model, those who have the opportunity to adjust prices, will respond more than the sticky information case. They really value this opportunity to change price. They will not wait for others, since if they wait, they may be stuck in that price in the future.

Dynamic New Keynesian Model–Optimal Monetary Policy

Macroeconomic Analysis Recitation 13

Yang Jiao*

1 Dynamic New Keynesian Model

1.1 Phillips Curve

Recall that in recitation 12, we already derived firm pricing under flexible price:

$$p_{it}^* = p_t + \mu + \alpha(y_t - a_{it})$$

Now assume that all firms are subject to the common productivity shock so that $a_t = a_{it}$. Furthermore, let's assume there is government spending g_t as well, so we have

$$p_{it}^* = p_t + \mu + \alpha(y_t - a_{it} - \eta g_{it}) \tag{1}$$

Remark. When introducing government spending, let's repeat similar procedures on page 2-3 of recitation 12.

Labor supply

$$c_t + l_{it}/\epsilon = w_{it} - p_t$$

$\rightarrow$

$$y_t - g_t + (y_{it} - a_{it})/\epsilon = w_{it} - p_t$$

$\rightarrow$

$$y_t - g_t + [y_t - \theta(p_{it} - p_t) - a_{it}]/\epsilon = w_{it} - p_t$$

Use

$$p_{it} = \log \frac{\theta}{(\theta-1)(1+\tau)} + w_{it} - a_{it}$$

to substitute w_{it} to get

$$p_{it} = p_t + \mu + \alpha(y_t - a_{it} - \eta g_t)$$

where $\alpha = \frac{\epsilon+1}{\epsilon+\theta}$ and $\eta = \frac{\epsilon}{\epsilon+1}$, $\mu = \log \frac{\theta}{(\theta-1)(1+\tau)}/(1+\frac{\theta}{\epsilon})$. With common shocks to productivity

*Please email me if you find errors or typos to yj2279@columbia.edu. Your help will be greatly appreciated. All comments and suggestions are especially welcome.

$a_t = a_{it}$.

$$p_{it} = p_t + \mu + \alpha(y_t - a_t - \eta g_t) \tag{2}$$

■

Natural rate of output is defined as the equilibrium output when prices are flexible and markups (price markup and wage markup) are constant. Integrate both sides of equation (1) to get

$$y_t^n = a_t + \eta g_t - \frac{\mu}{\alpha}$$

As before, when we introduce sticky price, we will have a Phillips Curve. Recall that in recitation note 12, when we derive the Phillips curve, we assume $a_t = 0$. Now suppose, $a_t + \eta g_t$ is not a constant 0, then the Phillips curve will change to (repeat similar procedures as before, in fact we will show the derivation in the Remark below.)

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa \hat{y}_t$$

with **output gap** $\hat{y}_t = y_t - y_t^n$.

Also, we will introduce exogenous markup shock (price markup or wage markup) so that the Phillips curve will have an additional shocks u_t:

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa \hat{y}_t + u_t$$

Remark.

Suppose the elasticity of substitution θ is no longer a constant but a stochastic process. Go back to page 6 of recitation 12, the equation just below equation (1). Substitute θ by θ_t. Define $\chi_t = \log(\frac{\theta_t}{\theta_t - 1})$, and steady state $\chi = \log(\frac{\theta}{\theta - 1})$. After log-linearization, we will have

$$p_{it} = [1 - \beta(1-\lambda)] \sum_{k=0}^{+\infty} [\beta(1-\lambda)]^k E_t(mc_{i,t+k} + \chi_t)$$

Notice that with shocks to θ_t, under flexible price $p^*_{i,t+k} = mc_{i,t+k} + \chi_t \rightarrow$

$$p_{it} = [1 - \beta(1-\lambda)] \sum_{k=0}^{+\infty} [\beta(1-\lambda)]^k E_t(p^*_{i,t+k})$$

Follow the similar steps as in recitation 12 to get (a same equation as in recitation 12)

$$(1-\lambda)\pi_t + \lambda p_t = \lambda[1 - \beta(1-\lambda)]p_t^* + \beta(1-\lambda)E_t[(1-\lambda)\pi_{t+1} + \lambda p_{t+1}] \tag{3}$$

For simplicity, let's assume $\epsilon \rightarrow +\infty$ so that labor supply is not differentiated. Set tax rate $\tau = 0$ or τ is a constant. Now the flexible price $p_t^* = p_t + \log(\frac{\theta_t}{\theta_t - 1}) + y_t - y_t^n = p_t + \chi_t + y_t - y_t^n - \chi$ (refer to equation (2), use θ_t to substitute θ and let $\epsilon \rightarrow +\infty$).

Substitute p_t^* into equation (3), we will have the Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + \frac{\lambda[1 - \beta(1-\lambda)]}{(1-\lambda)}(y_t - y_t^n + \chi_t - \chi)$$

i.e.

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(y_t - y_t^n) + u_t$$

where $\kappa = \frac{\lambda[1-\beta(1-\lambda)]}{(1-\lambda)}$ and $u_t = \kappa(\chi_t - \chi)$.

$\rightarrow$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t + u_t$$

with $\hat{y}_t = y_t - y_t^n$. Now we can see that with markup shocks, the Phillips curve will have an additional shock term u_t. The Phillips curve comes from firms optimization problem, so we sometimes also call u_t supply shocks.

There are also other ways to introduce shocks to the Phillips curve, e.g. shocks to wage markup will also introduce additional shocks to the Phillips curve. ■

1.2 IS Curve

Now we turn to the demand side. The Euler equation for nominal bonds

$$u'(c_t) = \beta(1+i_t)E_t(u'(c_{t+1})\frac{P_t}{P_{t+1}})$$

We will introduce preference shocks as well: $u(c_t, h_{it}) = \xi_t[\log C_t - \frac{\gamma\epsilon}{1+\epsilon}\int_0^1 h_{it}^{1+\frac{1}{\epsilon}} di]$, then we will get

$$c_t = E_t(c_{t+1}) - (i_t - E_t(\pi_{t+1})) + E_t(\xi_{t+1}) - \xi_t$$

With government spending, market clearing condition is $y_t = c_t + g_t \rightarrow$

$$y_t = E_t(y_{t+1}) - (i_t - E_t(\pi_{t+1})) + E_t(\xi_{t+1}) - \xi_t + E_t(g_{t+1}) - g_t$$

Define **the natural rate of interest** as

$$r_t^n = E_t(y_{t+1}^n - y_t^n) + E_t(\xi_{t+1}) - \xi_t + E_t(g_{t+1}) - g_t$$

It is also sometimes called **Wicksellian rate**.

Therefore, the **IS curve** is

$$\hat{y}_t = E_t(\hat{y}_{t+1}) - (i_t - E_t(\pi_{t+1}) - r_t^n)$$

We can substitute $i_t - E_t(\pi_{t+1})$ by r_t:

$$\hat{y}_t = E_t(\hat{y}_{t+1}) - (r_t - r_t^n)$$

Iterate forward to get

$$\hat{y}_t = -\sum_{k=0}^{+\infty} E_t(r_{t+k} - r_{t+k}^n)$$

which establishes the relationship between today's output gap and the long term interest rate.

Note the Phillips curve and the IS curve are not enough to nail down the equilibrium, since we have three variables $\pi_t, \hat{y}_t$ and i_t. We will use a policy rule below as an additional equation.

1.3 Taylor Rule

The Taylor rule is

$$i_t = r_t^n + \phi_\pi \pi_t + \phi_y \hat{y}_t + v_t$$

or

$$i_t = \pi_t + r_t^n + (\phi_\pi - 1)\pi_t + \phi_y \hat{y}_t + v_t$$

In the long run, we want $\pi_t = 0$ (zero inflation steady state). The above rule satisfies this property since when $r_t = r_t^n$, $\hat{y}_t = 0$, $v_t = 0$, we will get $\pi_t = 0$.

Assume $\phi_\pi > 1$ to ensure a determinate price or inflation (refer to recitation 11).

We also assume $\phi_y > 0$ so that when output gap is positive, the policy will increase nominal interest rate (thus real interest rate, since price is sticky, inflation doesn't change much), so that demand for today's consumption is lower (compared to the future), leading to lower output today. It provides stabilization of output.

In sum, we obtain the Dynamic New Keynesian model: three equations for three variables,

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t + u_t$$

$$\hat{y}_t = E_t(\hat{y}_{t+1}) - (i_t - E_t(\pi_{t+1}) - r_t^n)$$
$$i_t = r_t^n + \phi_\pi \pi_t + \phi_y \hat{y}_t + v_t$$

The solution will be $(\hat{y}_t, \pi_t, i_t)$ as functions of exogenous process (v_t, u_t, r_t^n). In fact, one can put the last equation into the second equation to eliminate i_t, so that we have

$$\hat{y}_t = E_t(\hat{y}_{t+1}) + E_t(\pi_{t+1}) - (\phi_\pi \pi_t + \phi_y \hat{y}_t + v_t)$$

Together with the NK Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t + u_t$$

Two equations for two variables: $\{\pi_t, \hat{y}_t\}$. To guarantee stability, we have to restrict the parameter space. For discussions about stability, one can refer to the Blanchard-Khan condition after writing the system in matrix representation form.

2 Optimal Monetary Policy

The previous section takes monetary policy as given. This section will look at how to design optimal monetary policy. So we will keep the Phillips curve and IS curve as constraints but drop the monetary policy, compared to the previous section. The two constraints are:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t + u_t$$
$$\hat{y}_t = E_t(\hat{y}_{t+1}) - (i_t - E_t(\pi_{t+1}) - r_t^n)$$

The objective function is to minimize a loss function

$$L = E\sum_{t=0}^{+\infty}\beta^t[\pi_t^2 + \lambda(\hat{y}_t - \hat{y^*})^2]$$

by choosing $\{\pi_t\}$ and $\{\hat{y}_t\}$, where λ and $\hat{y^*}$ are constants (here λ does not represent the Calvo changing price probability, it is only a constant). $\hat{y^*}$ is non-zero when there is monopoly distortion in steady state (remember we have monopolistic competitive firms). See the Remark below for how to derive the loss function (basically, second order approximation) and the details about the two constants.

Intuitively, inflation brings about price dispersion which means resource misallocation (first term) but inflation can also push up output to correct monopolistic distortions (second term).

Remark. (You don't need to know the following derivation to prepare for exams, just let you know how it works) For simplicity, we again **assume labor supply is not differentiated**, so that $H_t = \int_0^1 H_{it}di$ (then the utility function is $U(C,H) = \log C - \gamma H$, you can change it to more complicated $U(C,H) = \frac{C^{1-\sigma}}{1-\sigma} - \gamma\frac{H^{1+\varphi}}{1+\varphi}$) and since labor is homogeneous, $w_t = w_{it}$ as well. We will **shut down preference shock and ignore government spending** as well (but you can add them for exercise).

A representative household's utility function is $U(C,H)$. Do a second order Taylor expansion around steady state (C,H).

$$U_t - U \approx U_cC\frac{C_t - C}{C} + U_hH\frac{H_t - H}{H} + \frac{1}{2}U_{cc}C^2(\frac{C_t - C}{C})^2 + \frac{1}{2}U_{hh}H^2(\frac{H_t - H}{H})^2$$

Note that we have assumed utility is seperable in consumption and leisure so that $U_{ch} = 0$. We will employ the following second order approximation for variable X_t:

$$\frac{X_t - X}{X} = \frac{Xe^{\bar{x}_t} - X}{X} = e^{\bar{x}_t} - 1 \approx \bar{x}_t + \frac{1}{2}\bar{x}_t{}^2$$

where the log-deviation $\bar{x}_t = \log(X_t) - \log X$. Therefore, up to second order, we get

$$U_t - U \approx U_cC(\bar{c}_t + \frac{1-\sigma}{2}\bar{c}_t^2) + U_hH(\bar{h}_t + \frac{1+\varphi}{2}\bar{h}_t^2)$$

On the other hand,

$$H_t = \int_0^1 H_{it}di = \int_0^1 \frac{Y_{it}}{A_t}di = \frac{Y_t}{A_t}\int_0^1 (\frac{P_{it}}{P_t})^{-\theta}di$$

$\rightarrow$ (take log)

$$\bar{h}_t = \bar{y}_t - a_t + \log(\int_0^1 (\frac{P_{it}}{P_t})^{-\theta}di) \approx \bar{y}_t - a_t + \frac{\theta}{2}var_i\{p_{it}\}.$$

(for the price dispersion term above, see e.g. Gali (2008) book page 62) and $\bar{c}_t = \bar{y}_t$, $\rightarrow$

$$U_t - U = U_c C(\bar{y}_t + \frac{1-\sigma}{2}\bar{y}_t^2) + U_h H[\bar{y}_t - a_t + \frac{\theta}{2}var_i\{p_{it}\} + \frac{1+\varphi}{2}(\bar{y}_t - a_t)^2$$
$$+(1+\varphi)(\bar{y}_t - a_t)\theta var_i\{p_{it}\} + \frac{1+\varphi}{8}\theta var_i^2\{p_{it}\}]$$

Ignore terms higher than second order:

$$U_t - U = U_c C(\bar{y}_t + \frac{1-\sigma}{2}\bar{y}_t^2) + U_h H[\bar{y}_t - a_t + \frac{\theta}{2}var_i\{p_{it}\} + \frac{1+\varphi}{2}(\bar{y}_t - a_t)^2]$$

Policy cannot affect a_t term, so ignore it as well,

$$U_t - U = U_c C(\bar{y}_t + \frac{1-\sigma}{2}\bar{y}_t^2) + U_h H[\bar{y}_t + \frac{\theta}{2}var_i\{p_{it}\} + \frac{1+\varphi}{2}(\bar{y}_t - a_t)^2]$$

$\rightarrow$

$$\frac{U_t - U}{U_c C} = \bar{y}_t + \frac{1-\sigma}{2}\bar{y}_t^2 + \frac{U_h H}{U_c C}[\bar{y}_t + \frac{\theta}{2}var_i\{p_{it}\} + \frac{1+\varphi}{2}(\bar{y}_t - a_t)^2]$$

Since $-\frac{U_h}{U_c} = W/P$ and $C = Y = \frac{\theta}{\theta-1}W/PH$ (recall θ is the elasticity of substitution across varieties),

$$\frac{U_t - U}{U_c C} = \bar{y}_t + \frac{1-\sigma}{2}\bar{y}_t^2 - \frac{\theta-1}{\theta}[\bar{y}_t + \frac{\theta}{2}var_i\{p_{it}\} + \frac{1+\varphi}{2}(\bar{y}_t - a_t)^2]$$

$$\frac{U_t - U}{U_c C} = \frac{1}{\theta}\bar{y}_t + \frac{1-\sigma}{2}\bar{y}_t^2 - \frac{\theta-1}{\theta}[\frac{\theta}{2}var_i\{p_{it}\} + \frac{1+\varphi}{2}(\bar{y}_t - a_t)^2]$$

Again ignore terms that policy cannot affect (policy cannot affect a_t but can affect $\bar{y}_t$) to get

$$\frac{U_t - U}{U_c C} = \frac{1}{\theta}\bar{y}_t - \frac{\theta-1}{2}var_i\{p_{it}\} - \frac{\sigma-1}{2}\bar{y}_t^2 - \frac{\varphi+1}{2}\frac{\theta-1}{\theta}\bar{y}_t^2 + (\varphi+1)\frac{\theta-1}{\theta}\bar{y}_t a_t$$

$\rightarrow$

$$\frac{U_t - U}{U_c C} = -\frac{1}{2}\{(\theta-1)var_i\{p_{it}\} + [\sigma - 1 + \frac{(\varphi+1)(\theta-1)}{\theta}]\bar{y}_t^2 - \frac{2 + 2(\varphi+1)(\theta-1)a_t}{\theta}\bar{y}_t\}$$

To be consistent with what we have used before, we take $U(C, H) = \log C - \gamma H$, so that $\sigma = 1$ and $\varphi = 0$ (caution: if we want to use a more general utility function form, e.g. $U(C, H) = \frac{C^{1-\sigma}}{1-\sigma} - \gamma\frac{H^{1+\varphi}}{1+\varphi}$, we have to re-derive the natural rate of output, it is not necessarily $y_t^n = a_t - \log\frac{\theta}{\theta-1}$). Then

$$\frac{U_t - U}{U_c C} = -\frac{1}{2}\{(\theta-1)var_i\{p_{it}\} + \frac{\theta-1}{\theta}\bar{y}_t^2 - \frac{2 + 2(\theta-1)a_t}{\theta}\bar{y}_t\}$$

$\rightarrow$

$$\frac{U_t - U}{U_c C} = -\frac{1}{2}\{(\theta-1)var_i\{p_{it}\} + \frac{\theta-1}{\theta}(\bar{y}_t - a_t - \frac{1}{\theta-1})^2\}$$

Denote $\bar{x}_t = \bar{y}_t - a_t$, the above turns to

$$\frac{U_t - U}{U_c C} = -\frac{1}{2}\{(\theta - 1)var_i\{p_{it}\} + \frac{\theta - 1}{\theta}(\bar{x}_t - \frac{1}{\theta - 1})^2\}$$

From Woodford (2003) book Chapter 6, $\sum_0^\infty \beta^t var_i\{p_{it}\} = \frac{1-\lambda}{(1-\beta(1-\lambda))\lambda}\sum_0^\infty \beta^t \pi_t^2$

The welfare gain (as a fraction of steady state consumption) is

$$WG = E_0 \sum_0^\infty \beta^t \frac{U_t - U}{U_c C}$$

$\rightarrow$

$$WG = -\frac{1}{2}E_0 \sum_0^\infty \beta^t [\frac{(1-\lambda)(\theta - 1)}{(1 - \beta(1-\lambda))\lambda}\pi_t^2 + \frac{\theta - 1}{\theta}(\bar{x}_t - \frac{1}{\theta - 1})^2]$$

Then the loss function $L = -WG$. Now let $\hat{y^*} = \frac{1}{\theta - 1}$ and notice that $\bar{x}_t = \bar{y}_t - a_t = \bar{y}_t - \log\frac{\theta}{\theta-1} - (a_t - \log\frac{\theta}{\theta-1}) = y_t - y_t^n$ which is in fact, the **output gap** $\hat{y}_t$ we defined on page 2. ■

2.1 No cost-push shocks and $\hat{y^*} = 0$

Consider a case without markup shocks, i.e. we shut down u_t. The problem becomes:

$$\min E \sum_{t=0}^{+\infty} \beta^t [\pi_t^2 + \lambda(\hat{y}_t - \hat{y^*})^2]$$

s.t.

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t$$

$$\hat{y}_t = E_t(\hat{y}_{t+1}) - (i_t - E_t(\pi_{t+1}) - r_t^n)$$

Notice that in the objective function i_t doesn't appear and it only appears in the second constraint. Therefore, we can drop the second constraint.

If $\hat{y^*} = 0$ (for example, tax already corrects the distortion), the solution is $\pi_t = 0$ and $\hat{y}_t = 0$. We obtain **the divine coincidence**: the optimal solution shows there is no trade-off between stabilizing inflation and stabilizing output gap. Note that $\hat{y}_t = 0$ doesn't mean $y_t = 0$. With $\hat{y}_n = 0$, we get $y_t = y_t^n$, that is output moves one-to-one with natural level of output, which can be affected by e.g. technology. As with nominal interest rate, we go back to the second constraint to see that $i_t = r_t^n$, which implies the policy rule should give a coefficient of 1. Taylor rule satisfies this property.

In sum, the optimal policy is to stabilize inflation and let output fluctuate with the natural level. Although we have a Phillips curve which features trade-off between inflation and output gap, the optimal policy is not to exploit it.

2.2 With cost-push shocks and $\hat{y^*} > 0$

With cost-push shocks u_t and $\hat{y^*} > 0$, the Lagrangean problem is to minimize

$$E\sum_{t=0}^{+\infty}\{[\pi_t^2 + \lambda(\hat{y}_t - \hat{y^*})^2 + 2\varphi_t[\pi_t - \beta\pi_{t+1} - \kappa\hat{y}_t - u_t]]\}$$

First order conditions are

$$\pi_t + \varphi_t - \varphi_{t-1} = 0$$
$$\lambda(\hat{y}_t - \hat{y^*}) - \kappa\varphi_t = 0$$

Combining the above two gives

$$\pi_t + \frac{\lambda}{\kappa}(\hat{y}_t - \hat{y}_{t-1}) = 0$$

with initial $\pi_0 + \frac{\lambda}{\kappa}(\hat{y}_0 - \hat{y^*}) = 0$. Note that $\varphi_{-1} = 0$ since policy hasn't kicked in (assume the policy design is conducted from $t = 0$).

Since $\pi_t = p_t - p_{t-1}$, we get

$$p_t + \frac{\lambda}{\kappa}\hat{y}_t = p_{t-1} + \frac{\lambda}{\kappa}\hat{y}_{t-1} = ... = p_{-1} + \frac{\lambda}{\kappa}\hat{y^*}$$

which is a constant. This rule is **price level targeting**.

Plugging the F.O.C.s into the Phillips curve, we have

$$\varphi_t = \mu\varphi_{t-1} - \mu\sum_{j=0}^{+\infty}\beta^j\mu^j(\kappa\hat{y}^* + E_t u_{t+j})$$

where $0 < \mu < 1$ is a constant.

Without shocks, φ_t converges to a constant. So $\pi_t = \varphi_{t-1} - \varphi_t$ converges to 0. Notice that it gradually converges to 0 instead of a flat line.

We have assumed commitment so far, i.e. the optimal policy plan is set at time 0 and the central bank sticks to the original plan. Next we will consider the case without commitment.

2.3 Discretion

Policy will re-optimize each time, given its expectation $E_t(\pi_{t+1}) = \pi_t^e$ for tomorrow. The problem can be written as

$$\min \pi_t^2 + \lambda(\hat{y}_t - \hat{y^*})^2$$

s.t.

$$\pi_t = \beta\pi_t^e + \kappa\hat{y}_t + u_t$$

F.O.C. is

$$\pi_t = \frac{\lambda}{\lambda + \kappa^2}(\kappa\hat{y}^* + \beta\pi_t^e + u_t)$$

Since $E_t(\pi_{t+1}) = \pi_t^e$, we iterate forward the above to get

$$\pi_t = \frac{\lambda}{\lambda + \kappa^2} \sum_{j=0}^{+\infty} \beta^j (\frac{\lambda}{\lambda + \kappa^2})^j (\kappa \hat{y}^* + E_t u_{t+j})$$

Without any shocks, it is a flat line with positive value.

Discretion gives higher inflation than the commitment case in the long run or under a positive cost-push shock. The intuition is that under discretion, the policy maker doesn't internalize their choice of high inflation will push up previous period's inflation. The temptation to inflate comes from the incentive to get closer to $\hat{y}^*$ or to offset markup shocks u_t (u_t increase means markup increase, i.e. more monopoly distortions, which requires higher inflation to correct the distortion).