# Ordinary Differential Equations

Macroeconomic Analysis Recitation 1

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## 1 Introduction

We will cover some basics of ordinary differential equations (ODE). Within this class, we deal with differential equations, whose variable of interest takes derivative with respect to time $t$. Denote $\dot{Y}_t = \frac{dY_t}{dt}$, where $Y_t$ can be a scalar or vector. A general explicit form of ODE is

$$\dot{Y}_t = f(Y_t, t) \tag{1}$$

## 2 First-Order Differential Equations

- Autonomous equation: $\dot{y}_t = f(y_t)$, an equation is autonomous when it depends on time only through the variable itself. Example: $k_t = sk_t^\alpha - \delta k_t$, where $s, \alpha$ and $\delta$ are constants.

- Linear equation: $\dot{y}_t = a_t y_t + b_t$, where $a_t$ and $b_t$ are taken as given. Example: $\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}(r_t - \rho)$, where $\gamma$ and $\rho$ are parameters, while $r_t$ is a given function of $t$.

- Homogeneous: set the above linear differential equation $b_t = 0$. This terminology also applies to high-order differential equations: e.g. $\ddot{y}_t = g_t \dot{y}_t + h_t y_t$.

Autonomous equation can be solved (illustrated) graphically, while linear equation admits analytical solution.

### 2.1 Analytical Solution

#### A homogeneous differential equation

$$\dot{y}_t = a_t y_t \tag{2}$$

Divide both sides by $y_t$,

$$\frac{\dot{y}_t}{y_t} = a_t \tag{3}$$

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$$\Rightarrow \frac{dlog(y_t)}{dt} = a_t \tag{4}$$

Therefore,

$$y_t = C\exp(\int_0^t a_s ds) \tag{5}$$

where $C$ is determined by boundary condition.

**A linear differential equation**

$$\dot{y}_t = a_t y_t + b_t \tag{6}$$

Rearrange the above equation as

$$\dot{y}_t - a_t y_t = b_t \tag{7}$$

Multiply both sides by $\exp(-\int_0^t a_s ds)$, we obtain

$$\dot{y}_t \exp(-\int_0^t a_s ds) - a_t y_t \exp(-\int_0^t a_s ds) = b_t \exp(-\int_0^t a_s ds) \tag{8}$$

That is

$$\frac{d[y_t \exp(-\int_0^t a_s ds)]}{dt} = b_t \exp(-\int_0^t a_s ds) \tag{9}$$

$$\Rightarrow y_t = \exp(\int_0^t a_s ds)(\int_0^t b_u \exp(-\int_0^u a_s ds)du + C) \tag{10}$$

where $C$ is pinned down by boundary condition.

**Example**

$$\dot{k}_t = sk_t^{\alpha} - \delta k_t \tag{11}$$

Define $z_t = k_t^{1-\alpha}$, then

$$\dot{z}_t = (1-\alpha)k_t^{-\alpha}\dot{k}_t \tag{12}$$

Substitute $\dot{k}_t, k_t$ by $\dot{z}_t, z_t$, we arrive at

$$\dot{z}_t = s(1-\alpha) - \delta(1-\alpha)z_t \tag{13}$$

Let $a_t = -\delta(1-\alpha)$ and $b_t = s(1-\alpha)$, and use the solution we already get in the linear differential equation.

$$z_t = \frac{s}{\delta} + (z_0 - \frac{s}{\delta})\exp[-\delta(1-\alpha)t] \tag{14}$$

$$\Rightarrow k_t = \{\frac{s}{\delta} + (k_0^{1-\alpha} - \frac{s}{\delta})\exp[-\delta(1-\alpha)t]\}^{\frac{1}{1-\alpha}} \tag{15}$$

## 2.2 Graphical Solution

An autonomous equation

$$\dot{y}_t = f(y_t) \tag{16}$$

**Solution Steps:**

- Plot $f(y)$ in the space of $(y, \dot{y})$. Put $y$ on the x-axis and $\dot{y}$ on the y-axis.
- Draw rightward arrows when $f > 0$, and leftward arrows when $f < 0$.
- Given an initial point, follow the arrows to track the dynamics of $y$.

**Equilibrium and Stability:** When $\dot{y}_t = 0$, we have $f(y) = 0$. The solutions to $f(y_t) = 0$ are equilibrium points. An equilibrium point $y^*$ is said to be stable if $f'(y^*) < 0$ (or equivelently, write it as $\frac{\partial \dot{y}}{\partial y}|_{y^*} < 0$). Intuitively, the arrows around the equilibrium point direct to the stable equilibrium point. Or roughly speaking, after a small perturbation, $y_t$ will finally go back to the equilibrium point.

**Examples:**

- $\dot{y}_t = ay_t + b$, where $a < 0$. The equilibrium point is $y^* = -\frac{b}{a}$. Since $\frac{\partial \dot{y}}{\partial y}|_{y^*} = a < 0$, this equilibrium point is stable. See Figure 1. (Figures are on the last two pages.)
- $\dot{y}_t = ay_t + b$, where $a > 0$. The equilibrium point is $y^* = -\frac{b}{a}$. Since $\frac{\partial \dot{y}}{\partial y}|_{y^*} = a > 0$, this equilibrium point is unstable. See Figure 2.
- Go back to our old friend, $\dot{k}_t = sk_t^\alpha - \delta k_t$, with $0 < \alpha < 1$. We have two equilibrium points: $k^* = (\frac{s}{\delta})^{\frac{1}{1-\alpha}}$, $k^{**} = 0$. Then $\frac{\partial \dot{k}}{\partial k}|_{k^*} = \delta(\alpha - 1) < 0$ and $\frac{\partial \dot{k}}{\partial k}|_{k^{**}+} = +\infty$. Therefore, $k^*$ is stable and $k^{**}$ is unstable. See Figure 3.

## 2.3 Linearization

Suppose we are interested in the dynamics around the equilibrium $y^*$.

$$\dot{y}_t = f(y_t) \approx f(y^*) + f'(y^*)(y_t - y^*) = f'(y^*)(y_t - y^*) \tag{17}$$

Applying it to the above $\dot{k}_t = sk_t^\alpha - \delta k_t$, we obtain

$$\dot{k}_t = (\alpha s k^{*\alpha-1} - \delta)(k_t - k^*) \tag{18}$$

# 3 Systems of Differential Equations

## 3.1 Analytical Solution

Consider the linear system of differential equation $\dot{Y}_t = A_t Y_t + B_t$ with $A_t = A, B_t = 0$. $A$ is a $n \times n$ matrix with constant elements.

A simple case is when $A$ has $n$ linearly independent eigenvectors $v_1, v_2....v_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. It is equivalent to say $A$ is diagonalizable $D = P^{-1}AP$, where the diagonal elements of $D$ are $\lambda_1, \lambda_2, ..., \lambda_n$ and the columns of $P$ are $v_1, v_2....v_n$.

$$\dot{Y}_t = AY_t = PDP^{-1}Y_t \tag{19}$$

$$\Rightarrow P^{-1}\dot{Y}_t = P^{-1}AY_t = DP^{-1}Y_t \tag{20}$$

Denote $Z_t = P^{-1}Y_t$, we have

$$\dot{Z}_t = DZ_t \tag{21}$$

$$\Rightarrow \dot{z_{it}} = \lambda_i z_{it}, i = 1, 2, ..., n \tag{22}$$

$$\Rightarrow z_{it} = c_i \exp(\lambda_i t), i = 1, 2, ..., n \tag{23}$$

Given $Z_t$, we immediately get $Y_t = PZ_t$.

One can follow the same steps above to show that $\dot{Y}_t = AY_t + B_t$ has analytical solution as well.

**Remark 1.** If $A$ is not diagnonalizable (for example, in a two-dimension case, we may only have one linearly independent eigenvector), we can use another decomposition $T = U^{-1}AU$, where $T$ is an upper triangular matrix. Denote $W_t = U^{-1}Y_t$, we have $\dot{W}_t = TW_t + U^{-1}B_t$, then solve $w_{it}$ by the order of $w_{nt}, w_{n-1,t}, ..., w_{1t}$, where $w_{it}$ is the $ith$ element of $W_t$.

**Remark 2.** It is possible that eigenvalues are complex numbers, and eigenvectors are complex vectors thus we obtain complex solutions. However, we want real solutions instead of complex solutions. Apply the following observation: if $p_t + iq_t$ is a solution to $\dot{Y}_t = AY_t$ ($p_t - iq_t$ should also be a solution), where $p_t$ and $q_t$ are real vectors, then $p_t$ and $q_t$ are also the solutions. This is because $\dot{Y}_t = \dot{p}_t + i\dot{q}_t = A(p_t + iq_t) = Ap_t + iAq_t$, then we arrive at $\dot{p}_t = Ap_t$ and $\dot{q}_t = Aq_t$.

## 3.2 Graphical Solution (Phase Diagram)

Here we focus on two dimensions of system of differential equations. That is we have two variables $y_1$ and $y_2$ of interest.

$$\dot{y_{1t}} = f(y_{1t}, y_{2t}) \tag{24}$$

$$\dot{y_{2t}} = g(y_{1t}, y_{2t}) \tag{25}$$

**Solution Steps:**

- In the space $(y_1, y_2)$, draw the lines of $\dot{y_{1t}} = 0$ and $\dot{y_{2t}} = 0$ respectively. Or equivalently to say, draw both $f(y_1, y_2) = 0$ and $g(y_1, y_2) = 0$
- Draw rightward arrows when $\dot{y_{1t}} > 0$, and leftward arrows when $\dot{y_{1t}} < 0$
- Draw upward arrows when $\dot{y_{2t}} > 0$, and downward arrows when $\dot{y_{2t}} < 0$
- The intersections of $\dot{y_{1t}} = 0$ and $\dot{y_{2t}} = 0$ are equilibrium points. These points can be stable, unstable or saddle path stable.

- Given an initial point, follow the direction of arrows to track the dynamics of $(y_{1t}, y_{2t})$

**Remark.** In macroeconomics,we usually have initial values of state varibles (instead of all the varibles) and a transversality condition to uniquely determine the initial point and thus the whole dynamics. For example, given $y_{10}$ ( $y_{1t}$ is a state variable), we will pick up $y_{20}$ ($y_{2t}$ is a control variable). After picking up $y_{20}$, we follow arrows and we need to ensure that we will reach an equilibrium point which satisfies the transversality condition.

**Example 1** Consider the following system of differential equations of $c_t$ and $k_t$.

$$\dot{k_t} = k_t^\alpha - \delta k_t - c_t \tag{26}$$

$$\dot{c_t} = \frac{1}{\gamma}(\alpha k_t^{\alpha-1} - \delta - \rho)c_t \tag{27}$$

Boundary condition: initial state $k_0$ and transversality condition $lim_{t\to+\infty}\lambda_t k_t = 0$. You will see what transversality condition is in class, and it basically guarantees that the economy will not explode or converge to a non-sense point (no-ponzi scheme). Note we will have a unique saddle path. See Figure 4.

**Example 2** Now we turn to a two dimension linear case $\dot{Y_t} = AY_t$, where $Y_t = (y_{1t}, y_{2t})'$ and $A$ is a 2 by 2 matrix with constant elements. We first write out the analytical solutions (see the above Section 3.1 for how to solve it.). Suppose the eigenvalues of matrix $A$ are $\lambda_1$ and $\lambda_2$.

- If $\lambda_1 \neq \lambda_2$ and the two linearly independent eigenvectors are $v_1, v_2$, the solution is of the form:
$$Y_t = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t} \tag{28}$$

- If $\lambda_1 = \lambda_2$ (must be a real number) and we have two linearly independent eigenvectors $v_1, v_2$ , the solution is of the form:
$$Y_t = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_1 t} \tag{29}$$

- If $\lambda_1 = \lambda_2$ (must be a real number) and we only have one linearly independent eigenvector $v_1$, the solution is of the form:
$$Y_t = C_1 v_1 e^{\lambda_1 t} + C_2(v_1 t e^{\lambda_1 t} + v_2 e^{\lambda_1 t}) \tag{30}$$
where $v_2$ is the solution of $(A - \lambda_1 I)v_2 = v_1$

Constants $C_1$ and $C_2$ are determined by boundary conditions.

**Eigenvalues and Stability (Example 2)**

- If the eigenvalues are both positive real numbers, the equilibrium is unstable. See Figure 5.

- If the eigenvalues are both negative real numbers, the equilibrium is stable. See Figure 6.

- If the real parts of the eigenvalues are of opposite sign (it also implies that there is no complex part), the equilibrium is saddle path stable. Note this case is of special interest in this class. See Figure 7.

- If the eigenvalues are both complex numbers and the real parts are both positive, the system is unstable and oscillating. See Figure 8.

- If the eigenvalues are both complex numbers and the real parts are both negative, the system converges to the steady state in an oscillating manner. See Figure 9.

## 3.3 Linearization

Suppose we are interested in the dynamics around equilibrium point. We proceed with a two-dimension case.

$$\dot{y_{1t}} = f(y_{1t}, y_{2t}) \approx f(y_1^*, y_2^*) + f_1(y_1^*, y_2^*)(y_{1t} - y_1^*) + f_2(y_1^*, y_2^*)(y_{2t} - y_2^*) \tag{31}$$

$$\dot{y_{2t}} = g(y_{1t}, y_{2t}) \approx g(y_1^*, y_2^*) + g_1(y_1^*, y_2^*)(y_{1t} - y_1^*) + g_2(y_1^*, y_2^*)(y_{2t} - y_2^*) \tag{32}$$

Since we linearize around the equilibrium point, $f(y_1^*, y_2^*) = 0$ and $g(y_1^*, y_2^*) = 0$. That implies

$$\dot{Y}_t = A(Y_t - Y^*) \tag{33}$$

where $Y_t = (y_{1t}, y_{2t})'$ or

$$\dot{Z}_t = AZ_t \tag{34}$$

where $Z_t = Y_t - Y^*$

Then the above formula goes back to the aforementioned homogeneous system of differential equation with constant coefficient matrix.

**Example**

$$\dot{k}_t = k_t^\alpha - \delta k_t - c_t \approx (\alpha k^{*\alpha-1} - \delta)(k_t - k^*) - (c_t - c^*) \tag{35}$$

$$\dot{c}_t = \frac{1}{\gamma}(\alpha k_t^{\alpha-1} - \delta - \rho)c_t \approx \frac{\alpha(\alpha-1)}{\gamma} c^* k^{*\alpha-2}(k_t - k^*) \tag{36}$$

Pick up the equilibrium point with positive $(k^*, c^*)$ and write the above in a more compact form:

$$\dot{k}_t = \beta(k_t - k^*) - (c_t - c^*) \tag{37}$$

$$\dot{c}_t = -\tau(k_t - k^*) \tag{38}$$

with $\beta > 0$ and $\tau > 0$. The coefficient matrix is (assume parameter $0 < \alpha < 1$)

$$A = \begin{pmatrix} \beta & -1 \\ -\tau & 0 \end{pmatrix}$$

One can show the eigenvalues of the above have opposite sign and are real numbers. Therefore, the equilibrium is saddle path stable.

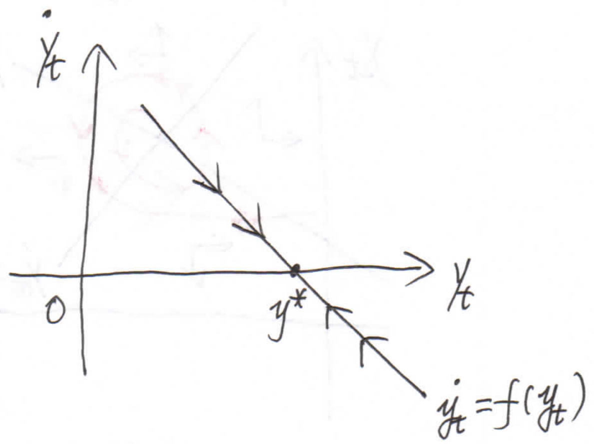


Figure 1.

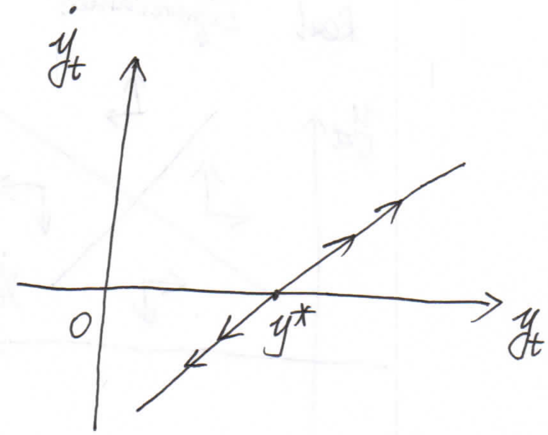


Figure 2.

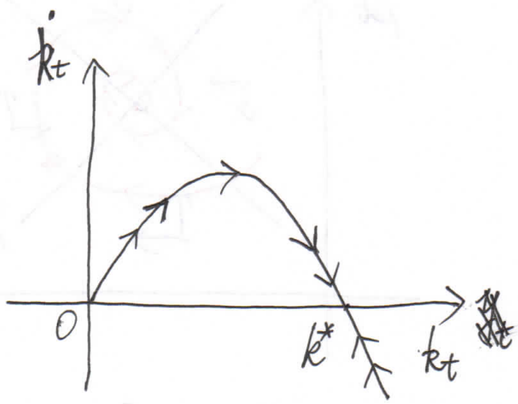


Figure 3.

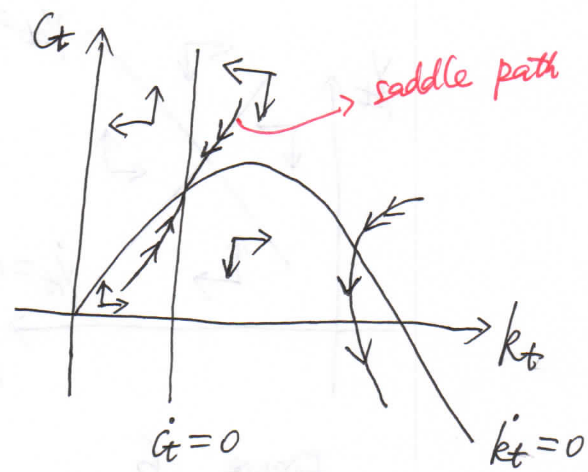


Figure 4

| | Real Eigenvalues | Complex Eigenvalues |
|---|---|---|

**Real parts are positive**

Real Eigenvalues:

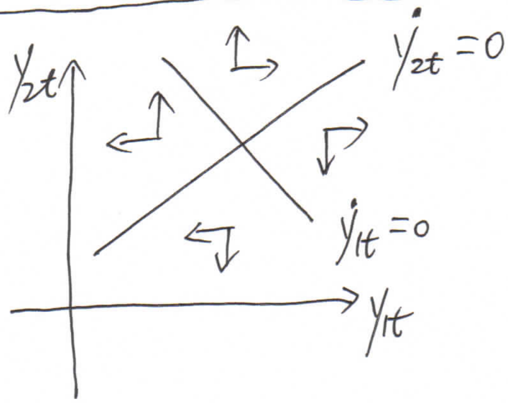


Figure 5

Complex Eigenvalues:

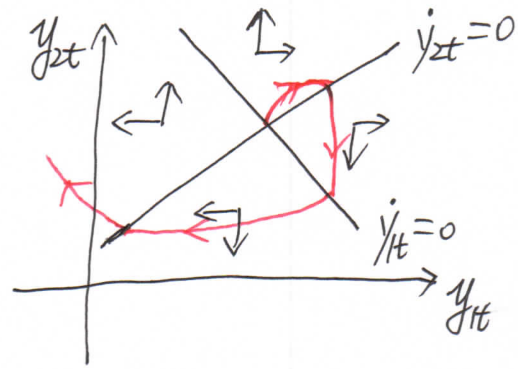


Figure 8

**Real parts are negative**

Real Eigenvalues:

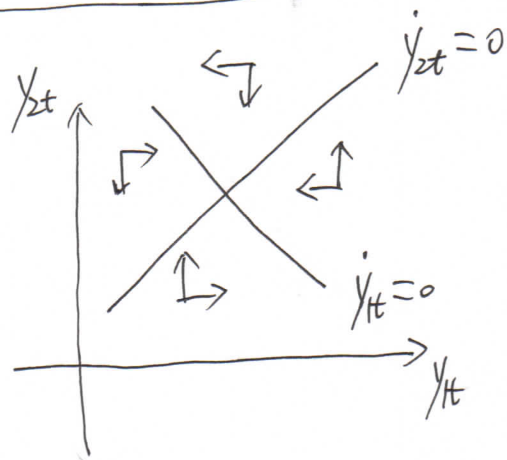


Figure 6

Complex Eigenvalues:

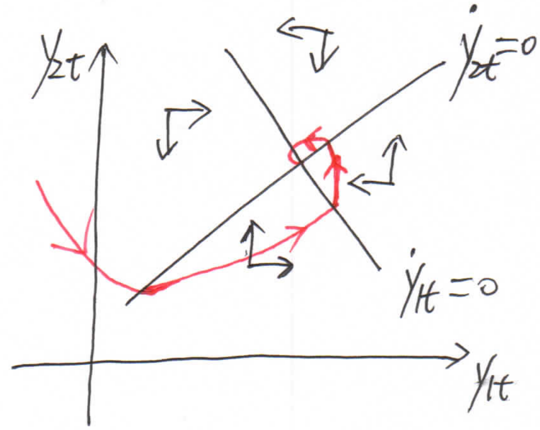


Figure 9

**Real parts have different signs.**

Real Eigenvalues:

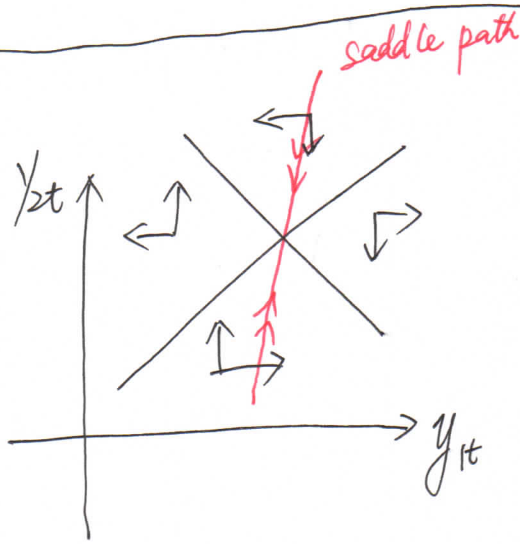


Figure 7

Complex Eigenvalues:

No this case.

# Investment Theory

Macroeconomic Analysis Recitation 2

Yang Jiao*

In this note, our focus is 1). Linearization of the internal adjustment cost model. 2). External adjustment cost model.

# 1 Internal Adjustment Cost Model

## 1.1 Model Setup

Internal adjustment cost means in order to invest $I_t$, firms have to themselves forego additional resources $I_t\varphi(\frac{I_t}{K_t})$. Assume $\varphi(0) = 0$, $\varphi'(\cdot) > 0$ and $2\varphi' + \frac{I}{K}\varphi'' > 0$. For simplicity, assume depreciation rate $\delta = 0$.

$$V_0 = \max_{K_t,L_t,I_t} \int_0^{+\infty} e^{-rt}[AF(K_t, L_t) - w_tL_t - I_t(1 + \varphi(\frac{I_t}{K_t}))]dt \tag{1}$$

s.t.

$$\dot{K}_t = I_t \tag{2}$$

$$\text{Non-ponzi scheme: } \lim_{t\to\infty} e^{-rt}K_t \geq 0 \tag{3}$$

$$K_0 \text{ is given.} \tag{4}$$

## 1.2 Solve the Model

We first set up the Hamiltonian:

$$\mathscr{H} = e^{-rt}[AF(K_t, L_t) - w_tL_t - I_t(1 + \varphi(\frac{I_t}{K_t}))] + \lambda_tI_t \tag{5}$$

First order conditions are:

$$AF_L = w_t \tag{6}$$

$$e^{-rt}\{-[1 + \varphi(\frac{I_t}{K_t}) + \frac{I_t}{K_t}\varphi'(\frac{I_t}{K_t})]\} + \lambda_t = 0 \tag{7}$$

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$$e^{-rt}[AF_K + (\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})] = -\dot{\lambda}_t \tag{8}$$

$$TVC: \lim_{t\to+\infty} \lambda_t K_t = 0 \tag{9}$$

Denote current shadow price $q_t = e^{rt}\lambda_t$, then F.O.C.s change to

$$AF_L = w_t \tag{10}$$

$$q_t = 1 + \varphi(\frac{I_t}{K_t}) + \frac{I_t}{K_t}\varphi'(\frac{I_t}{K_t}) \tag{11}$$

$$\dot{q}_t = rq_t - [AF_K + (\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})] \tag{12}$$

$$TVC: \lim_{t\to+\infty} e^{-rt}q_t K_t = 0 \tag{13}$$

Equation (11) establishes the relationship between $q_t$ and $\frac{I_t}{K_t}$. It is a one-to-one mapping (by $2\varphi' + \frac{I}{K}\varphi'' > 0$, we know $q_t$ is an increasing function of $\frac{I_t}{K_t}$).

$$\frac{I_t}{K_t} = h(q_t) \tag{14}$$

From equation (11), when $\frac{I}{K} = 0$, $q = 1$, thus $h(1) = 0$ in equation (14).

To solve the model, capital accumulation equation has to be used as well (don't foreget it, since it is not listed in the first order conditions.).

$$\dot{K}_t = I_t = h(q_t)K_t \tag{15}$$

From the first order conditions, we also have

$$\dot{q}_t = rq_t - [AF_K + h^2(q_t)\varphi'(h(q_t)] \tag{16}$$

Now we have got a system of differential equations for $q_t, K_t$ with initial condition $K_0$ given and TVC $\lim_{t\to+\infty} e^{-rt}q_tK_t = 0$.

**Remark.** In fact, the problem is complicated because labor input $L_t$ will depend on $K_t$, see equation (10). And $L_t$ shows up in equation (16) $F_K(K_t, L_t)$ as well. Denote the functional relationship derived from equation (10) as $L_t = g(K_t)$. Then in equation (16), one needs to substitute $F_K(K, L)$ by $F_K(K, g(K))$. However, to simplify the analysis, we assume labor demand $L_t$ keeps fixed at $L^*$ for the moment.

### 1.3 Graphical Solution (Phase Diagram)

- We first need to find the steady state. We will focus on the steady state with positive $(k^*, q^*)$.

$$h(q^*) = 0 \tag{17}$$

$$rq^* = [AF_K(K^*, L^*) + h^2(q^*)\varphi'(h(q^*)] \tag{18}$$

Since $h(1) = 0$ and $h$ is monotonic, we immediately have $q^* = 1$. And then $K^* =$

$F_K^{-1}(r/A)$.

- It is easy to draw $\dot{K}_t = 0$, that is $h(q) = 0$ thus $q = 1$, a horizontal line in the space of $(K, q)$.

  Since we will concentrate on the dynamics around the steady state, we would like to know the slope of $\dot{q}_t = 0$ in a small neighborhood of $(K^*, q^*)$. Denote the line $\dot{q}_t = 0$ as $m(K, q) = 0$,

  $$m(K, q) = rq - [AF_K(K, L^*) + h^2(q)\varphi'(h(q))] \tag{19}$$

  For the line $m(K, q) = 0$, at point $(K^*, q^*)$, by the implicit function theorem, the slope is

  $$\frac{dq}{dK}|_{(K^*,q^*)} = -\frac{\partial m/\partial K}{\partial m/\partial q} = \frac{AF_{KK}(K^*, L^*)}{r - 2h(q^*)\varphi'(h(q^*)) - h^2(q^*)\varphi''(h(q^*))h'(q^*)} = \frac{AF_{KK}(K^*, L^*)}{r} < 0 \tag{20}$$

  By continuity, in a small neighborhood of $(K^*, q^*)$, the slope is also negative.

  **Remark 1.** If we forget about the assumption that $L^*$ is fixed, what we need for $\frac{dq}{dK}|_{(K^*,q^*)} < 0$ to hold is $F_{KK}(K^*, g(K^*)) + F_{K,L}(K^*, g(K^*))g'(K^*) < 0$. In fact, to justify the fixed $L^*$, we may consider a general equilibrium model with fixed labor supply $L^*$, so wages will adjust to guarantee that labor demand equals labor supply $L_t = L^*$.

  **Remark 2.** When we have capital depreciation rate $\delta > 0$ in this internal adjustment cost model, the steady state $q^*$ will not be 1. Please check by yourself how depreciation rate will affect steady state level of $q^*$.

See Figure 1 for the phase diagram. Note the role of TVC is to guarantee that firms will choose to stay on the saddle path. When $q_t > 1$, firms invest, thus capital stock increases, while when $q_t < 1$, firms disinvest thus capital stock decreases. That is to say investment depends on $q_t$.

### 1.4 Linearization

Around the steady state $(K^*, q^*)$:

$$\dot{K}_t \approx h(q^*)(K_t - K^*) + K^* h'(q^*)(q_t - q^*) \tag{21}$$

$$\dot{q}_t \approx -AF_{KK}(K^*, L^*)(K_t - K^*) + [r - 2h(q^*)\varphi'(h(q^*)) - h^2(q^*)\varphi''(h(q^*))h'(q^*)](q_t - q^*) \tag{22}$$

Substituting the steady state property that $h(q^*) = h(1) = 0$, we obtain

$$\dot{K}_t \approx K^* h'(q^*)(q_t - q^*) \tag{23}$$

$$\dot{q}_t \approx -AF_{KK}(K^*, L^*)(K_t - K^*) + r(q_t - q^*) \tag{24}$$

Write the above in a more compact form

$$\dot{Z}_t = GZ_t \tag{25}$$

where $Z_t = (K_t - K^*, q_t - q^*)'$ and the coefficient matrix is

$$G = \begin{pmatrix} 0 & K^* h'(q^*) \\ -AF_{KK}(K^*, L^*) & r \end{pmatrix}$$

The eigenvalues $\lambda_1$ and $\lambda_2$ satisty that

$$\lambda_1 * \lambda_2 = det(G) = K^* h'(q^*) * AF_{KK}(K^*, L^*) < 0 \tag{26}$$

We conclude that eigenvalues must have opposite signs and they are real numbers thus this system is saddle path stable.

Solving the eigenvalues explicitly, we have

$$\lambda = \frac{r \pm \sqrt{r^2 - 4AF_{KK} h'(q^*) K^*}}{2} \tag{27}$$

Let $\lambda_1$ be the eigenvalue smaller than 0, and $\lambda_2$ larger than 0. The solution to the above system of differential equation is

$$Z_{1t} = K_t - K^* = \Psi_{11} e^{\lambda_1 t} + \Psi_{12} e^{\lambda_2 t} \tag{28}$$

$$Z_{2t} = q_t - q^* = \Psi_{21} e^{\lambda_1 t} + \Psi_{22} e^{\lambda_2 t} \tag{29}$$

(Recall the result from recitation 1, a two-dimension linear system of differential equation with two different eigenvalues should have the solution form $Z_t = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$, where $C_1$ and $C_2$ are constants, $v_1$ and $v_2$ are eigenvectors, and $\lambda_1$ and $\lambda_2$ are eigenvalues. Therefore, $\Psi_{11} = C_1 v_{11}$, $\Psi_{21} = C_1 v_{12}$, $\Psi_{12} = C_2 v_{21}$, and $\Psi_{22} = C_2 v_{22}$)

To let the system converge to $(K^*, q^*)$, we must have $\Psi_{12} = \Psi_{22} = 0$ ( just set $C_2$=0 ). Otherwise, the solution would have a term $e^{\lambda_2 t}$ going to infinity and TVC will be violated. Now we are on the saddle path.

To determine $\Psi_{11}$ and $\Psi_{21}$, we need two conditions. First, the initial $K_0$ is given

$$K_0 - K^* = \Psi_{11} \tag{30}$$

Second, $(\Psi_{11}, \Psi_{21})'$ is the eigenvector of $\lambda_1$($v_1$ is an eigenvalue, then $C_1 v_1$ is also an eigenvector when $C_1 \neq 0$),

$$\Psi_{21} = \frac{\lambda_1 \Psi_{11}}{K^* h'(q^*)} = \frac{\lambda_1 (K_0 - K^*)}{K^* h'(q^*)} \tag{31}$$

In sum, we have determined all the dynamics of this linearized system analytically.

Additionally, initial $q$ is given by

$$q_0 = q^* + \Psi_{21} = q^* + \frac{\lambda_1 (K_0 - K^*)}{K^* h'(q^*)} \tag{32}$$

# 2 External Adjustment Cost

In the above internal adjustment cost model, capital is owned by firms and firms bear the adjustment cost. Now consider an alternative setting in which final goods producers purchase capital from capital goods producers. Final goods producers bear no adjustment cost while capital good producers incur adjustment cost (modeled as a convex cost of production, i.e. decreasing return to scale production technology). One can think that installation of capital is done by capital good producers instead of final good producers.

## 2.1 Model Setup

### 2.1.1 Capital Goods Firms

To produce $I_t$ capital goods, capital goods firms have to incur cost $C(I_t)$. We assume the cost function satisfies $C(0) = 0$, $C'(I) > 0$ for $I > 0$, $C''(I) > 0$ for $I > 0$ and $\lim_{I\to+\infty} C'(I) = +\infty$. In short, cost function is convex and marginal cost goes to infinity when producing infinite capital.

These firms are price takers, and their problem is

$$\max_{I_t} \int_0^{+\infty} e^{-rt}[P_{It}I_t - C(I_t)]dt \tag{33}$$

This is a static problem. It is equivalent to maximize $P_{It}I_t - C(I_t)$ at each moment. The first order condition is

$$P_{I_t} = C'(I_t) \tag{34}$$

Since $C'' > 0$, $P_I$ is a strictly increasing function of $I$ and the reverse is true as well:

$$I = h(P_I) \tag{35}$$

with $h'(\cdot) > 0$. It is a supply function of capital goods.

### 2.1.2 Final Goods Producers

We assume final goods producers directly buy capital from capital good producers and they don't pay adjustment cost.

$$V_0 = \max_{K_t,L_t,I_t} \int_0^{+\infty} e^{-rt}[P_tAF(K_t, L_t) - w_tL_t - P_{It}I_t]dt \tag{36}$$

s.t.

$$\dot{K}_t = I_t - \delta K_t \tag{37}$$

$$\text{Non-ponzi scheme: } \lim_{t\to\infty} e^{-rt}K_t \geq 0 \tag{38}$$

$$K_0 \text{ is given.} \tag{39}$$

## 2.2 Solve the Model

We set up the Hamiltonian of final goods producers:

$$\mathscr{H} = e^{-rt}[P_tAF(K_t, L_t) - w_tL_t - P_{It}I_t] + \lambda_t(I_t - \delta K_t) \tag{40}$$

First order conditions are:

$$AF_L = w/P \tag{41}$$

$$-e^{-rt}P_{It} + \lambda_t = 0 \tag{42}$$

$$e^{-rt}PAF_K - \lambda_t\delta = -\dot{\lambda}_t \tag{43}$$

$$TVC: \lim_{t\to+\infty} \lambda_t K_t = 0 \tag{44}$$

Denote current shadow price $q_t = e^{rt}\lambda_t$, then F.O.C.s change to

$$AF_L = w_t/P_t \tag{45}$$

$$P_{It} = q_t \tag{46}$$

$$\dot{P}_{It} = (r+\delta)P_I - P \cdot AF_K \tag{47}$$

$$TVC: \lim_{t\to+\infty} e^{-rt}q_t K_t = 0 \tag{48}$$

Remember we also have the capital accumulation equation, and plugging in the solution of capital goods producers yields

$$\dot{K}_t = h(P_{It}) - \delta K_t \tag{49}$$

Combining

$$\dot{P}_{It} = (r+\delta)P_{It} - P \cdot AF_K, \tag{50}$$

initial condition $K_0$ and TVC $\lim_{t\to+\infty} e^{-rt}P_{It}K_t = 0$, we are ready to solve a two-dimension differential equation system.

## 2.3 Housing Market Interpretation

- One can think of the capital goods producers as firms in construction sector who build new houses. Final goods producers are real estate agents who will hire labor to provide housing services.

- Capital stock is housing stock and real estate agents buy newly built houses from the construction sector. Real estate agents are the owners of housing stock.

- We can view the newly produced capital goods $I_t$ as residential investment. $PAF(K, L)$ are the rent income of real estate agents. $P_I$ is the price of a unit of newly built house. We take final goods price $P$ as given, but one can have a household side utility function to derive a demand function for final goods, then $P$ can be endogenized.

- Why capital goods production has convex cost? Here is a simple interpretation: since land supply is limited, building one house on the top of a skyscraper will be more difficult than building a house on the ground.

- Interest rate

$$r = \frac{\dot{P}_I + P \cdot AF_K - \delta P_I}{P_I} \tag{51}$$

given by equation (50). It is a no-arbitrage condition: return from investing in a unit of house is equal to the interest rate.

- Rearrange equation (50):

$$(r+\delta)P_{It} - \dot{P}_{It} = P \cdot AF_K \tag{52}$$

Multiply both sides by $e^{-(r+\delta)t}$, and take integral from 0 to $+\infty$ on both sides,

$$P_{I0} = \int_0^{+\infty} e^{-(r+\delta)t} P \cdot AF_K(K_t, L^*) dt \tag{53}$$

The price of a house is equal to the present value of all future rent income from the house. Notice we have an addtional depreciation rate $\delta$ in the equation, because in the setup we assume $\delta \neq 0$.

**Remark.**You can use similar steps to get $q_0$ in the **internal adjustment cost** model:

$$q_0 = \int_0^{+\infty} e^{-rt}[AF_K + (\frac{I_t}{K_t})^2 \varphi'(\frac{I_t}{K_t})] dt \tag{54}$$

The term $(\frac{I_t}{K_t})^2\varphi'(\frac{I_t}{K_t})$ captures the learning by doing benefit from installing capital today.

## 2.4 Graphical Solution (Phase Diagram)

Setting $\dot{K}_t = 0$ and $\dot{P}_{It} = 0$ leads to

$$h(P_I) = \delta K \tag{55}$$

$$P_I = \frac{P \cdot AF_K(K, L^*)}{r+\delta} \tag{56}$$

One is a positive relationship between $K$ and $P_I$, and the other is a negative relationship. The steady state is given by the intersection of the above two lines. See Figure 2 for the phase diagram.

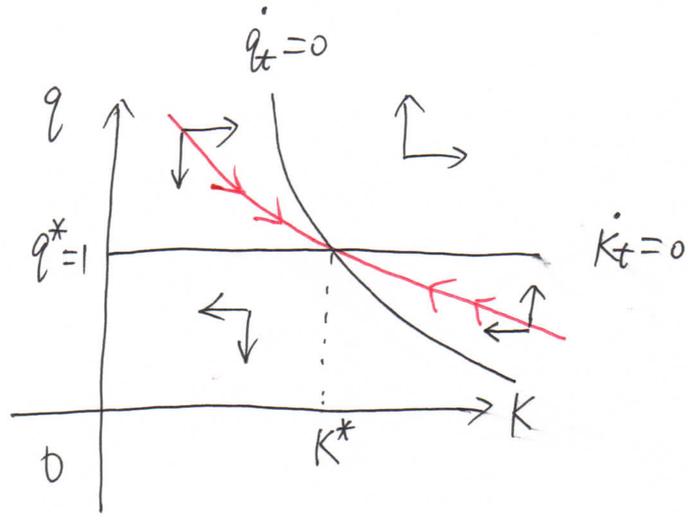



Figure 1.

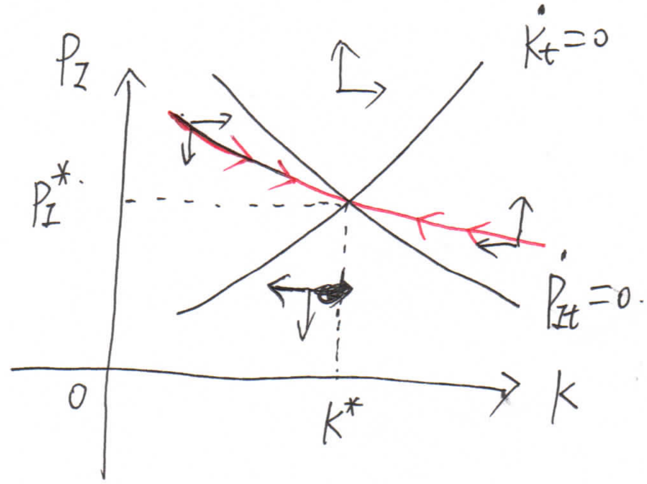



Figure 2

# Investment Theory (continued)- Neoclassical Production Functions

Macroeconomic Analysis Recitation 3

Yang Jiao*

## 1 Comparison of Internal and External Adjustment Cost Models

Consider a permanent positive productivity shock to both models. See Figure 1 (internal cost model) and Figure 2 (external cost model). In the external adjustment cost model, long run $q^* = P_I^*$ will increase, while in the internal adjustment cost model, steady state $q^*$ is fixed at 1.

One may wonder are these two models really different? Adjustment cost is just adjustment cost after all. Note the difference we introduced in these two models: in the external adjustment cost model, we add depreciation rate of capital, and assume the cost of producing new capital is $C(I)$ instead of $I(1+\varphi(\frac{I}{K}))$ as in the internal adjustment cost model. Now the question is whether these two differences in modeling generate the different predictions as shown in Figure 1 and Figure 2.

In the external adjustment cost model, in steady state, when $K^*$ changes, $I^* = \delta K^*$ will change as well since we have depreciation rate $\delta > 0$. Then $q^* = P_I^* = C'(I^*)$ will also change. In order to make $q^* = P_I^*$ not change with $K^*$ (i.e. a flat $\dot{K}_t = 0$ line), we need to revise the cost function in the external adjustment cost model so that the marginal cost of capital goods will not differ when we have a different $K^*$. Here are two ways: 1) let the marginal cost of new capital goods be a function of $I - \delta K$ (in steady state $I^* - \delta K^*$ will always be 0). 2) let the marginal cost of new capital goods be a function of $\frac{I}{K}$ (in steady state $\frac{I^*}{K^*}$ will always be $\delta$).

### 1.1 Capital Depreciation

Assume in the external adjustment cost model, cost function takes the form $\bar{C}(I) = C(I - \delta K)$, therefore, replacing depreciated capital costs nothing: $\bar{C}(I^*) = C(I^* - \delta K^*) = C(0) = 0$. Resolving the model shows:

$$P_{It} = \bar{C}'(I) = C'(I_t - \delta K_t) \tag{1}$$

*Please email me if you find errors or typos to yj2279@columbia.edu. All comments and suggestions are welcome and appreciated.

Capital accumulation function becomes

$$\dot{K}_t = I_t - \delta K_t = h(P_{It}) \tag{2}$$

In this way, you have a flat $\dot{K}_t = 0$ schedule: $P_I = h^{-1}(0)$.

### 1.2 $\frac{I}{K}$ and $I$

In the internal adjustment cost model, the cost function is related to capital stock to capture the learning by doing effect, while in the external adjustment cost, we ignore it. Now suppose in the external adjustment cost model, we also take into account the learning by doing effect and assume the margial cost of producing capital goods is a function of $\frac{I}{K}$: $g(\frac{I}{K})$. For example, if $C(I,K) = I(1+\varphi(I/K))$, we have $g(I/K) = \frac{\partial C(I,K)}{\partial I} = 1 + \varphi(I/K) + \frac{I}{K}\varphi'(I/K)$. Resolving the external adjustment cost model shows:

$$P_{It} = g(\frac{I_t}{K_t}) \tag{3}$$

The above establishes a relation

$$\frac{I_t}{K_t} = h(P_{It}) \tag{4}$$

Capital accumulation function becomes

$$\dot{K}_t = I_t - \delta K_t = [h(P_{It}) - \delta]K_t \tag{5}$$

We reach a flat $\dot{K}_t = 0$ as well: $P_I = h^{-1}(\delta)$.

Conclusion: these two investment models are essentially the same if we revise the cost function in the external adjustment cost model to eliminate the differences caused by different modeling strategies in the two models.

**Remark.** We have discussed the line of $\dot{K}_t = 0$ for the two models, but what about the line of $\dot{q}_t = 0$ ? In fact, there is externality that is not internalized. In the external adjustment cost model, final goods producers don't take into account that their purchasing of newly built capital goods can decrease the cost of capital goods producers in the future. That's why when you set $\delta = 0$ and $C(I,K) = I(1+\varphi(I/K))$ in the external adjustment cost model, you will get a different line for $\dot{q}_t = 0$ when comparing to the internal adjustment cost model (please check what I am saying is correct).

## 2 Neoclassical Production Functions $Y = F(K, L, A)$

### 2.1 Basic Properties

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}$ is twice continuously differentiable to its three arguments. A production function is called a neoclassical production function if the following properties are satistied.

- **Constant returns to scale:**

$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A) \tag{6}$$

That is $F(K, L, A)$ is homogeneous of degree one in $K$ and $L$. Note $A$ is non-rivalry, so the replication principle doesn't apply to $A$.

- **Positive and diminishing returns to $K$ and $L$:**

$$\frac{\partial F}{\partial K} > 0, \qquad \frac{\partial F}{\partial L} > 0 \tag{7}$$

$$\frac{\partial^2 F}{\partial K^2} < 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0 \tag{8}$$

If we increase the amount of one input, output will increase, but marginal product will decrease as the input increases.

- **Inada Conditions**

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty \tag{9}$$

$$\lim_{K\to +\infty} \frac{\partial F}{\partial K} = \lim_{L\to +\infty} \frac{\partial F}{\partial L} = 0 \tag{10}$$

Inada conditions can help us nail down interior solutions.

- **Essentiality**

$$F(K, 0, A) = F(0, L, A) = 0 \tag{11}$$

Therefore, to produce a positive amount of output, a positive amount of each input is required.

The first three properties imply the last essentiality property, so we don't need to write down the last essentiality property. See the following proof of this argument:
**Proof.** Recall **L' Hôpital's Rule:**
If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ around $I$ which is a small neighborhood of $c$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$
Apply it to the following:

$$\lim_{K\to +\infty} \frac{Y}{K} = \lim_{K\to\infty} \frac{\frac{\partial Y}{\partial K}}{\frac{\partial K}{\partial K}} = \lim_{K\to\infty} \frac{\partial Y}{\partial K} = 0 \tag{12}$$

The last equality comes from Inada conditions.
Note that by CRS (constant returns to scale)

$$\lim_{K\to +\infty} \frac{Y}{K} = \lim_{K\to +\infty} F(1, \frac{L}{K}, A) = F(1, 0, A) \tag{13}$$

which leads to

$$F(1, 0, A) = 0 \tag{14}$$

And $F(L,0,A) = L * F(1,0,A) = 0$ follows immediately. Similarly, one can prove that $F(0,K,A) = 0$ as well. ■

**Examples** A constant elasticity of substitution (CES) production function

$$Y = F(K,L) = (aK^{\frac{\sigma-1}{\sigma}} + bL^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \tag{15}$$

where $a \geq 0$, $b \geq$ and $\sigma \geq 0$ are constants. $\sigma = -\frac{d\ln(L/K)}{d\ln(F_L/F_K)}$, where $F_K$ and $F_L$ are partial derivatives.

- When $\sigma \to +\infty$, $Y = aK + bL$, perfect substitutability
- When $\sigma \to 0$, $Y = \min\{K,L\}$, perfect complementarity, Leontief. When $Y = F(K,L) = [(aK)^{\frac{\sigma-1}{\sigma}} + (bL)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$, and $\sigma \to 0$, we have $Y = \min\{aK, bL\}$. Note the slight difference.
- When $\sigma = 1$, $Y = K^\alpha L^{1-\alpha}$ with $\alpha = \frac{a}{a+b}$, Cobb-Douglas.

We can use L' Hôpital's Rule to show when $\sigma \to 0$ and $\sigma \to 1$ , we are approaching Leontief production function and Cobb-Douglas production function respectively.

## 2.2 Constant Returns to Scale and Zero Profit

**Euler Theorem** Suppose $f : \mathbb{R}^M \to \mathbb{R}$ is continuously differentiable and homogeneous of degree $\alpha$, i.e.

$$\forall x \in \mathbb{R}^M, f(\lambda x) = \lambda^\alpha f(x) \tag{16}$$

then

$$\sum_{i=1}^{M} \frac{\partial f(x)}{\partial x_i} x_i = \alpha f(x) \tag{17}$$

**Proof.** Differentiate both sides of equation (11) with respective to $\lambda$ and set $\lambda = 1$. ■

For a constant returns to scale production function $F(K,L,A)$, we know $\alpha = 1$. Apply Euler Theorem,

$$F_K K + F_L L = F \tag{18}$$

A profit maximization firm takes input price $R$ and $W$ and output price $P$ as given and first order conditions are $PF_K = R$ and $PF_L = W$. Therefore, $RK + WL = PF$, or firms' profit $PF - RK - WL = 0$. Note the implicit assumption is that firms are price takers of input and output.

One may wonder that in investment theory, firms are price takers but they do have profits. The reason is that in investment theory we study, firms are capital good owners themselves (so they don't rent capital), and they start with a capital stock $K_0 > 0$.

## 2.3 Technological Progress

### 2.3.1 Three Production Function Forms

Three forms of production functions:

- Hicks neutral: $Y = A_t F(K, L)$;
- Harrod neutral: $Y = F(K, A_t L)$;
- Solow neutral: $Y = F(A_t K, L)$.

Cobb-Douglas production function can be written as all of the above three forms.

#### 2.3.2 What Form of Technological Progress?

Suppose there is a production function:

$$Y_t = F(K_t, L_t, A_t), \tag{19}$$

if

- $F$ exhibits constant returns to scale in $K$ and $L$
- Resource constraint: $\dot{K}_t = Y_t - C_t - \delta K_t$ and saving rate is constant $s$
- Labor grows at a constant rate $\frac{\dot{L}_t}{L_t} = n$
- Capital stock grows at a constant rate $\frac{\dot{K}_t}{K_t} = \gamma_K$

then the production function must be labor augmenting, i.e. Harrod neutral
**Proof.**

From the resource constraint

$$\frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} - \delta \tag{20}$$

Since the left hand side is a constant $\gamma_K$, we immediately conclude that $\frac{Y}{K}$ is a constant.

$$Y_t = F(B_t K_t, A_t L_t) \tag{21}$$

Note when $B_t$ and $A_t$ grow at the same rate, we will go to Hicks neutral case by CRS, therefore, this form includes all three possible production function forms.

We additionally assume that technology will grow at a constant rate. Assume that $\frac{\dot{B}_t}{B_t} = z$ and $\frac{\dot{A}_t}{A_t} = x$. Without loss of generality, we let $B_0 = 1$ and $A_0 = 1$ so that $B_t = e^{zt}$ and $A_t = e^{xt}$. Then

$$\frac{Y_t}{K_t} = \frac{F(B_t K_t, A_t L_t)}{K_t} = F(B_t, A_t \frac{L_t}{K_t}) = B_t F(1, \frac{A_t}{B_t}\frac{L_t}{K_t}) = e^{zt} F(1, e^{(x-z)t}\frac{L_t}{K_t}) \tag{22}$$

Since labor grows at a constant rate $n$ and capital grows at a constant rate of $\gamma_K$ (again, for simplicity assume $L_0 = 1$ and $K_0 = 1$), we get

$$\frac{Y_t}{K_t} = e^{zt} F(1, e^{(x-z+n-\gamma_K)t}) \tag{23}$$

Define $\varphi(\cdot) = F(1, \cdot)$ to obtain

$$\frac{Y_t}{K_t} = e^{zt}\varphi(e^{(x-z+n-\gamma_K)t}) \tag{24}$$

We have proved that $\frac{Y_t}{K_t}$ is a constant, and now we discuss two scenarios:

1). If $x - z + n - \gamma_K = 0$ thus $x = \gamma_K - n$, we need to have $z = 0$, which means $A_t$ grows at a constant rate and $B_t$ is a constant . So the production function is labor augmenting.

2). If $x - z + n - \gamma_K \neq 0$, we still need to have $\frac{\partial[e^{zt}\varphi(e^{(x-z+n-\gamma_K)t})]}{\partial t} = 0$ which implies

$$\frac{\varphi'(\chi)\chi}{\varphi(\chi)} = \frac{-z}{n + x - z - \gamma_K} \tag{25}$$

where $\chi = e^{(x-z+n-\gamma_K)t}$. Notice that we need to require $n + x - z - \gamma_K$ is non-zero here.

Solve the above differential equation to reach

$$\varphi(\chi) = constant \cdot \chi^{1-\alpha} \tag{26}$$

where $\alpha$ is a constant. Substitute back to the production function to finally write out

$$F(B_tK_t, A_tL_t) = B_tK_t \cdot F(1, \frac{A_tL_t}{B_tK_t}) = B_tK_t(\frac{A_tL_t}{B_tK_t})^{1-\alpha} = constant \cdot K_t^{\alpha}(L_te^{\nu t})^{1-\alpha} \tag{27}$$

where $\nu = \frac{z\alpha + x(1-\alpha)}{1-\alpha}$. This Cobb-Douglas production function also belongs to labor augmenting production function.

We then conclude that production function takes the labor augmenting form. ■

**Remark.** We can use other ways to prove the labor augmenting production function. First, since $Y/K$ is constant and $K$ grows at a constant rate of $\gamma_K$, $Y$ will also grow at a constant rate $\gamma_Y$ and $\gamma_Y = \gamma_K$. At time 0 (pick up an arbitrary time $T$ should work as well), the production function is

$$Y_0 = F(K_0, L_0, A_0) \tag{28}$$

Multiply both sides by $e^{\gamma_K t}$, we obtain

$$e^{\gamma_K t}Y_0 = F(e^{\gamma_K t}K_0, e^{\gamma_K t}L_0, A_0) \tag{29}$$

$$Y_t = F(K_t, e^{(g_K - n)t}L_t, A_0) \tag{30}$$

Denote $\bar{A}_t = e^{(g_K-n)t}$ and re-write

$$Y_t = F(K_t, e^{(\gamma_K-n)t}L_t, A_0) = F(K_t, \bar{A}_tL_t, A_0) = \bar{F}(K_t, \bar{A}_tL_t) \tag{31}$$

where $\bar{A}_t = e^{(\gamma_K-n)t}$. The above $\bar{F}(K_t, \bar{A}_tL_t)$ is already a labor augmenting production function.

### 2.3.3 Kaldor Facts

Kaldor facts about growth:

- Per capita output $\frac{Y}{L}$ grows over time, and its growth rate doesn't tend to diminish.
- Physical capital per worker $\frac{K}{L}$ grows over time
- Return of capital $R$ keeps nearly constant
- Ratio of physical capital to output $\frac{K}{Y}$ keeps nearly constant
- Labor share $\frac{WL}{Y}$ and capital share $\frac{RK}{Y}$ are nearly constant
- The growth rate of output per worker differs substantially across countries

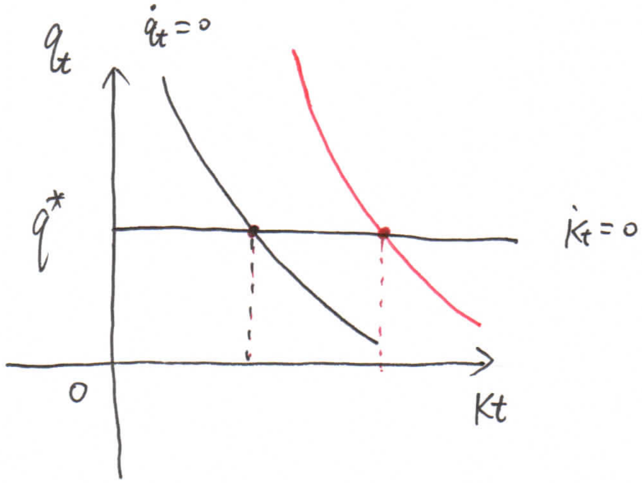



Figure 1

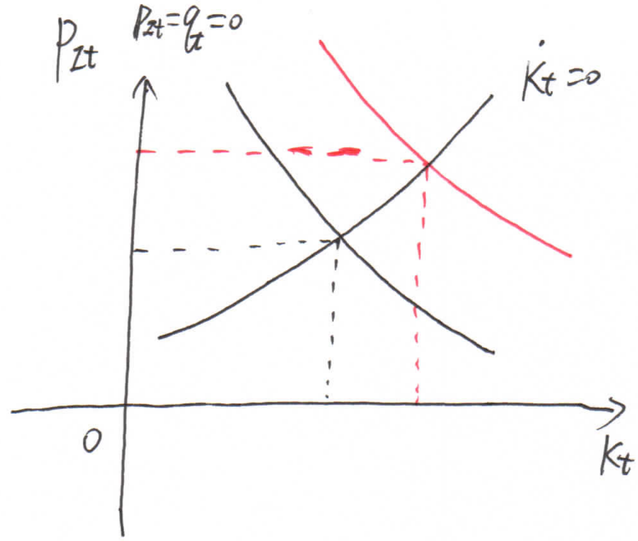



Figure 2.

after a permanent technology shock,

$\dot{q}_t = 0$ will shift

# Growth Model with Exogenous Saving Rate

Macroeconomic Analysis Recitation 4

Yang Jiao*

# 1 Extensions of Solow-Swan Model

## 1.1 Solow-Swan Model with Technology Progress

Now we introduce technology progress as well. Assume we have a neoclassical production function $Y = F(K_t, A_tL_t)$, where technology $A_t$ grows at a constant rate $\gamma_A$ and labor $L_t$ grows at $n$. All else equal as in the Solow-Swan model we studied in class.

$$\frac{\dot{A}_t}{A_t} = \gamma_A \tag{1}$$

$$\frac{\dot{L}_t}{L_t} = n \tag{2}$$

Define lowercase letter variable $x_t = \frac{X_t}{A_tL_t}$ where $X = Y, K$.

$$y_t = \frac{Y_t}{A_tL_t} = F(k_t, 1) = f(k_t) \tag{3}$$

The last equality is just a definition of $f(\cdot)$.

Capital accumulation equation

$$\dot{K}_t = I_t - \delta K_t = sY_t - \delta K_t \tag{4}$$

Divide both sides by $K_t$ to obtain

$$\frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} + \delta \tag{5}$$

First,

$$\frac{Y_t}{K_t} = \frac{y_t}{k_t} = \frac{f(k_t)}{k_t} \tag{6}$$

Second,

$$k_t = \frac{K_t}{A_tL_t} \to \frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} \tag{7}$$

*Please email me if you find errors or typos to yj2279@columbia.edu. All comments and suggestions are welcome and appreciated.

$$\rightarrow \frac{\dot{K}_t}{K_t} = \frac{\dot{k}_t}{k_t} + \gamma_A + n \tag{8}$$

Substitute equation (6) and (8) to equation (5) to get

$$\frac{\dot{k}_t}{k_t} = s\frac{f(k_t)}{k_t} - (\delta + n + \gamma_A) \tag{9}$$

Note the interpretation of $k_t$ or $y_t$ is no longer per capita variables. $k_t A_t$ and $y_t A_t$ are per capita terms instead.

## 1.2 A Model with Poverty Trap (for your reference)

Suppose the economy has access to two possible technologies,

$$Y_A = AK^\alpha L^{1-\alpha} \tag{10}$$

$$Y_B = BK^\alpha L^{1-\alpha} - bL \tag{11}$$

where $B > A$, and $b > 0$.

In per capita terms, the production functions become

$$y_A = Ak^\alpha \tag{12}$$

$$y_B = Bk^\alpha - b \tag{13}$$

The economy will compare and decide which technology to use and it will depend on the level of $k$. There is a level of capital $\tilde{k} = (\frac{b}{B-A})^{1/\alpha}$ such that when $k > \tilde{k}$, the economy chooses technology B to produce, otherwise when $k \leq \tilde{k}$, the economy chooses technology A .

From the Solow-Swan model, we have

$$\frac{\dot{k}}{k} = sf(k)/k - (n + \delta) \tag{14}$$

Here $f(k) = Ak^\alpha$ when $k \leq \tilde{k}$, and $f(k) = Bk^\alpha - b$ when $k > \tilde{k}$.

It is easy to draw $sf(k)/k$ when $k \leq \tilde{k}$, since it is strictly decreasing. However, when $k > \tilde{k}$, function $h(k) := sf(k)/k = Bk^{\alpha-1} - \frac{b}{k}$ may not be a monotonic function. There is a cutoff $\bar{k} = (\frac{b}{B(1-\alpha)})^{\frac{1}{\alpha}}$ such that when $k < \bar{k}$, $h(k)$ is decreasing in $k$, and when $k \geq \bar{k}$, $h(k)$ is increasing in $k$.

If $\bar{k} > \tilde{k}$, $sf(k)/k$ will first decrease then increase and finally decrease. To determine the steady state, we also need to plot a horizontal line $n + \delta$. One particularly interesting case is shown in Figure 1. There are three steady states. And two of them are stable. It means the initial level $k_0$ matters for the long run steady state. A country with very little $k_0$ will end up with a steady state of lower output per capita (poverty trap).

If $\bar{k} \leq \tilde{k}$, $sf(k)/k$ will always decrease with $k$. There is only one stable steady state. See Figure 2.

### 1.3 A Model with Physical Capital $K$ and Human Capital $H$ (for your reference)

Now we introduce both physical capital and human capital into the Slow-Swan model. Let's assume a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} H_t^{\eta} (A_t L_t)^{1-\alpha-\eta}$$

where $A_t$ is technology which grows at rate $\gamma_A$ and $L_t$ is labor force which grows at rate $n$. Parameters satisfy $0 < \alpha + \eta < 1$.

Divide both sides by $A_t L_t$:

$$y_t = k_t^{\alpha} h_t^{\eta}$$

There are two possible ways to introduce exogenous saving rates. 1) exogenous saving rate $s$ for the sum of physical capital and human capital, and the economy will decide how to allocate between physical capital and human capital 2) exogenous saving rates $s_k$ and $s_h$ for physical capital and human capital respectively.

For the first case, the law of motion of the sum of physical and human capital is

$$\dot{k} + \dot{h} = sk^{\alpha}h^{\eta} - (\delta + n + \gamma_A)(k + h)$$

Additionally, households allocate between human capital and physical capital so that the marginal returns are equalized:

$$MPK - \delta = MPH - \delta$$

That is $\alpha h = \eta k$. Substitute back to the law of motion equation to eliminate $h$, we can get

$$\dot{k} = s(\frac{\eta^{\eta}\alpha^{1-\eta}}{\alpha + \eta})k^{\alpha+\eta} - (\delta + n + \gamma_A)k$$

For the second case, the law of motions of physical and human capital are

$$\dot{k} = s_k k^{\alpha} h^{\eta} - (\delta + n + \gamma_A)k$$

$$\dot{h} = s_h k^{\alpha} h^{\eta} - (\delta + n + \gamma_A)h$$

Notice that in the second case, we cannot impose the condition that $MPK - \delta = MPH - \delta$. This is because, once we have exogenous saving rates for both physical and human capital, then given $k_0, h_0$, the path for $k_t$ and $h_t$ are pinned down by the two law of motions. There is no freedom to adjust $k_t$ or $h_t$ to equalize marginal returns of physical and human capital.

# 2 Golden Rule

In the Solow-Swan model, consumption is

$$c_t = (1-s)f(k_t) \tag{15}$$

In the steady state $k^*$ is a function of saving rate $s$ and higher $s$ leads to higher $k^*$ thus higher output per capita $f(k^*)$. However, higher saving rate also means smaller fraction of output per capita will go to consumption. So there is a trade-off here if one wants to achieve higher consumption. The question is what value of saving rate $s$ can deliver the highest steady state consumption $c^*$.

We want to maximize the following by choosing $s$

$$c^* = (1-s)f[k^*(s)] \tag{16}$$

In steady state, $sf(k^*) = (n+\delta)k^*$, therefore, our objective is to maximize

$$c^* = (1-s)f[k^*(s)] = f[k^*(s)] - (n+\delta)k^*(s) \tag{17}$$

First order condition:

$$\{f'[k^*(s)] - (n+\delta)\}\frac{dk^*(s)}{ds} = 0 \tag{18}$$

As $\frac{dk^*(s)}{ds} > 0$, we have

$$f'[k^*(s)] = (n+\delta) \tag{19}$$

Denote the corresponding $k^*$ as $k_{gold}$, saving rate $s$ as $s_{gold}$.

$$f'(k_{gold}) = n+\delta \tag{20}$$

Once we have $k_{gold}$, we can use the relationship $k^*(s)$ to infer $s_{gold}$.

**Remark.** With Cobb-Douglas production function (capital share $\alpha$), the above condition becomes

$$\alpha k_{gold}^{\alpha-1} = n+\delta \tag{21}$$

and we also know that in the Solow-Swan model, in steady state

$$sf(k_{gold}) = (n+\delta)k_{gold} \tag{22}$$

i.e.

$$sk_{gold}^{\alpha} = (n+\delta)k_{gold} \tag{23}$$

Comparing equation (21) and (22) we conclude that $s_{gold} = \alpha$ for this special case.

# 3 Absolute and Conditional Convergence

## 3.1 Absolute Convergence

In the Solow-Swan model,

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta+n) \tag{24}$$

So the growth rate $\frac{\dot{k}}{k}$ depends on the level of $k$.

$$\frac{d(\frac{\dot{k}}{k})}{dk} = s\frac{f'(k) - kf(k)}{k^2} < 0 \tag{25}$$

This implies higher the level of capital per capita, the slower the growth rate of capital per capita: poor countries should grow faster than rich countries. Note the underlying assumption is that poor countries and rich countries share the similar characteristics (parameters). The hypothesis that poor countries tend to grow faster than rich ones without conditioning on any other characteristics of economies is called absolute convergence (i.e. a negative relationship between the level of output per capita and the growth rate of output per capita).

In the data, when we have a broad set of countries (countries from both poor countries and rich countries), the hypothesis in fact fails. While if we look at more homogeneous groups, e.g. only OECD countries or states within U.S., evidence accepts the hypothesis instead.

## 3.2 Conditional Convergence

Next we drop the assumption that countries share similar characteristics. We will proceed with a simple example. Suppose two countries only differ in their saving rates. So they will also differ in their steady state $k^*$. The country with higher saving rate will have higher $k^*$. Again the growth rate of capital per capita is

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n) \tag{26}$$

and in steady state

$$s\frac{f(k^*)}{k^*} = (\delta + n) \tag{27}$$

Substitute equation (27) to equation (26) to eliminate $s$.

$$\frac{\dot{k}}{k} = (\delta + n)[\frac{f(k)/k}{f(k^*)/k^*} - 1] \tag{28}$$

Assume the production function is in Cobb-Douglas form, then

$$\frac{\dot{k}}{k} = (\delta + n)[(\frac{k}{k^*})^{\alpha-1} - 1] \tag{29}$$

Then it is clear that the growth rate depends on the distance of $k$ to steady state $k^*$ ($\frac{k}{k^*}$ matters). It becomes possible that a richer country can grow faster than a poorer country as they have different steady states.

Therefore, in order to account for the difference in these two countries' growth rate, one has to look at both the current level of $k_t$ and the steady state level $k^*$ which depends on country characteristics, i.e. country characteristics matter. After controlling variables that proxy for differences in steady state positions, we can get a significantly negative relationship between per capita growth rate and the log of initial real per capita GDP. In short, data supports the conditional convergence hypothesis.

# 4 The Speed of Convergence

We have discussed roughly about the speed of convergence. Now we formally define this concept speed of convergence as

$$\beta = -\frac{\partial(\dot{k}/k)}{\partial logk} \tag{30}$$

We assume that countries are near to their steady state (in order to use approximations). We will also adopt Cobb-Douglas production function.

$$\frac{\dot{k}}{k} = sAk^{\alpha-1} - (\delta + n + \gamma_A) = sAe^{(\alpha-1)log(k)} - (\delta + n + \gamma_A) \tag{31}$$

Around the steady state

$$\beta^* = -\frac{\partial(\dot{k}/k)}{\partial logk}|_{k^*} = -\frac{\partial(sAe^{(\alpha-1)logk} - (\delta + n + \gamma_A))}{\partial logk}|_{k^*} = (1-\alpha)sAk^{*\alpha-1} \tag{32}$$

Recall that with technology progress in steady state $sAk^{*\alpha-1} = \delta + n + \gamma_A$.

$$\beta^* = (1-\alpha)(\delta + n + \gamma) \tag{33}$$

The above procedures are equivalent (the same thing) to first log-linearize the right hand side of equation (31) and take the coefficient directly

$$\frac{\dot{k}}{k} = sAk^{\alpha-1} - (\delta+n) \approx -\frac{\partial(sAe^{(\alpha-1)logk} - (\delta + n + \gamma_A))}{\partial logk}|_{k^*}(logk - logk^*) = -\beta^*(logk - logk^*) \tag{34}$$

**Remark.** A more general form of log-linearization is the following. Assume we want to log-linearize a multivariate function $y = f(x_1, ..., x_n)$ around $x^*$.

$$y = f(x_1, ..., x_n) = f(e^{log(x_1)}, ..., e^{log(x_n)}) \tag{35}$$

Define $z_i = log(x_i)$, then the first order approximation is

$$y = f(x_1, ..., x_n) = f(e^{z_1}, ..., e^{z_n}) \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}|_{x^*} \cdot e^{z_i^*} \cdot (z_i - z_i^*) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}|_{x^*} \cdot x_i^* \cdot [log(x_i) - log(x_i^*)] \tag{36}$$

Notice for the left hand side, we didn't do any thing. In the future (the second half of this semester), we will do log-linearization on both sides. We are interested in the percentage deviation of variables from its steady state. You can ignore the following now.

$$y = f(x_1, ..., x_n) \tag{37}$$

Then

$$log(y) - log(y^*) = \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* \cdot [log(x_i) - log(x_i^*)] \tag{38}$$

We will come back to this later in the second half of the semester.

# 5 Some Measurement Notes

## 5.1 Capital Stock Measurement

Perpetual Inventory Method (PIM) is so far perhaps the most popular way to compute gross capital stock. In discrete time, capital accumulation is

$$K_{t+1} = (1-\delta)K_t + I_t \tag{39}$$

$$\rightarrow K_t = \sum_{i=0}^{+\infty} (1-\delta)^i I_{t-(i+1)} \tag{40}$$

But unfortunately in the data, we don't have an infinite series of investment.

$$K_t = (1-\delta)^{t-1} K_0 + \sum_{i=0}^{t-1} (1-\delta)^i I_{t-(i+1)} \tag{41}$$

How to get the initial capital stock $K_0$?
In neoclassical growth model, under balanced growth path (constant growth rate)

$$g_{GDP} = g_K = \frac{K_t - K_{t-1}}{K_{t-1}} = \frac{I_t}{K_{t-1}} - \delta \tag{42}$$

Therefore,

$$K_0 = \frac{I_1}{g_{GDP} + \delta} \tag{43}$$

But the economy may not be on a balanced growth path, we can use $g_I$ instead of $g_{GDP}$, it is still an approximation anyway.

$$K_0 = \frac{I_1}{g_I + \delta} \tag{44}$$

To obtain $g_I$, we can choose a three-year average or more years' average.

## 5.2 Growth Accounting

Assume the production function is

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{45}$$

Define

$$SR_t = \frac{Y_t}{K_t^{\alpha} L_t^{1-\alpha}} \tag{46}$$

Then we get the data of $SR$.

$$\frac{\dot{SR}_t}{SR_t} = \frac{\dot{Y}_t}{Y_t} - \alpha\frac{\dot{K}_t}{K_t} - (1-\alpha)\frac{\dot{L}_t}{L_t} \tag{47}$$

$sr_t = \frac{\dot{SR}_t}{SR_t}$ is the Solow residual: the growth of output that cannot be accounted by the growth of capital and labor.

$$\frac{\dot{Y}_t}{Y_t} = sr_t + \alpha\frac{\dot{K}_t}{K_t} + (1-\alpha)\frac{\dot{L}_t}{L_t} \tag{48}$$

or write it in discrete time

$$\frac{Y_t - Y_{t-1}}{Y_{t-1}} = \frac{SR_t - SR_{t-1}}{SR_{t-1}} + \alpha\frac{K_t - K_{t-1}}{K_{t-1}} + (1-\alpha)\frac{L_t - L_{t-1}}{L_{t-1}} \tag{49}$$

This equation can be used to find out the sources of a particular economy's economic growth. Alwyn Young (1995, QJE) uses this method (not exactly the same, e.g. they use a more complicated production function and take human capital into account as well etc., but the main idea is similar) to show that the East Asian growth miracles are largely due to capital accumulation $\frac{\dot{K}_t}{K_t}$ and increasing labor force participation $\frac{\dot{L}_t}{L_t}$.

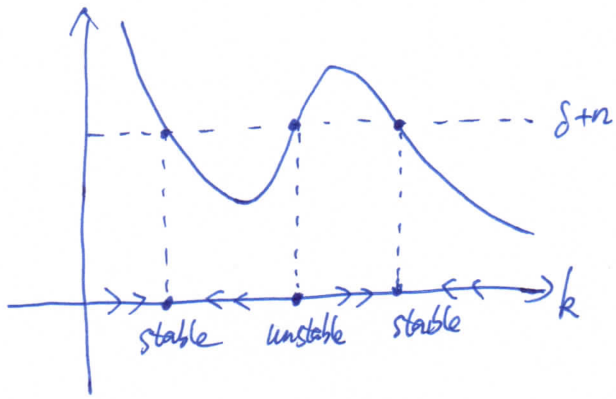



Figure 1.

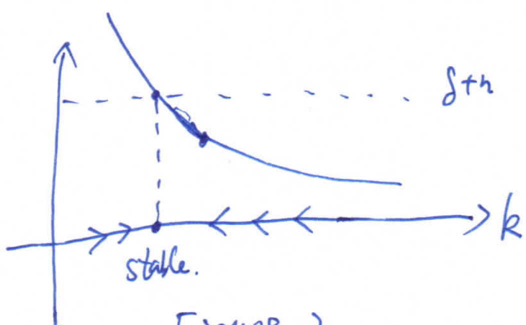



Figure 2

# Endogenous Growth Models

Macroeconomic Analysis Recitation 5

Yang Jiao*

## 1 A Congestion Model

In class, we learned a growth model with government spending. In that model, production function is $Y_i = AL_i^{1-\alpha}K_i^{\alpha}G^{1-\alpha}$, where $G$ is government spending. In this setup, government spending is a public good. However, in reality, many government spending is not, such as highways. Now we introduce a congestion model with a continuum of firms, where the production function for firm $i \in [0, 1]$ is

$$Y_i = AK_i^{\alpha}L_i^{1-\alpha}f(G/K) \tag{1}$$

where $G$ is government spending and $K$ is aggregate capital: $K = \int_0^1 K_i di$. We assume that $f$ satisfies $f' > 0$ and $f'' < 0$.

### 1.1 Decentralized Economy

**Firms' problem** is static as before

$$\max_{K_{it}, L_{it}} AK_{it}^{\alpha}L_{it}^{1-\alpha}f(G/K) - w_t L_{it} - R_t K_{it} \tag{2}$$

F.O.C.s are

$$\alpha AK_{it}^{\alpha-1}L_{it}^{1-\alpha}f(G/K) = R_t \tag{3}$$

$$(1-\alpha)AK_{it}^{\alpha}L_{it}^{-\alpha}f(G/K) = w_t \tag{4}$$

In equilibrium $L = \int_0^1 L_i di$ as well, and all firms are symmetric, so $L_{it} = L$, $K_{it} = K$ for $i \in [0, 1]$.

In per capita terms:

$$\alpha Ak_t^{\alpha-1}f(g_t/k_t) = R_t \tag{5}$$

$$(1-\alpha)Ak_t^{\alpha}f(g_t/k_t) = w_t \tag{6}$$

**Households' problem** is quite standard as in Ramsey model,

*Please email me if you find errors or typos to yj2279@columbia.edu. All comments and suggestions are welcome and appreciated.

$$\max_{a_t, c_t} \int_0^{+\infty} e^{-(\rho-n)t} \frac{c_t^{1-\theta} - 1}{1-\theta} dt \tag{7}$$

s.t.

$$\dot{a}_t = r_t a_t + w_t - c_t - n a_t - \tau_t \tag{8}$$

$$\lim_{t \to +\infty} e^{-rt} a_t \geq 0 \tag{9}$$

with $a_0 > 0$ given. $\tau_t$ is a lump sum tax in per capita term $\tau_t = \frac{T_t}{L_t}$.

Set up the Hamiltonian

$$\mathscr{H} = e^{-(\rho-n)t} \frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t (r_t a_t + w_t - c_t - n a_t - \tau_t) \tag{10}$$

F.O.C.s are

$$e^{-(\rho-n)t} c_t^{-\theta} - \lambda_t = 0 \tag{11}$$

$$-\dot{\lambda}_t = \lambda_t (r_t - n) \tag{12}$$

$$\lim_{t \to +\infty} \lambda_t a_t = 0 \tag{13}$$

Therefore,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) \tag{14}$$

The rate of return from investing in risk free bond is $r_t$ and the rate of return from investing in physical capital is $R_t - \delta$. By **no arbitrage condition**, they are equal

$$r_t = R_t - \delta \tag{15}$$

**Market clearing condition** gives

$$a_t = k_t + b_t \tag{16}$$

and $b_t = 0$.

**Balanced government budget**:

$$G_t = T_t \tag{17}$$

or in per capita term

$$g_t = \tau_t \tag{18}$$

We conclude that equation (14) and (15) gives the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\alpha A k_t^{\alpha-1} f(g_t/k_t) - \delta - \rho) \tag{19}$$

The law of motion of capital comes from equation (3)(4)(8)(15)

$$\dot{k}_t = A k_t^{\alpha} f(g_t/k_t) - c_t - (n+\delta)k_t - g_t \tag{20}$$

which is in fact the resource constraint for the economy.

Notice we haven't chosen the sequence of $g_t$ yet to maximize household utility. That's going to be a complicated problem. Perhaps what we can do is to let the government set a constant $g$ and compare steady states to choose an optimal fiscal policy $g$. For simplicity, let's just take $g_t$ as exogenous policy.

We are interested in whether the above decentralization problem is socially optimal or if not efficient, is there too much or too little capital in the decentralized economy.

### 1.2 Social Planner

Social planner maximizes household utility subject to resource constraint.

$$\max_{a_t, c_t} \int_0^{+\infty} e^{-(\rho-n)t} \frac{c_t^{1-\theta}-1}{1-\theta} dt \tag{21}$$

s.t. resource constraint

$$\dot{k}_t = Ak_t^{\alpha-1} f(g_t/k_t) - c_t - (n+\delta)k_t - g_t \tag{22}$$

From F.O.C.s, we derive Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\alpha Ak_t^{\alpha-1}[f(g_t/k_t) - \frac{g_t}{k_t} f'(\frac{g_t}{k_t})] - \delta - \rho) \tag{23}$$

What we find is that the decentralized case doesn't have the same formula of the Euler equation as that of the social planner's problem. The externality comes from the fact that each individual firm doesn't consider their capital choice's congestion effect on other firms, so they over employ capital: a traditional Tragedy of Commons problem.

To have a rough idea why we think decentralized case will over accumulate capital, we can assume a common steady state policy $g$ for these two cases. Then we compare the steady state $k$, which comes from the Euler equation. For decentralized case, in steady state

$$\alpha Ak^{\alpha-1} f(g/k) = \delta + \rho$$

For social planner, in steady state

$$\alpha Ak^{\alpha-1} f(g/k) = \frac{g}{k} f'(g/k) + \delta + \rho > \delta + \rho$$

Comparing the above two equations, it is easy to verify that decentralized case have higher steady state $k$.

**Remark.** We deliberately choose a lump sum tax in order not to impose additional externality so that we can isolate the aformentioned externality which firms don't internize.

# 2 A Constant R&D Cost Model (Social Planner)

In class, you have learned how to solve a constant *R&D* cost model in the decentralized case. Now we move to social planner's problem and see whether there is any inefficiency that needs to be corrected in the decentralized economy.

The social planner's problem is to

$$\max_{C_t, N_t, X_t} \int_0^{+\infty} \frac{C_t^{1-\theta} - 1}{1-\theta} dt \tag{24}$$

s.t. resource constraint

$$AL^{1-\alpha} N_t X_t^{\alpha} = C_t + \eta \dot{N}_t + N_t X_t \tag{25}$$

Re-write the resource constraint as

$$\dot{N}_t = \frac{1}{\eta}(AL^{1-\alpha} N_t X_t^{\alpha} - C_t - N_t X_t) \tag{26}$$

The corresponding Hamiltonian is

$$\mathscr{H} = e^{-\rho t} \frac{C_t^{1-\theta} - 1}{1-\theta} + \lambda_t \frac{1}{\eta}(AL^{1-\alpha} N_t X_t^{\alpha} - C_t - N_t X_t) \tag{27}$$

First order conditions are

$$\mathscr{H}_C = e^{-\rho t} C_t^{-\theta} - \frac{\lambda_t}{\eta} = 0 \tag{28}$$

$$\mathscr{H}_X = AL^{1-\alpha} N_t \alpha X_t^{\alpha - 1} - N_t = 0 \tag{29}$$

$$\mathscr{H}_N = \lambda_t \frac{1}{\eta}(AL^{1-\alpha} X_t^{\alpha} - X_t) = -\dot{\lambda}_t \tag{30}$$

We obtain

$$-\frac{\dot{\lambda}_t}{\lambda_t} = \frac{1}{\eta}(AL^{1-\alpha} X_t^{\alpha} - X_t) = \frac{1}{\eta}\left(A^{\frac{1}{1-\alpha}} L \alpha^{\frac{1}{1-\alpha}} (\frac{1}{\alpha} - 1)\right). \tag{31}$$

And finally

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(-\frac{\dot{\lambda}_t}{\lambda_t} - \rho) = \frac{1}{\theta}\left[\frac{1}{\eta}\left(A^{\frac{1}{1-\alpha}} L \alpha^{\frac{1}{1-\alpha}} (\frac{1}{\alpha} - 1)\right) - \rho\right] \tag{32}$$

Notice that in order to fully solve the model, one needs to use consumption growth rate equation (32) and also the resource constraint equation (26) to pin down the dynamics of $N_t$ and initial $C_0$, given $N_0$.

The consumption growth rate in the social planner case is higher than that of the decentralized economy case which you came across in class. The reason is that in the decentralized case $R\&D$ firms have monopoly power, so they under-produce. In order to achieve efficiency, one can subsidize $R\&D$ firms or final good firms. However, although subsidizing $R\&D$ cost can deliver the same growth rate of consumption as in the social planner case, the level of consumption (that is $C_0$) is not optimal. Intuitively, the monopoly power of $R\&D$ firms has not been corrected yet (given the number of intermediate good firms $N_t$). One can check by employing equation (32) and equation (26).

# 3 A Rising Cost of R&D model (Decentralized Economy)

In class, we learned a expanding variety model of growth with constant $R\&D$ cost. Now let's turn to a setting with increasing cost of $R\&D$: $\eta'(N) > 0$. Specifically, let's assume that

$$\eta(N) = \phi N^{\sigma} \tag{33}$$

Free entry condition says

$$V_t = \eta(N_t) = \phi N_t^{\sigma} \tag{34}$$

**Remark.** Free entry condition above holds only when there is positive entry. In general, it should be a complementary slackness condition $\dot{N}_t[\eta(N_t) - V_t] = 0$, $\dot{N}_t \geq 0$ and $\eta(N_t) \geq V_t$. So if $\dot{N}_t$ is strictly greater than 0, we will have $\eta(N_t) = V_t$.

Interest rate is

$$r_t = \frac{\pi + \dot{V}_t}{V_t} = \frac{\pi}{\phi N_t^{\sigma}} + \sigma \frac{\dot{N}_t}{N_t} \tag{35}$$

Households' problem is

$$\int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\theta} - 1}{1-\theta} dt \tag{36}$$

s.t.

$$\dot{a}_t = w_t L + r_t a_t - C_t \tag{37}$$

The Euler equation from the above households' problem is

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}(\frac{\pi}{\phi N_t^{\sigma}} + \sigma \frac{\dot{N}_t}{N_t} - \rho) \tag{38}$$

where we have substituted $r_t$ from equation (35).

When $\sigma = 0$, we go back to the case where innovation cost is constant, and this equation alone can tell us the growth rate of $C_t$. However, now we have to deal with the dynamics of $N_t$ as well.

Note in this model, the asset held by households is the market value of all firms $a = \eta \cdot N$. Cobb-Douglas final good production function means $wL = (1-\alpha)Y$. Plug these two equations and also equation (35) into equation (37) to generate

$$\eta(N_t)\dot{N}_t = N_t \pi + (1-\alpha)Y_t - C_t \tag{39}$$

Since $Y = AL^{1-\alpha}\sum_{j=1}^{N} X_j^{\alpha}$, $X_j = A^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}L$ and $\pi = LA^{\frac{1}{1-\alpha}}\frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}} = LA^{\frac{1}{1-\alpha}}(\alpha - \alpha^2)\alpha^{\frac{2\alpha}{1-\alpha}}$(One can check that by some manipulation, we can express the above as $\eta\dot{N}_t = Y_t - C_t - N_t X_t$ which is in fact the resource constraint), we obtain

$$\eta(N_t)\dot{N}_t = N_t \pi + (1-\alpha)Y_t - C_t = (1-\alpha^2)A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}LN_t - C_t \tag{40}$$

That is

$$\frac{\dot{N}_t}{N_t} = \frac{\psi}{\phi} N_t^{-\sigma} - \frac{C_t}{\phi} N_t^{-(1+\sigma)} \tag{41}$$

with $\psi = (1-\alpha^2)A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}L$

Substitute equation (41) to equation (38) to yield

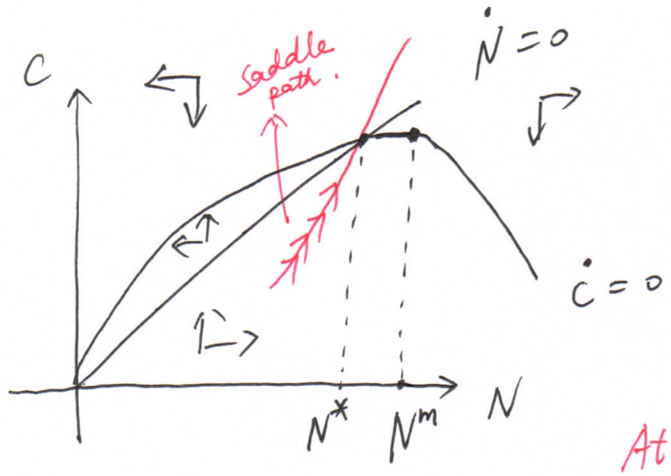
$$\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}(\frac{\pi}{\phi} N_t^{-\sigma} + \sigma\frac{\psi}{\phi} N_t^{-\sigma} - \sigma\frac{C_t}{\phi} N_t^{-(1+\sigma)} - \rho) \tag{42}$$

Equation (41) and (42) define a differential equation system.

To sum up, the main procedure is, first we start from Euler equation and households' budget constraint, and then we try to substitute variables except $C_t$ and $N_t$.

We then display the phase diagram. See Figure 1. $N^m = (\frac{\pi+\sigma\psi}{\rho\phi(1+\sigma)})^{1/\sigma}$ and $N^* = (\frac{\pi}{\rho\phi})^{1/\sigma}$. Notice $N^m > N^*$ because of $\psi > \pi$.

In this rising cost of $R\&D$ setting, there is no long run growth. While with constant $R\&D$ cost, we do have long run growth.




At $N^m$, $\dot{c}=0$ schedule achieves the maximum.

Figure 1

# Suggested Solution-Problem Set 1

Yang Jiao*

## 1 Cobb Douglas Technology

**a).**

1. Constant returns to scale

$$F(\lambda K, \lambda L, A) = A(\lambda K)^{\alpha}(\lambda L)^{1-\alpha} = \lambda A K^{\alpha} L^{1-\lambda} = \lambda F(K, L, A)$$

2. Positive marginal product and diminishing marginal returns

$$F_K = \alpha A K^{\alpha-1} L^{1-\alpha} > 0$$

$$F_L = (1-\alpha) A K^{\alpha} L^{-\alpha} > 0$$

$$F_{KK} = \alpha(\alpha-1) A K^{\alpha-2} L^{1-\alpha} < 0$$

$$F_{LL} = -\alpha(1-\alpha) A K^{\alpha} L^{-\alpha-1} < 0$$

Some books also list the negative semi-definite requirement for Hessian matrix. One can check Cobb-Douglas production function satisfies this property as well since

$$F_{KK} F_{LL} - F_{KL}^2 = 0$$

3. Inada conditions:

$$\lim_{K \to 0} F_K = \lim_{K \to 0} \alpha A K^{\alpha-1} L^{1-\alpha} = \infty$$

$$\lim_{L \to 0} F_L = \lim_{L \to 0} (1-\alpha) A K^{\alpha} L^{-\alpha} = \infty$$

$$\lim_{K \to \infty} F_K = \lim_{K \to \infty} \alpha A K^{\alpha-1} L^{1-\alpha} = 0$$

$$\lim_{L \to \infty} F_L = \lim_{L \to \infty} (1-\alpha) A K^{\alpha} L^{-\alpha} = 0$$

**b).**

*Please email me if you find errors or typos to yj2279@columbia.edu

Set up the Hamiltonian:

$$\mathscr{H} = e^{-rt}(AK_t^{\alpha}L_t^{1-\alpha} - w_tL_t - I_t) + \lambda_t(I_t - \delta K_t)$$

Then the first order conditions are:

$$\mathscr{H}_L = 0 : w_t = (1-\alpha)AK_t^{\alpha}L_t^{-\alpha}$$

$$\mathscr{H}_I = 0 : \lambda_t = e^{-rt}$$

$$\mathscr{H}_K = 0 : -\dot{\lambda}_t = e^{-rt}\alpha AK_t^{\alpha-1}L_t^{1-\alpha} - \lambda_t\delta$$

$$TVC : \lim_{t\to\infty} \lambda_t K_t = 0$$

From $\lambda_t = e^{-rt}$, we have $\dot{\lambda}_t = -re^{-rt}$. Combine with the above equation $\mathscr{H}_K = 0$, we get

$$r + \delta = \alpha AK_t^{\alpha-1}L_t^{1-\alpha}$$

Therefore, given that labor supply is fixed $L_t = L$, capital is

$$K_t = (\frac{\alpha A}{r+\delta})^{\frac{1}{1-\alpha}}L$$

**c).**
Define $q_t = e^{rt}\lambda_t$, so $\lambda_t = e^{-rt}q_t \to \dot{\lambda}_t = e^{-rt}(\dot{q}_t - rq_t)$. Then the first order conditions are:

$$\mathscr{H}_L = 0 : w_t = (1-\alpha)AK_t^{\alpha}L_t^{-\alpha}$$

$$\mathscr{H}_I = 0 : q_t = 1$$

$$\mathscr{H}_K = 0 : -(\dot{q}_t - rq_t) = \alpha AK_t^{\alpha-1}L_t^{1-\alpha} - q_t\delta$$

$$TVC : \lim_{t\to\infty} e^{-rt}q_tK_t = 0$$

We then have the same formula for capital

$$K_t = (\frac{\alpha A}{r+\delta})^{\frac{1}{1-\alpha}}L$$

**d).** See Figure 1
**e).** See Figure 2
**f).** See Figure 3. But change of $I = \delta K = (\frac{\alpha A\delta^{1-\alpha}}{r+\delta})^{\frac{1}{1-\alpha}}$ becomes ambiguous after time $t_0$ since it is not a monotonic function of $\delta$.
**g).** The same as f) except that the jump happens at time $t_1$.

# 2 Accelerator Theory of Investment

**a).**

$$F_K = \alpha\frac{Y_t}{K_t} = r + \delta$$

We can express $K$ as a function of $Y$

$$K_t = \frac{\alpha Y_t}{r+\delta}$$

**b).**

$$\rightarrow \dot{K}_t = \frac{\alpha \dot{Y}_t}{r+\delta}$$

It is consistent with the accelerator theory of investment. Higher growth in output will lead to higher growth in capital stock which in turn pushes up output growth.

**c).**

Yes. Within neoclassical production function (and we also assume it is continuously differentiable), it is still true.

We first introduce one mathematical property. If $F(K, L, A)$ is homogeneous of degree $\beta$ in $(K, L)$, then $F_K$ and $F_L$ are homogeneous of degree $\beta - 1$ in $(K, L)$. The following is the proof:

$$F(\lambda K, \lambda L, A) = \lambda^\beta F(K, L, A)$$

Differentiate both sides with respect to $K$:

$$\lambda F_K(\lambda K, \lambda L, A) = \lambda^\beta F_K(K, L, A)$$

$$\rightarrow F_K(\lambda K, \lambda L, A) = \lambda^{\beta-1} F_K(K, L, A)$$

The fact that $F(K, L, A)$ is a neoclassical production function implies $\beta = 1$, then $F_K$ is homogeneous of degree 0. Employing this property, since $F_K(K, L, A) = r + \delta$, we will also have $F_K(\frac{K}{L}, 1, A) = r + \delta$ which shows a relationship between $\frac{K}{L}$ and $r + \delta, A$. Additionally, because of $F_{KK} < 0$, we know $\frac{K_t}{L_t}$ will be a monotonic function of $r+\delta$. Note $\frac{K_t}{L_t}$ is also going to be a function of $A$. Denote this function as $h(\cdot, \cdot)$.

$$\frac{K_t}{L_t} = h(A, r+\delta)$$

Also $Y_t = F(K_t, L_t, A) = K_t F(1, \frac{L_t}{K_t}, A) = K_t F[1, h^{-1}(A, r+\delta), A]$. That is, we still have $K_t$ and $Y_t$ are proportional thus they satisfy the accelerator theory of investment.

# 3 Q-Theory Without Learning-by-Doing

**a).**

Set up the Hamiltonian:

$$\mathscr{H} = e^{-rt}[F(K_t, L_t) - w_t L_t - I_t(1 + \varphi(I_t))] + \lambda_t(I_t - \delta K_t)$$

First order conditions are

$$F_L = w_t$$

$$1 + \varphi + I\varphi' = \lambda_t e^{rt}$$

$$-\dot{\lambda}_t = e^{-rt} F_K - \delta \lambda_t$$
$$\lim_{t \to \infty} \lambda_t K_t = 0$$

**b).**

Current value $q_t = e^{rt} \lambda_t$. Re-write the first order conditions:

$$\begin{aligned} F_L &= w_t \\ 1 + \varphi + I\varphi' &= q_t \qquad (1) \\ rq_t - \dot{q}_t &= F_K - \delta q_t \\ \lim_{t \to \infty} e^{-rt} q_t K_t &= 0 \end{aligned}$$

**c).**

Equation (1) establishes a relationship between $I$ and $q$: $I = h(q)$ with $h(1) = 0$. The two differential equations we have here are:

$$\dot{K}_t = I_t - \delta K_t = h(q_t) - \delta K_t$$
$$\dot{q}_t = (r + \delta) q_t - F_K$$

Set $\dot{K}_t = 0$ and $\dot{q}_t = 0$ respectively.

$$h(q) = \delta K$$
$$(r + \delta) q = F_K$$

See Figure 4 for the phase diagram.

**d).**

The steady state $q^*$ is the solution of

$$(r + \delta) q = F_K\left(\frac{h(q)}{\delta}, L\right)$$

$q^* > 1$ since $h(q^*) = \delta K^* > 0$ and $h(1) = 0$, $h(\cdot)$ is an increasing function.

**e).**

See Figure 4 as well for the dynamics after the interest rate shock. It will shift down $\dot{q} = 0$ scehdule. The steady state $q^*$ will be smaller. This is because a positive interest rate shock will raise the opportunity cost of investment (or equivalently to say, the future is more heavily discounted), therefore, firms will dis-invest, leading to less capital stock in steady state. Then firms only need to replace less capital by new investment. The less demand for new investment will drive down capital price (here it is shadow price $q^*$).

# 4 The Baby Boom

**i).**

Equaton (a) is the demand function for housing. Higher rents lead to lower housing demand.

Equation (b) is a no-arbitrage condition. Returns from investing in housing market is equal to the return from investing in the risk free bond. Equation (c) is the law of motion of housing stock: increase in the housing stock equals newly build housing minus depreciation of existing housing stock.

**ii).**

From equation (c):

$$\dot{H} = \varphi(P)N - \delta H$$

$$\rightarrow \frac{\dot{H}}{N} = \varphi(P) - \delta\frac{H}{N} \tag{2}$$

Because of $h = \frac{H}{N}$, we can get $\dot{h} = \frac{\dot{H}}{N} - \frac{\dot{N}H}{N^2} = \frac{\dot{H}}{N} - nh$. Substitute it back to equation (2):

$$\dot{h} = \varphi(P) - (\delta + n)h$$

Equation (b) tells us

$$\dot{P} = rP - R(h)$$

See Figure 5 for the phase diagram.

**iii).**

See Figure 5 for the dynamics.

The intuition is in year 1960, due to the expectation that in the future housing demand will shift up, house price goes up and firms build more houses. When the baby boom arrives, per capita housing will be dragged down (new houses are built but slower than population growth, since they understand this baby boom will not last forever). However, when the baby boom retires, per capita housing stock will go up again. And as a result of this demand decrease, prices will also go down.

**iv).**

Once the baby boom retires, from the phase diagram, we know that prices will go down from 2006. In fact, housing prices start to drop earlier than 2006. However, intuitively, it may drop faster after 2006.

# 5 Housing Rent Controls

**i).**

See Figure 6 for the dynamics. Prices will drop, because buying houses to rent is going to be less profitable.

**ii).**

Both price and quantity will decline. The rent control will decrease the demand for investing in buying houses since this investment becomes less profitable.

**iii).**

Suppose poor people rent houses. Therefore, they care about both rental rate $R(h)$ and the available supply of $h$. Initially, rent control will benefit them since housing stock does not change immediately after the policy but rental price will drop. However, in the long run, the market adjusts its supply of housing stock, which will increase the rental price. Since the quantity of housing (demand needs to equal supply) enters into poor people's utility function, the decrease in the available housing stock in the market may hurt them eventually. That is although they would like to consume more housing under cheaper rental price, the supply of housing may be depressed by the rental control policy.

# 6 Diminishing or Increasing Returns to Learning-by-Doing

**i).ii).iii).**

If the firm does not split, the total cost of installing $I$ is $I\varphi(\frac{I}{K^\beta})$. If we split the firm into $N$ identical parts ($N \geq 2$), the total cost of installing $I$ is

$$\sum_{n=1}^{N} \frac{I}{N}\varphi(\frac{I/N}{(K/N)^\beta}) = I\varphi(\frac{I}{K^\beta}N^{\beta-1})$$

If $\beta < 1$, $I\varphi(\frac{I}{K^\beta}N^{\beta-1}) < I\varphi(\frac{I}{K^\beta})$, so the firm wants to split itself. In fact, the larger $N$ is, the lower cost of installing new investment.

If $\beta = 1$, $I\varphi(\frac{I}{K^\beta}N^{\beta-1}) = I\varphi(\frac{I}{K^\beta})$, so the firm is indifferent

If $\beta > 1$, $I\varphi(\frac{I}{K^\beta}N^{\beta-1}) > I\varphi(\frac{I}{K^\beta})$, so the firm doesn't want to split.

**iv).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-rt}[F(K_t, L_t) - w_t L_t - I_t(1 + \varphi(\frac{I_t}{K_t^\beta}))] + \lambda_t I_t$$

First order conditions are

$$F_L = w_t$$
$$\lambda_t = e^{-rt}[1 + \varphi(\frac{I_t}{K_t^\beta}) + \frac{I_t}{K_t^\beta}\varphi'(\frac{I_t}{K_t^\beta})]$$
$$-\dot{\lambda}_t = e^{-rt}[F_K + \beta\frac{I_t^2}{K_t^{\beta+1}}\varphi'(\frac{I_t}{K_t^\beta})]$$
$$TVC: \lim_{t\to\infty} \lambda_t K_t = 0$$

Re-write the above with current shadow price $q_t$:

$$F_L = w_t$$
$$q_t = 1 + \varphi(\frac{I_t}{K_t^\beta}) + \frac{I_t}{K_t^\beta}\varphi'(\frac{I_t}{K_t^\beta}) \tag{3}$$
$$rq_t - \dot{q}_t = F_K + \beta\frac{I_t^2}{K_t^{\beta+1}}\varphi'(\frac{I_t}{K_t^\beta})$$
$$TVC: \lim_{t\to\infty} e^{-rt}q_t K_t = 0$$

Equation (3) defines a relationship between $q$ and $\frac{I}{K^\beta}$:

$$\frac{I_t}{K_t^\beta} = h(q_t)$$

with $h(1) = 0$

Finally, we write down the system of differential equations

$$\dot{K}_t = K^\beta h(q_t)$$

$$\dot{q}_t = rq_t - F_K - \beta \frac{h(q_t)^2}{K_t^{1-\beta}} \varphi'(h(q_t))$$

$\dot{K}_t = 0$ is a flat schedule. As with $\dot{q}_t = 0$, one can show that $\frac{dq}{dK}|_{(K^*,q^*)} = \frac{F_{KK}(K^*,L)}{r} < 0$.

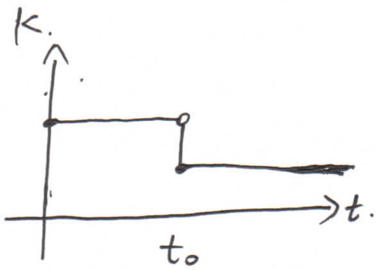
Qualitatively, the phase diagram (around the steady state) does not depend on whether $\beta$ is larger, equal or smaller than 1. See Figure 7 for the phase diagram.

# 7 Housing

Please refer to your recitation note 2.


figure 1.


Figure 2.




I ambiguous, you can draw two possible cases.

Figure 3.

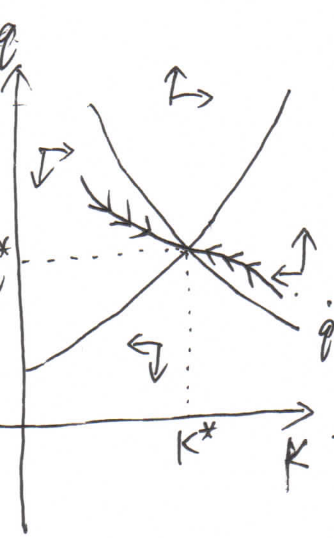



Figure 4.

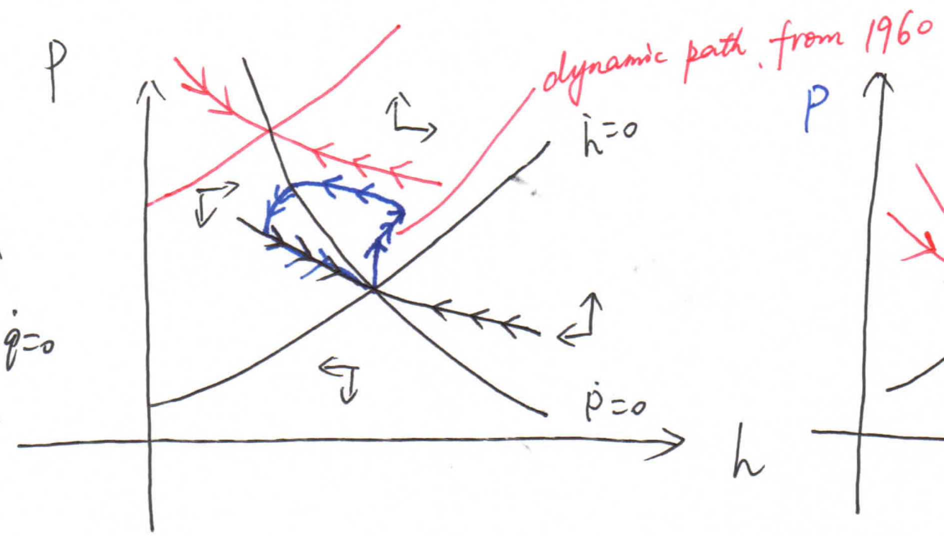



Figure 5

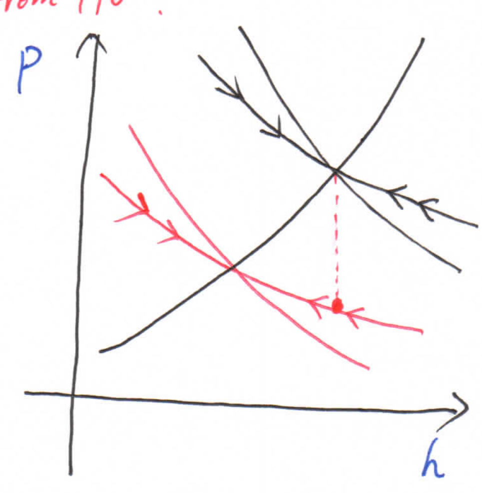



Figure 6.

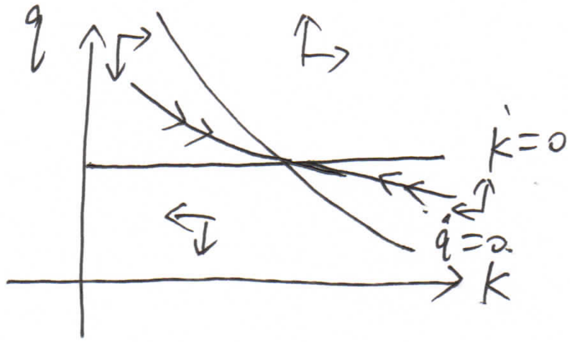



Figure 7

# Suggested Solution-Problem Set 2

Yang Jiao*

## 1 Increasing Savings Rate

**a).**

$$\frac{\dot{k}_t}{k_t} = s\frac{f(k_t)}{k_t} - (\delta + n)$$

With Cobb-Douglas production function as specified in the question

$$\frac{\dot{k}_t}{k_t} = sAk_t^{\alpha-1} - (\delta + n)$$

**b).**
See Figure 1. There are two stable steady states.

**c).**
In steady state

$$\frac{\dot{k}}{k} = sAk^{\alpha-1} - (\delta + n) = 0$$

Then

$$k^* = (\frac{\delta + n}{sA})^{\frac{1}{\alpha-1}}$$

Substitute the corresponding values to yield $k^* = 400$.

**d).** Substitute the corresponding values to yield $k^* = 4900$.

**e).** After receiving the donation, the economy's capital stock is still below 1000, therefore, the economy will return to the low steady state.

**f).** After receiving the donation, the economy's capital stock is above 1000, therefore, the economy will converge to the high steady state gradually.

**g).** Small donations may not be able to push the economy out of their trap since the economy may need to have a large initial capital stock to converge to a higher steady state.

## 2 Diminishing Population Growth Rate

**a).**

$$\frac{\dot{k}_t}{k_t} = sAk_t^{\alpha-1} - (\delta + n) = \frac{2}{\sqrt{k_t}} - (0.08 + n)$$

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**b).c)**
See Figure 2. There are two steady states.
**d).e)**
In steady state

$$\frac{2}{\sqrt{k}} - (0.08 + n) = 0$$

When $k < 200$, $n = 0.12$, we have $k^* = 100$, and $y = A\sqrt{k^*} = 100$. When $k \leq 200$, $n = 0.02$, we have $k^* = 400$, and $y = A\sqrt{k^*} = 200$.
**f).**
The donation is not able to lift the economy to cross $k = 200$, therefore, the economy will return to the low steady state.
**g).**
The donation is able to lift the economy to cross $k = 200$, therefore, the economy will converge to the high steady state.

# 3 Harrod-Domar

**a).**
See Figure 3.
**b).**

$$y = \min\{Ak, B\}$$

**c).**

$$y/k = \min\{A, B/k\}$$

**d).**

$$\dot{k} = s\min\{Ak, B\} - \delta k$$

**e).**

$$\frac{\dot{k}}{k} = s\min\{A, \frac{B}{k}\} - \delta$$

**f).**
See Figure 4. The problem is that there is an excess of capital since in steady state $k^* > \frac{B}{A}$, which means $Ak^* > B$. Notice that this economy's production function is in Leontif form. In order to produce $y = B$, the economy doesn't need to use so much capital. It is going to be a waste of capital.
**g).**
See Figure 5.

$$\frac{\dot{k}}{k} = s\min\{A, \frac{B}{k}\} - \delta \leq sA - \delta < 0$$

The economy will gradually converge to $k^* = 0$.
**h).**

See Figure 6. When initially $k_0 \geq \frac{B}{A}$, the economy will converge to $k^* = \frac{B}{A}$. Otherwise, when $k_0 < \frac{B}{A}$, the economy will stay there.

**i).**

Yes, In f), there is excess of capital for production. In g), the steady state delivers 0 capital per capita. In $h$), steady states are not stable and the economy doesn't reach its potential capacity except the point $k^* = \frac{B}{A}$.

# 4 Inada Or Diminishing Returns

**a).**

Yes, since

$$F(\lambda K, \lambda L) = A\lambda K + B(\lambda K)^{1/2}(\lambda L)^{1/2} = \lambda[AK + BK^{1/2}L^{1/2}] = \lambda F(K, L)$$

**b).**

Yes, because

$$\frac{\partial F}{\partial K} = A + \frac{1}{2}BK^{-\frac{1}{2}}L^{\frac{1}{2}} > 0$$

$$\frac{\partial^2 F}{\partial K^2} = -\frac{1}{4}BK^{-\frac{3}{2}}L^{\frac{1}{2}} < 0$$

**c).**

No. When $K \to \infty$, we get $\frac{\partial F}{\partial K} \to A \neq 0$

**d).**

No, since Inada conditions are not fully satisfied.

**e).**

$$y = Ak + B\sqrt{k}$$

**f).**

$$\frac{\dot{k}}{k} = s\frac{Ak + B\sqrt{k}}{k} - (\delta + n) = s(A + \frac{B}{\sqrt{k}}) - (\delta + n)$$

**g).**

See Figure 7. There is one stable steady state, and there is no long run growth.

**h).**

See Figure 8. There is long run growth and in the long run, growth rate converges to $sA - \delta - n$.

**i).**

$sA - (n + \delta) = 0.2 > 0$, the growth rate will converge to 0.2.

**j).**

No. Let $Y = AK + BL$, and $sA < n + \delta$, then the long run growth rate is zero. (Notice this production function doesn't satisfy Inada conditions.)

**k).**

No. See h). Although diminishing returns to capital is satisfied, we have long run growth.

# 5 Convexities in the Production Function

Please refer to your recitation note 4: a model with poverty trap. Make sure to move up and down of the $n+\delta$ line to discuss all possible cases.

The effective production function is not concave. Large enough donations are possible to drag the economy out of the trap.

# 6 Convergence in Solow-Swan with Physical and Human Capital

**a).**

$$\gamma_k = sk^{\alpha-1} - (\delta+n) = se^{(\alpha-1)log(k)} - (\delta+n)$$

**b).**

Since $y = k^\alpha$, we obtain

$$\gamma_y = \alpha\gamma_k = \alpha se^{(\alpha-1)log(k)} - \alpha(\delta+n) = \alpha se^{\frac{\alpha-1}{\alpha}log(y)} - \alpha(\delta+n)$$

**c).**

Do the first order approximation.

$$\gamma_y \approx 0 + \alpha s\frac{\alpha-1}{\alpha}e^{\frac{\alpha-1}{\alpha}logy^*}(\log y - \log y^*) = (\alpha-1)(n+\delta)(\log y - \log y^*)$$

The speed of convergence is $\beta^* = -\frac{\partial\gamma_y}{\partial\log y} = (1-\alpha)(n+\delta)$

**d).**

Substitute numbers to yield $\beta^* = 0.077$

**e).**

$$\frac{\dot{k}}{k} = s_k Ak^{\alpha-1}h^\eta - (\delta+n)$$

$$\frac{\dot{h}}{h} = s_h Ak^{\alpha}h^{\eta-1} - (\delta+n)$$

**f).**

$y = k^\alpha h^\eta$ implies

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \eta\frac{\dot{h}}{h} \tag{1}$$

From e), we implement the log-linearization

$$\gamma_k \approx 0 + (\alpha-1)s_k Ak^{*\alpha-1}h^{*\eta}[\log k - \log k^*] + \eta s_k Ak^{*\alpha-1}h^{*\eta}[\log h - \log h^*]$$

$$\gamma_h \approx 0 + \alpha s_h Ak^{*\alpha}h^{*\eta-1}[\log k - \log k^*] + (\eta-1)s_h Ak^{*\alpha}h^{*\eta-1}[\log h - \log h^*]$$

Substitute steady state properties $s_k Ak^{*\alpha-1}h^{*\eta} = n+\delta$, $Ak^{*\alpha}h^{*\eta-1} = n+\delta$ to the above two equations. And then go back to equation (1):

$$\gamma_y = \alpha(\alpha+\eta-1)(n+\delta)(\log k - \log k^*) + \eta(\alpha+\eta-1)(n+\delta)(\log h - \log h^*) = (\alpha+\eta-1)(n+\delta)(\log y - \log y^*)$$

So the speed of convergence is

$$\beta^* = (1-\alpha-\eta)(n+\delta) = 0.022$$

See the textbook page 59, $\beta$ is around 0.015 to 0.03 in the data. The above number fits well with the data. However, if you doesn't account for human capital, as in d), you probably get a too small speed of convergence compared to the data.

**g).**

In order to find $\hat{\beta}$, we will run a cross section (different countries) regression of $\frac{\dot{y}}{y}$ on $y_0$, where each observation includes a country's initial $log(y_0)$ and a country's average growth rate $\frac{\dot{y}}{y}$ through years.

Possible econometric issues: 1) Omitted variables issue: one needs to control for country heterogeneity in order to avoid biased estimates of speed of convergence. 2)Measurement error: e.g. measurement error in $y$, and $y$ appears on both sides of the regression ($\frac{\dot{y}}{y}$ on the left and $\log y_0$ on the right, estimate $(X'X)^{-1}X'y$ will be biased if we have correlated measurement error for X and y ). It will bias the estimate. 3) Reverse causality: higher growth leads to higher level of $log(y_0)$, which will make the estimate of $\beta$ downward biased 4)One can also cover the importance of accounting for human capital in the discussion.

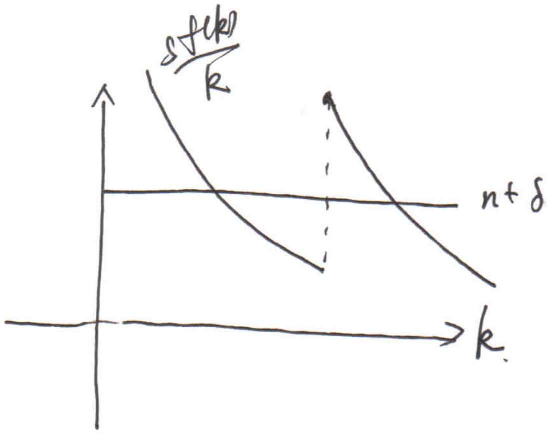



Figure 1

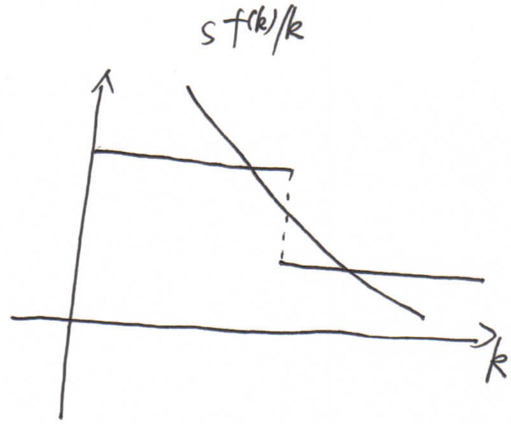
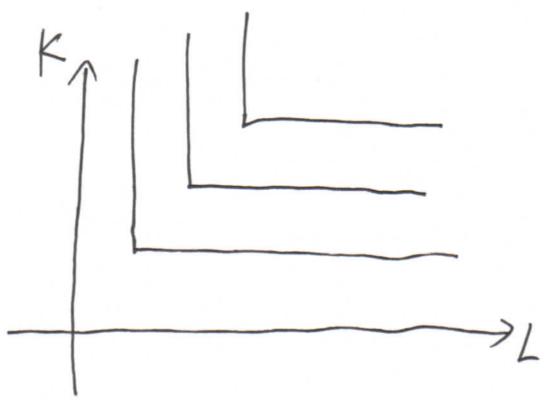



Figure 2

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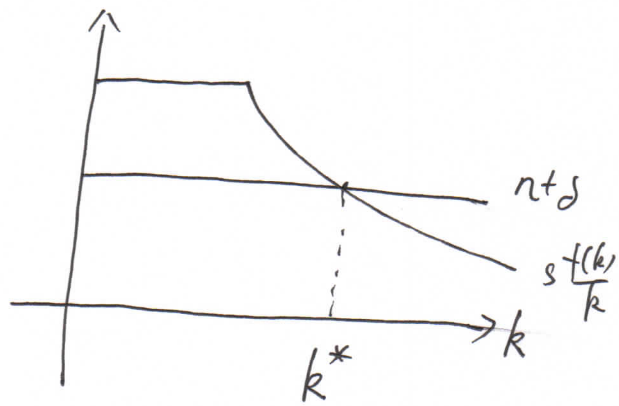


figure 4.

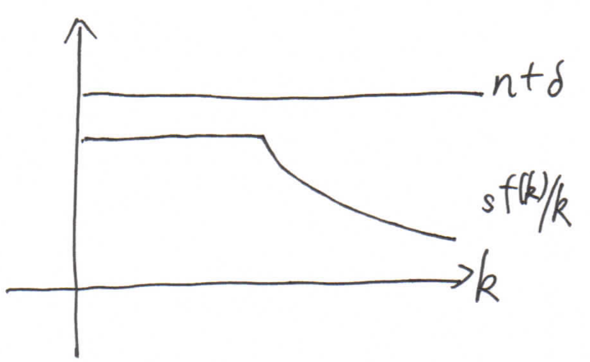


Figure 5

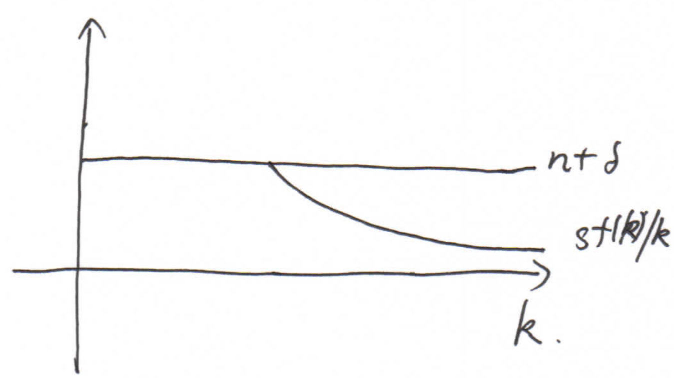


Figure 6

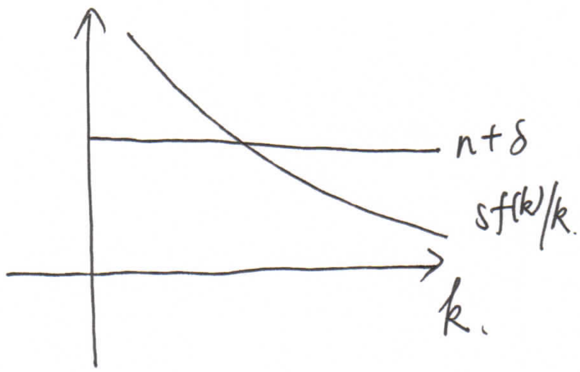


Figure 7

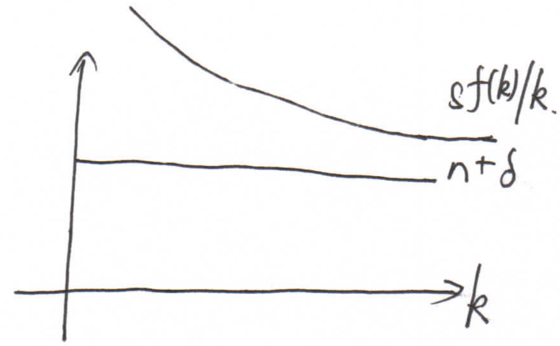


Figure 8.

# Suggested Solution-Problem Set 3

Yang Jiao*

## 1 Increasing Savings Rate

**a).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-\rho t} \frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t (Ak_t - c_t - \delta k_t)$$

F.O.C.s are

$$e^{-\rho t} c_t^{-\theta} = \lambda_t$$

$$-\dot{\lambda}_t = (A - \delta)\lambda_t$$

$$\lambda_T k_T = 0$$

**b).**

The first two conditions imply

$$\frac{\dot{c}_t}{c_t} = \frac{A - \delta - \rho}{\theta}$$

The budget constraint is

$$\dot{k}_t = (A - \delta)k_t - c_t$$

See Figure 1 for phase diagram.

**c).**

Solving the two equations in $b$), we have

$$c_t = c_0 e^{\frac{A-\delta-\rho}{\theta} t}$$

$$k_t = \frac{c_0}{A - \delta - \frac{1}{\theta}(A - \delta - \rho)} e^{\frac{A-\delta-\rho}{\theta} t} + \psi e^{(A-\delta)t}$$

We have $k_0$ and $k_T = 0$ to determine $c_0$ and $\psi$.

$$k_0 = \frac{c_0}{A - \delta - \frac{1}{\theta}(A - \delta - \rho)} + \psi$$

$$\frac{c_0}{A - \delta - \frac{1}{\theta}(A - \delta - \rho)} e^{\frac{A-\delta-\rho}{\theta} T} + \psi e^{(A-\delta)T} = 0$$

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Then the solution to $c_0$ is

$$c_0 = \frac{\frac{1}{\theta}(A-\delta-\rho)-(A-\delta)}{e^{[\frac{1}{\theta}(A-\delta-\rho)-(A-\delta)]T}-1}k_0$$

$$\psi = (1+\frac{1}{e^{[\frac{1}{\theta}(A-\delta-\rho)-(A-\delta)]T}-1})k_0$$

No matter whether $A-\delta > \frac{1}{\theta}(A-\delta-\rho)$ or $A-\delta < \frac{1}{\theta}(A-\delta-\rho)$, $c_0$ is decreasing in $T$.

When $A$ increases, $c_0$ will decrease, so that the econoy can take more time to finally reach the vertical axis.

Now assume $A-\delta > \frac{1}{\theta}(A-\delta-\rho) = \gamma_c$ which is consistent with the assumption in class.

**d).** When $T \to \infty$, we get $c_0 \to [A-\delta-\frac{1}{\theta}(A-\delta-\rho)]k_0$, $\psi \to 0$, and $k_t \to \frac{c_t}{A-\delta-\gamma_c}$, we go to the $AK$ model with permanent consumption growth.

# 2 Spending and Taxes in the Neoclassical Model

**i).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-(\rho-n)t}\frac{c_t^{1-\theta}-1}{1-\theta} + \lambda_t((1-\tau_y)Ak_t^\alpha - \tau_L - c_t - (\delta+n)k_t + \nu_t)$$

F.O.C.s are

$$e^{-(\rho-n)t}c_t^{-\theta} = \lambda_t$$

$$-\dot{\lambda}_t = ((1-\tau_y)\alpha A k_t^{\alpha-1} - \delta - n)\lambda_t$$

$$\lim_{t\to\infty} \lambda_t k_t = 0$$

**ii)**

The first two equations above give

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[(1-\tau_y)\alpha A k_t^{\alpha-1} - \rho - \delta)]$$

Additionally, the budget constraint is

$$\dot{k}_t = (1-\tau_y)Ak_t^\alpha - \tau_L - c_t - (n+\delta)k_t + \nu_t$$

Set $\dot{c}_t = 0$ and $\dot{k}_t = 0$ to obtain the two loci

$$c_t = (1-\tau_y)Ak_t^\alpha - \tau_L - (\delta+n)k_t + \nu_t$$

$$k_t = (\frac{\rho+\delta}{(1-\tau_y)A})^{\frac{1}{\alpha-1}}$$

See Figure 2 for phase diagram.

**iii).**

Both loci will shift. See Figure 3 for dynamics.

**iv).**

$\dot{k} = 0$ schedule will shift down. See Figure 4 for dynamics.

**v).**

If the government rebates taxes, the law of motion of capital is

$$\dot{k_t} = Ak_t^\alpha - c_t - (\delta + n)k_t$$

Change in tax rates will not affect $\dot{k_t} = 0$ line. The reason is that now when rebating taxes, the aggregate resource constraint will not be affected by taxes.

So repeat iii), we find that only $\dot{c_t} = 0$ line shift leftward now. Repeat iv), both loci will not shift.

# 3 Bicycles

**a).**

$$\dot{k_t} = Ak_t^\alpha - c_t - b_t - nk_t$$

**b).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-(\rho-n)t}\frac{(c_t^\beta b_t^{1-\beta})^{1-\theta} - 1}{1-\theta}dt + \lambda_t(Ak_t^\alpha - b_t - c_t - (\delta+n)k_t + \nu_t)$$

F.O.C.s are

$$e^{-(\rho-n)t}\beta(c_t^\beta b_t^{1-\beta})^{-\theta}c_t^{\beta-1}b_t^{1-\beta} = \lambda_t$$
$$e^{-(\rho-n)t}(1-\beta)(c_t^\beta b_t^{1-\beta})^{-\theta}c_t^\beta b_t^{-\beta} = \lambda_t$$
$$-\dot{\lambda_t} = (\alpha Ak_t^{\alpha-1} - n)\lambda_t$$
$$\lim_{t\to\infty}\lambda_t k_t = 0$$

**c).**

$$\frac{c_t}{b_t} = \frac{\beta}{1-\beta}$$

**d).**

$$\frac{\dot{c_t}}{c_t} = \frac{1}{\theta}(\alpha Ak_t^{\alpha-1} - \rho)$$

**e).**

Use result from c) to eliminate $b_t$ in the capital accumulation equation to yield

$$\dot{k_t} = Ak_t^\alpha - \frac{c_t}{\beta} - nk_t$$

**f).**

Let $\dot{c}_t = 0$ and $\dot{k}_t = 0$ respectively:

$$k = (\frac{\rho}{\alpha A})^{\frac{1}{\alpha-1}}$$

$$c = \beta(Ak^{\alpha} - nk)$$

See Figure 5 for phase diagram. There is no long run growth because of the diminishing returns to scale of capital.

**g).**

The Euler equation now shows

$$\frac{\dot{b}_t}{b_t} = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(A - \rho)$$

When $A > \rho$, there is long run growth.

**h).**

$$\dot{k}_t = Ak_t^{\alpha}(1-\tau) - c_t - nk_t$$

**i).**

$$\mathscr{H} = e^{-(\rho-n)t}\frac{(c_t^{\beta}b_t^{1-\beta})^{1-\theta} - 1}{1-\theta}dt + \lambda_t((1-\tau)Ak_t^{\alpha} - c_t - (\delta+n)k_t + \nu_t)$$

F.O.C.s are

$$e^{-(\rho-n)t}\beta(c_t^{\beta}b_t^{1-\beta})^{-\theta}c_t^{\beta-1}b_t^{1-\beta} = \lambda_t$$

$$-\dot{\lambda}_t = (\alpha(1-\tau)Ak_t^{\alpha-1} - n)\lambda_t$$

$$\lim_{t\to\infty}\lambda_t k_t = 0$$

**j).**

$$b_t = \tau A k_t^{\alpha}$$

**l).**

The act ensures the choice of bicycles to be consistent with the optimal allocation between $c_t$ and $b_t$ without government policy.

**m).**

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\alpha(1-\tau)Ak_t^{\alpha-1} - \rho)$$

$$\dot{k}_t = Ak_t^{\alpha} - \frac{c_t}{\beta} - nk_t$$

Now compare with the case without government policy, we have an additional $(1-\tau)$ in the Euler Equation.

**n).**

The aformentioned new tax term will shift $\dot{c}_t = 0$ to the left. See Figure 6. We won't have long run growth due to the diminishing returns to capital, either.

**o).**

As long as $A(1-\tau) > \rho$, we still have long run growth. But the constant growth rate is smaller due to the tax.

**p).**

No. Tax here is distortionary and welfare is decreased. The case without government policy is in fact equivalent to the social planner problem thus achieves the highest welfare.

**q).**

The same as in a). Therefore, r).s).t).u). follow as well.

**v).**

The proportional tax distorts people's behavior of saving.

# 4 Growth and Externalities

**i).**

$$\alpha K^{\alpha-1} L^{1-\alpha} \hat{K}^{\eta} = r + \delta$$

$$\alpha(1-\alpha) K^{\alpha} L^{-\alpha} \hat{K}^{\eta} = w$$

**ii).**

$$\mathscr{H} = e^{-\rho t} \frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t (r_t a_t - w_t - c_t)$$

First order conditions are

$$\lambda_t = e^{-\rho t} c_t^{-\theta}$$

$$-\dot{\lambda}_t = r_t \lambda_t$$

$$\lim_{t\to\infty} \lambda_t a_t = 0$$

**iii).**

From a), $k = \frac{\alpha w}{(1-\alpha)(\delta+r)}$.

Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[\alpha L^{\eta} k^{\eta+\alpha-1} - \delta - \rho]$$

$$\dot{k}_t = k_t^{\alpha+\eta} L^{\eta} - c_t - \delta k_t$$

**iv).**

See Figure 7. There is a steady state which is saddle path stable.

**v).**

Follow the saddle path to track the dynamics. The growth rate in the long run is 0.

**vi).**

Not really. The whole economy still display diminishing returns to scale of capital.

**vii).**

The $\dot{c}_t = 0$ schedule will shift rightward while $\dot{k}_t = 0$ will shift up. So both $k^*$ and $c^*$ go up as $\eta$ increases.

**viii).**

$k^*$ and $c^*$ go to infinity as $\alpha + \eta$ approaches 1.

**ix).**

Firms don't internalize the externality of their investment on other firms, therefore, they in fact underinvest relative to the optimal solution.

One can show that the social planner's problem gives the following Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[(\alpha+\eta)L^\eta - \delta - \rho] = \frac{1}{\theta}[L^\eta - \delta - \rho] > \frac{1}{\theta}[\alpha L^\eta - \delta - \rho]$$

Therefore, social planner is going to enjoy a strictly higher growth rate.

Therefore, a subsidy on capital is possible to achieve the optimality. Impose a proportional subsidy $\tau_t$ on capital.

$$\alpha\frac{Y_t}{K_t} + \tau_t = r_t + \delta$$

Let $\tau_t = \eta\frac{Y_t}{K_t}$, then

$$(\alpha+\eta)\frac{Y_t}{K_t} = r_t + \delta$$

which is the social planner's first order condition for $K$. Therefore, this capital subsidy can implement the social planner solution.

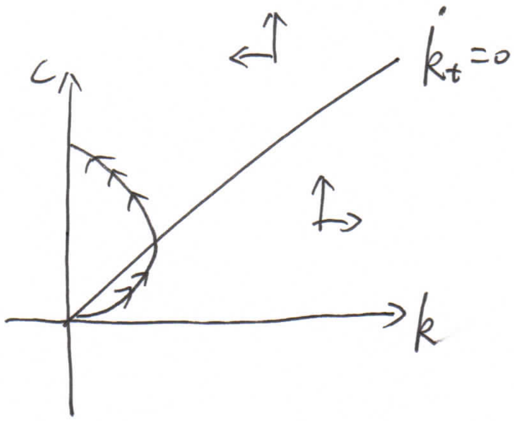


Figure 1

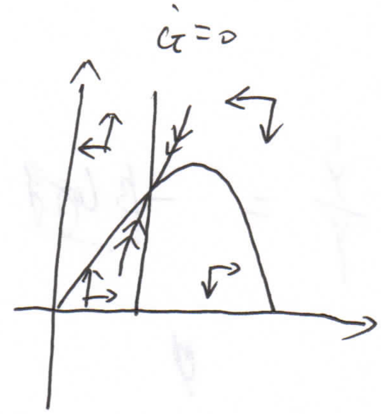


Figure 2

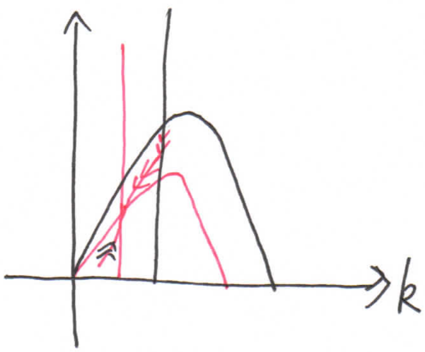


Figure 3.

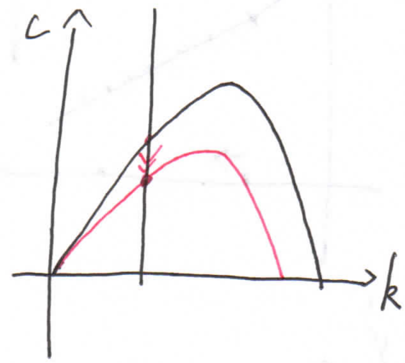


Figure 4


Figure 5

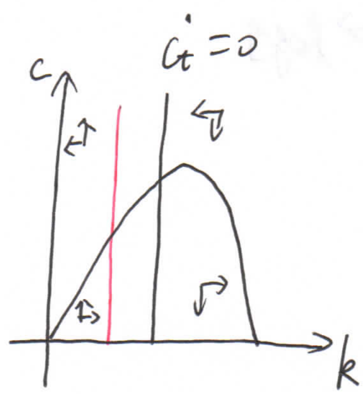


Figure 6

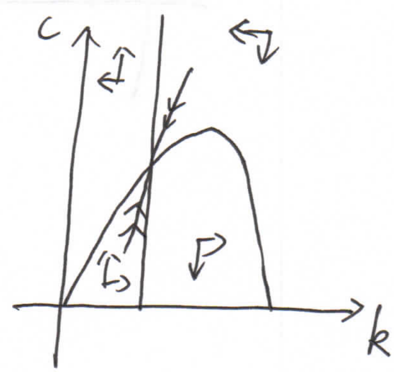


Figure 7.

# Suggested Solution-Problem Set 4

Yang Jiao*

## 1 Human Capital

**a).**

Define $z = k + h$ and set up the Hamiltonian

$$\mathscr{H} = e^{-(\rho-n)t}\frac{c_t^{1-\theta}-1}{1-\theta} + \lambda_t(Ak_t^\alpha(z_t-k_t)^\beta \bar{h_t}^\mu - (\delta+n)z_t - c_t)$$

F.O.C.s are

$$e^{-(\rho-n)t}c_t^{-\theta} = \lambda_t$$
$$\alpha A_t k_t^{\alpha-1}(z_t-k_t)^\beta \bar{h_t}^\mu = \beta A_t k_t^\alpha (z_t-k_t)^{\beta-1}\bar{h_t}^\mu$$
$$-\dot{\lambda_t} = (\beta A_t k_t^\alpha(z_t-k_t)^{\beta-1}\bar{h_t}^\mu - \delta - n)\lambda_t$$
$$\lim_{t\to\infty}\lambda_t z_t = 0$$

Substitute $z_t$ by $k_t + h_t$ of the above conditions:

$$e^{-(\rho-n)t}c_t^{-\theta} = \lambda_t$$
$$\alpha h_t = \beta k_t$$
$$-\dot{\lambda_t} = (\beta A_t k_t^\alpha h_t^{\beta-1}\bar{h_t}^\mu - \delta - n)\lambda_t$$
$$\lim_{t\to\infty}\lambda_t(k_t+h_t) = 0$$

**b).**

There is a typo in the question. $k$ should be $h$.

From a), we have Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[A\beta(\frac{\beta}{\alpha})^{\mu+\beta-1}k_t^{\alpha+\beta+\mu-1} - \rho - \delta]$$

and law of motion of capital

$$(1+\beta/\alpha)\dot{k_t} = (\frac{\beta}{\alpha})^{\mu+\beta}Ak_t^{\alpha+\beta-\mu} - (\delta+n)(1+\beta/\alpha)k_t - c_t$$

Since $\alpha+\beta+\mu<1$, we will have a steady state and the long run growth rate is 0.

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**c).**
See Figure 1.
**d).** First let $\bar{h} = h$ and $z = k + h$. Then set up the Hamiltonian

$$\mathscr{H} = e^{-(\rho-n)t}\frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t(Ak_t^{\alpha}(z_t - k_t)^{\beta+\mu} - (\delta+n)z_t - c_t)$$

Derive F.O.C.s and substitute $z$ by $k + h$ to yield:

$$e^{-(\rho-n)t}c_t^{-\theta} = \lambda_t$$

$$\alpha h_t = (\beta+\mu)k_t$$

$$-\dot{\lambda}_t = ((\beta+\mu)A_t k_t^{\alpha} h_t^{\beta+\mu-1} - \delta - n)\lambda_t$$

$$\lim_{t\to\infty} \lambda_t(k_t + h_t) = 0$$

We have Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[A(\beta+\mu)(\frac{\beta+\mu}{\alpha})^{\mu+\beta-1}k_t^{\alpha+\beta+\mu-1} - \rho - \delta]$$

and law of motion of capital

$$(1 + (\beta+\mu)/\alpha)\dot{k}_t = (\frac{\beta+\mu}{\alpha})^{\mu+\beta}Ak_t^{\alpha+\beta-\mu} - (\delta+n)(1 + (\beta+\mu)/\alpha)k_t - c_t$$

The social planner problem is not the same as the decentralized case. And one can check that social planner gives higher steady state consumption and capital.
**e).**
In decentralized case,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[A\beta(\frac{\beta}{\alpha})^{\mu+\beta-1} - \rho - \delta]$$

which is the growth rate.
**f).**
For social planner,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[A(\beta+\mu)(\frac{\beta+\mu}{\alpha})^{\mu+\beta-1} - \rho - \delta]$$

which displays higher growth rate than the decentralized case.

# 2 R&D Model with Intermediate Capital Goods

**a).**

$$\max AL^{1-\alpha}\sum_{j=1}^{N} k_{jt}^{\alpha} - w_t L - \sum_{j=1}^{N} R_{jt}k_{jt}$$

F.O.C.s are

$$w_t = (1-\alpha)\frac{Y_t}{L}$$

$$R_{jt} = \alpha A L^{1-\alpha} k_{jt}^{\alpha-1}$$

**b)**

Set up the Hamiltonian

$$\mathscr{H} = e^{-rt}(\alpha A L^{1-\alpha} k_{jt}^{\alpha} - \theta I_{jt}) + \mu_{jt} I_{jt}$$

F.O.C.s

$$e^{-rt}\theta = \mu_{jt}$$

$$-\dot{\mu_{jt}} = e^{-rt}\alpha^2 A L^{1-\alpha} k_{jt}^{\alpha-1}$$

**c).**

$q_{jt} = e^{rt}\mu_{jt}$, so $q_t = \theta$.

**d).**

$$k_{jt}^* = (\frac{\alpha^2 A L^{1-\alpha}}{r\theta})^{\frac{1}{1-\alpha}}$$

So rental rate is also constant by the second equation in a)

$$R_{jt}^* = \frac{r\theta}{\alpha}$$

**e).**

If $\eta \leq V$, firms will invest in $R\&D$. When there is positive entry of $R\&D$, $\eta = V$. Since $k$ is a constant, we know that $I = \dot{k} = 0$.

$$V = -\theta k^* + \int_0^{\infty} e^{-rt} R^* k^* dt = A^{1/(1-\alpha)} L r^{-1/(1-\alpha)} \theta^{-\alpha/(1-\alpha)} \alpha^{2/(1-\alpha)} (1/\alpha - 1)$$

and $V = \eta$.

Note up to now, we implicitly assume interest rate $r_t$ is a constant and we will verify later.

**f).**

Assume the CIES parameter is $\sigma$ (to distinguish from the above $\theta$).

$$\int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\sigma} - 1}{1-\theta} dt \tag{1}$$

s.t.

$$\dot{a_t} = w_t L + r_t a_t - C_t \tag{2}$$

**g).h).**

It is quite standard. I directly write down the Euler equation

$$\dot{C}_t / C_t = \frac{1}{\sigma}(r - \rho) = \frac{1}{\sigma}[\frac{A L^{1-\alpha} \theta^{-\alpha} \alpha^2 (1/\alpha - 1)^{1-\alpha}}{\eta^{1-\alpha}} - \rho]$$

**i).**

$$\int_0^{+\infty} e^{-\rho t}\frac{c_t^{1-\sigma}-1}{1-\theta}dt \tag{3}$$

s.t.

$$\eta\dot{N}_t + C_t = AL^{1-\alpha}\sum_{j=1}^{N_t} k_{jt}^{\alpha} - \sum_{j=1}^{N_t}\theta I_{jt}$$

$$\dot{k_{jt}} = I_{jt}$$

Set up the Hamiltonian

$$e^{-\rho t}\frac{C_t^{1-\sigma}-1}{1-\theta}dt + \lambda_t\frac{1}{\eta}(AL^{1-\alpha}\sum_{j=1}^{N_t} k_{jt}^{\alpha} - \sum_{j=1}^{N_t}\theta I_{jt} - C_t) + \sum_{j=1}^{N_t}\mu_{jt}I_{jt}$$

F.O.C.s w.r.t. $C$, $I$, $k$ and $N$:

$$\frac{\lambda_t}{\eta} = e^{-\rho t}C_t^{-\sigma}$$

$$\frac{\lambda_t}{\eta}\theta = \mu_{jt}$$

$$-\dot{\mu_{jt}} = \lambda_t\frac{1}{\eta}\alpha AL^{1-\alpha}k_{jt}^{\alpha-1}$$

$$-\dot{\lambda}_t = \frac{AL^{1-\alpha}}{\eta}k_{jt}^{\alpha}\lambda_t - \frac{\lambda_t}{\eta}\theta I_{jt} + \mu_{jt}I_{jt}$$

Combine the second and third equation to generate $\frac{\dot{\lambda}_t}{\lambda_t}$.

$$k_{jt} = A^{1/(1-\alpha)}L\alpha^{1/(1-\alpha)}\theta^{-1/(1-\alpha)}[\frac{-\dot{\lambda}_t}{\lambda_t}]^{-1/(1-\alpha)}$$

For the fourth equation of the F.O.C.s above, notice the last two terms on the right hand side cancel out. Therefore,

$$\frac{-\dot{\lambda}_t}{\lambda_t} = \frac{AL^{1-\alpha}}{\eta}k_{jt}^{\alpha}$$

The above two equations pin down $\frac{\dot{\lambda}_t}{\lambda_t}$ and $k_{jt}$.

$$\frac{-\dot{\lambda}_t}{\lambda_t} = AL^{1-\alpha}\alpha^{\alpha}\theta^{-\alpha}/\eta^{1-\alpha}$$

The first equation of F.O.C.s still give us the growthr rate of consumption

$$\dot{C}_t/C_t = \frac{1}{\sigma}[\frac{-\dot{\lambda}_t}{\lambda_t} - \rho] = \frac{1}{\sigma}[AL^{1-\alpha}\alpha^{\alpha}\theta^{-\alpha}/\eta^{1-\alpha} - \rho]$$

It is larger than the decentralized case, since $\alpha^\alpha > \alpha^\alpha \alpha(1-\alpha)^{1-\alpha} = \alpha^2(\frac{1-\alpha}{\alpha}^{1-\alpha})$

**j).**

Subsidy to the monopolistic R&D firms' production.

**k).**

No. Because the marginal cost and marginal benefit of these monopolistic firms are still not equalized. There is still under provision of R&D firms' good.

# 3 R&D and the Human Component of Research

**a).**

$\rho$ is subjective discount factor. $\theta$ is the inverse of the elasticity of intertemporal substitution.

**b).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-\rho t}\frac{c_t^{1-\theta}-1}{1-\theta}dt + \lambda_t(r_t a_t + w_t - c_t)$$

F.O.C.s are

$$\lambda_t = e^{-\rho t}c_t^{-\theta}$$

$$-\frac{\dot{\lambda}_t}{\lambda_t} = r_t$$

**c).**

$$\max Y_t - \sum_{j=1}^{N_t} p_{jt}x_{jt} - w_t L_t$$

F.O.C.s are

$$w_t = (1-\alpha)\frac{Y_t}{L_t}$$

$$p_{jt} = \alpha A L_t^{1-\alpha} x_{jt}^{\alpha-1}$$

**d).**

$$\max_{p_j} p_j x_j - x_j$$

where $p_j = \alpha A L^{1-\alpha} x_j^{\alpha-1}$.

The solution is standard constant markup $p_j = 1/\alpha$ and $x_j = \alpha^{2/(1-\alpha)}A^{1/(1-\alpha)}L$, profit $\pi_j = (1/\alpha - 1)\alpha^{2/(1-\alpha)}A^{1/(1-\alpha)}L$.

**e).**

$$Y_t = AL^{1-\alpha}\sum_{j=1}^{N_t} x_j^\alpha = A^{1/(1-\alpha)}\alpha^{2\alpha/(1-\alpha)}LN_t$$

Labor share is a constant fraction of final output

$$w_t L = (1-\alpha) Y_t$$

thus

$$w_t = (1-\alpha) A^{1/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)} N_t$$

**f).**

If firm value $\pi_j \int_t^{\infty} e^{-\int_0^s r_u du} ds > w_t \phi$, there will be new R&D entry until they are equalized.

**g).**

Free entry condition $V_t = \pi_j \int_t^{\infty} e^{-\int_0^s r_u du} ds = w_t \phi$

**h).**

In the market asset $La_t = \phi w_t N_t$. No arbitrage condition means

$$r_t = \frac{\pi_j + \dot{V}_t}{V_t} = \frac{\pi_j}{\phi w_t} + \frac{\dot{N}_t}{N_t}$$

It is not constant.

**i).**

From households problem, we still have Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}\left(\frac{\pi_j}{\phi w_t} + \frac{\dot{N}_t}{N_t} - \rho\right)$$

Then we substitute $w_t$ from e). to the above equation.

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}\left(\frac{\pi_j}{\phi(1-\alpha) A^{1/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)}} N_t^{-1} + \frac{\dot{N}_t}{N_t} - \rho\right)$$

Another equation comes from

$$\dot{a}_t = r_t a_t + w_t - c_t$$

i.e.

$$\dot{a}_t / a_t = r_t + w_t / a_t - c_t / a_t$$

Since we have $La_t = \phi w_t N_t$, and then take $r_t$ from h). and $w_t$ from e)., we will have another equation with only $c_t$ and $N_t$.

$$2\dot{N}_t / N_t = \left[\frac{\pi_j}{\phi(1-\alpha) A^{1/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)}} + L/\phi\right] N_t^{-1} + \dot{N}_t / N_t - c_t \frac{L}{\phi(1-\alpha) A^{1/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)}} N_t^{-2}$$

i.e.

$$\dot{N}_t / N_t = \left[\frac{\pi_j}{\phi(1-\alpha) A^{1/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)}} + L/\phi\right] N_t^{-1} - c_t \frac{L}{\phi(1-\alpha) A^{1/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)}} N_t^{-2}$$

Substitute the above back to the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}\Big(\frac{\pi_j}{\phi(1-\alpha)A^{1/(1-\alpha)}\alpha^{2\alpha/(1-\alpha)}}N_t^{-1} + \Big[\frac{\pi_j}{\phi(1-\alpha)A^{1/(1-\alpha)}\alpha^{2\alpha/(1-\alpha)}} + L/\phi\Big]N_t^{-1} - c_t\frac{L}{\phi(1-\alpha)A^{1/(1-\alpha)}\alpha^{2\alpha/(1-\alpha)}}N_t^{-2} - \rho\Big)$$

Let $\dot{N}_t = 0, \dot{c}_t = 0$, one can get a solution $(N^*, c^*) > 0$, which means there is a non-trivial steady state. Therefore, there is no long run growth.

The intuition is as the economy grows, wage also grows, which increases the cost of $R\&D$. This rising $R\&D$ cost will drag down the growth rate.

**j).**

It is not optimal. There is monopoly power of $R\&D$ firms. The policy to correct this distortion is to subsidize the production of intermediate inputs to resolve the under provision problem of these firms.

**Alternative R&D Cost:** Remember there is a linear relationship between $w$ and $N$, see e). Now the cost to $R\&D$ would be $w\phi/N$, which is again a constant. In this case, we will have long run growth as the standard model we learned in class.

# 4 Public Services, Public Investment and Growth

**A.i.).**

The marginal cost of government spending should be the marginal benefit of the government spending.

$$MPG = 1$$

i.e.

$$(1-\alpha)y/g = 1 \rightarrow \tau = g/y = 1-\alpha$$

**A.ii.).**

In decentralized case, the proportional tax distorts firms behavior. While social planner internalizes this distortion, individual firms don't.

**B.i.).**

$\rho$ is subjective discount factor. $\theta$ is the inverse of the elasticity of intertemporal substitution. A larger $\rho$ means agents weigh less of the future consumption. A larger $\theta$ means a larger incentive to smooth consumption across time.

To make $\theta = 1$ well defined and correspond to log-utility.

**B.ii.).iii).iv). v).**

(I solve the problem by picking up $c, k, g$ and $\tau$, otherwise, the problem seems very complicated. If one can only pick up $c, k, g$, then one cannot combine the two constraints.)

Define $z_t = g_t + k_t$. Sum the two budget constraint to get

$$\dot{z}_t = Ak_t^{\alpha}h_t^{1-\alpha} - c_t - \delta z_t$$

(Why can we combine the two constraints when we are allowed to pick up $\tau_t$? Intuitively, if one can adjust $\tau_t$, that means one can freely transform physical capital to public service or the reverse. More formally and strictly, first, combining the two constraints to one constraint will for sure deliver higher welfare. Second, one can pick up $\tau$ finally such that $\tau_t = \frac{\dot{g}_t + \delta g_t}{y_t}$ where the right hand side variables are given by the solution with combining the two constraints, which means the economy can in fact achieve the solution with combining the two constraints, given below. Then we conclude the solution is equivalent by using one combined constraint or with two constraints once we are allowed to pick up $\tau_t$.)

Setup the Hamiltonian

$$\mathscr{H} = e^{-\rho t}\frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t[Ak_t^{\alpha}(z_t - k_t)^{1-\alpha} - c_t - \delta z_t]$$

F.O.C.s

$$e^{-\rho t}c_t^{-\theta} = \lambda_t$$

$$\alpha A k_t^{\alpha-1}(z_t - k_t)^{1-\alpha} = (1-\alpha)Ak_t^{\alpha}(z_t - k_t)^{-\alpha}$$

$$-\frac{\dot{\lambda}_t}{\lambda_t} = (1-\alpha)Ak_t^{\alpha}(z_t - k_t)^{-\alpha} - \delta$$

and TVC.

Solving the above to yield $\alpha g_t = (1-\alpha)k_t$ and Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[(1-\alpha)^{1-\alpha}\alpha^{\alpha}A - \delta - \rho]$$

There is long run growth. The economy will immediately jump to $\alpha g_0 = (1-\alpha)k_0$ and we have a constant growth rate from that point on. After the earthquake, the growth rate remains but starts from a lower level since consumption is proportional to capital stock.

**C.i.).ii).iii).iv).v).vi)**

There may be different interpretations about what this question means. Here is my understanding. If you have better interpretation, please let me know.

Write down the constraints first

$$\dot{k}_t = \bar{\tau}_t y_t - c_t - \delta k_t = (1-\tau_t)^{1-\alpha} g_t^{-\eta} A k_t^{\alpha} g_t^{1-\alpha} - c_t - \delta k_t \tag{4}$$

$$\dot{g}_t = \tilde{\tau}_t y_t - \delta g_t = \tau_t(\frac{g_t}{k_t})^{\alpha} A k_t^{\alpha} g_t^{1-\alpha} - \delta g_t \tag{5}$$

where $\bar{\tau} = (1-\tau)^{1-\alpha}g^{-\eta}$. We solve the equilibrium by choosing $c_t, k_t, g_t, \tau_t$.

Setup the Hamiltonian for the social planner:

$$\mathscr{H} = e^{-\rho t}\frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t[(1-\tau_t)^{1-\alpha}g_t^{-\eta}Ak_t^{\alpha}g_t^{1-\alpha} - c_t - \delta k_t] + \mu_t[\tau_t(\frac{g_t}{k_t})^{\alpha}Ak_t^{\alpha}g_t^{1-\alpha} - \delta g_t]$$

First order conditions are:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(-\frac{\dot{\lambda}_t}{\lambda_t} - \rho) \tag{6}$$

$$\mathscr{H}_\tau = 0 = \lambda_t(1-\alpha)(1-\tau_t)^{-\alpha}(-1)Ak_t^\alpha g_t^{1-\alpha-\eta} + \mu_t A g_t \tag{7}$$

$$\mathscr{H}_k = -\dot{\lambda}_t = \lambda_t\left(\alpha(1-\tau_t)^{1-\alpha}Ak_t^{\alpha-1}g_t^{1-\alpha-\eta} - \delta\right) \tag{8}$$

$$\mathscr{H}_g = -\dot{\mu}_t = \lambda_t(1-\alpha-\eta)(1-\tau_t)^{1-\alpha}Ak_t^\alpha g_t^{1-\alpha-\eta} + \mu_t(\tau_t A - \delta) \tag{9}$$

In equation (9), substitute $\lambda_t$ by using equation (7) to obtain

$$-\frac{\dot{\mu}_t}{\mu_t} = \frac{1-\alpha-\eta}{1-\alpha}(1-\tau_t)A + \tau_t A - \delta \tag{10}$$

In steady state, $\tau_t$ is a constant. Therefore, $\mu_t$ grows at a constant rate in steady state.

$\frac{\dot{g}_t}{g_t} = \tau_t A - \delta$ is also a constant. One can guess that in steady state $\tau A = \delta$ so that there is a steady state value of $g_t = g^*$. Then equation (4), (6) and (8) give a steady state for $c^*$ and $k^*$, just as a standard Ramsey model. Note that equation (6) and (8) already show that in steady state $-\frac{\dot{\lambda}_t}{\lambda_t} = \rho$. Go back to equation (7), we know in steady state

$$-\frac{\dot{\lambda}_t}{\lambda_t} = -\frac{\dot{\mu}_t}{\mu_t}$$

That is, parameter values have to satisfy

$$\rho = \frac{1-\alpha-\eta}{1-\alpha}(1-\tau)A + \tau A - \delta$$

where $\tau = \frac{\delta}{A}$.

But suppose $\frac{1-\alpha-\eta}{1-\alpha}(1-\tau)A + \tau A > \rho + \delta$, then from the above, we see that cannot have a steady state featuring $c^*$, $k^*$, $g^*$ and $\tau^*$. We conclude that in steady state $g_t$ grows at a constant rate not equal to 0.

Equation (4), (6) and (8) effectively describe a standard Ramsey model with constant growth rate of technology.

$$\dot{k}_t = (1-\tau)^{1-\alpha}g_t^{1-\alpha-\eta}Ak_t^\alpha - c_t - \delta k_t \tag{11}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(-\frac{\dot{\lambda}_t}{\lambda_t} - \rho\right) \tag{12}$$

$$-\frac{\dot{\lambda}_t}{\lambda_t} = \alpha(1-\tau)^{1-\alpha}Ag_t^{1-\alpha-\eta}k_t^{\alpha-1} - \delta \tag{13}$$

Define $B_t^* = g_t^{\frac{1-\alpha-\eta}{1-\alpha}}$, which will grow at a constant rate $\gamma = \frac{1-\alpha-\eta}{1-\alpha}(\tau A - \delta)$. Denote variable $\hat{x}_t = \frac{x_t}{B_t^*}$. The above system can be re-written as

$$\dot{\hat{k}}_t = (1-\tau)^{1-\alpha}A\hat{k}_t^\alpha - \hat{c}_t - (\delta+\gamma)\hat{k}_t \tag{14}$$

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} + \gamma = \frac{1}{\theta}\left(\alpha(1-\tau)^{1-\alpha}A\hat{k}_t^{\alpha-1} - \rho - \delta\right) \tag{15}$$

Therefore, $\hat{k}_t$ and $\hat{c}_t$ will display a constant steady state while $c_t$ and $k_t$ grow at a rate of $\gamma$.

In order to nail down steady state $\tau$, one only needs to go back to equation (7), which says the growth rate of $\lambda_t B_t$ is equal to the growth rate of $\mu_t g_t$.

The model will display transitional dynamics. After an earthquake, the economy will gradually accumulate capital (in terms of $\hat{k}_t$) by reducing consumption. (Intuitively, tax rate should also decrease in order to quickly recover physical capital stock.)

# 5 Two Sector Growth

**a).**

$$\max \int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}-1}{1-\theta}dt$$

s.t.

$$c_t = A(u_t k_t)^\alpha \hat{k}_t^\psi$$

$$\dot{k}_t = B(1-u_t)k_t - (n+\delta)k_t$$

**b).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-\rho t}\frac{c_t^{1-\theta}-1}{1-\theta} + \mu_t[A(u_t k_t)^\alpha \hat{k}_t^\psi - c_t] + \lambda_t[B(1-u_t)k_t - \delta k_t]$$

F.O.C.s

$$\mu_t = e^{-\rho t}c_t^{-\theta}$$

$$\mu_t \alpha A(u_t k_t)^{\alpha-1} k_t \hat{k}_t^\psi = \lambda_t B k_t$$

$$-\dot{\lambda}_t = \lambda_t B(1-u_t) + \mu_t A\alpha(u_t k_t)^{\alpha-1}\hat{k}_t^\psi u_t - \delta\lambda_t$$

and TVC

$$\lim_{t\to\infty} \lambda_t k_t = 0$$

**c).**

The steady state growth rate of $u_t$ should be 0 because $u_t \in [0,1]$, so it cannot have long run growth rate, i.e. $u_t$ has a steady state value $u^*$.

In equilibrium $\hat{k} = k$, so

$$c_t = A(u_t k_t)^\alpha \hat{k}_t^\psi = A(u_t k_t)^\alpha k_t^\psi$$

Since $u_t = u^*$ in steady state,

The growth rate

$$\dot{c}_t / c_t = (\alpha+\psi)\dot{k}_t / k_t$$

When $\alpha + \psi = 1$, they are identical.

**d).**

From F.O.C.s in b), we get

$$\dot{c}_t/c_t = \frac{1}{\theta}[-\dot{\mu}_t/\mu_t - \rho]$$

When $\alpha + \psi = 1$, we obtain from the third equation in the F.O.C.s:

$$-\frac{\dot{\lambda}_t}{\lambda_t} = B(1-u^*) + A\alpha u^{*\alpha}\frac{\mu_t}{\lambda_t} - \delta$$

Since $\lambda_t$ grows at a constant rate in steady state, then $\mu_t/\lambda_t$ has to be a constant from the equation above, which means they have the same growth rate.

**e).**

Denote $x = \frac{\mu_t}{\lambda_t}$, so the growth rate of consumption

$$\gamma_c = \dot{c}_t/c_t = \frac{1}{\theta}[-\dot{\mu}_t/\mu_t - \rho] = \frac{1}{\theta}[-\dot{\lambda}_t/\lambda_t - \rho] = \frac{1}{\theta}\{[B(1-u^*) + A\alpha u^{*\alpha}x - \delta] - \rho\}$$

Notice that the law of motion of capital in fact gives the steady state growth rate of the economy.

$$\gamma_k = B(1-u^*) - \delta$$

**f).**

Additionally, the second equation of F.O.C.s in b) tells us

$$x = \frac{B}{\alpha A u^{*\alpha-1}}$$

Substitute the above to the $\gamma_c$ equation and let $\gamma_c = \gamma_k$ to finally solve $u^*$.

$$u^* = \frac{\rho}{B}$$

And growth rate of consumption and capital follow.

**g).**

Social planner takes the externality into consideration. The solution steps are similar to the above.

Check that the growth rate of social planner should be larger than the decentralized case.

# 6 Human Capital and Adjustment Costs

**a).**

$$\max_{I_K, I_H, H, K} \int_0^{+\infty} e^{-rt}[AK^\alpha H^\eta - I_K - I_H(1+\varphi)]dt$$

s.t.

$$\dot{K}_t = I_{Kt} - \delta_K K_t$$

$$\dot{H}_t = I_{Ht} - \delta_H H_t$$

$$\lim_{t\to\infty} e^{-rt}K \geq 0$$

$$\lim_{t\to\infty} e^{-rt}H \geq 0$$

**b).**

Set up the Hamiltonian

$$\mathscr{H} = e^{-rt}[AK^{\alpha}H^{\eta} - I_K - I_H - \frac{\zeta}{2}\frac{I_H^2}{H}] + \lambda(I_K - \delta_K K) + \mu(I_H - \delta_H H)$$

F.O.C.s are

$$e^{-rt} = \lambda_t$$

$$e^{-rt}(1 + \zeta\frac{I_{Ht}}{H_t}) = \mu_t$$

$$-\dot{\lambda}_t = e^{-rt}\alpha AK_t^{\alpha-1}H_t^{\eta} - \lambda_t\delta_K$$

$$-\dot{\mu}_t = e^{-rt}\eta AK_t^{\alpha}H_t^{\eta-1} - \mu_t\delta_H + e^{-rt}\frac{\zeta}{2}\frac{I_{Ht}^2}{H_t^2}$$

and TVC

$$\lim_{t\to\infty} \lambda_t K_t = 0$$

$$\lim_{t\to\infty} \mu_t H_t = 0$$

**c).**

The current shadow value of physical capital is $p_t = \lambda_t e^{-rt} = 1$. Then the relationship between $K$ and $H$ follows from the third equation of F.O.C.s.

$$K = (\frac{\alpha A}{r + \delta_K})^{\frac{1}{1-\alpha}} H_t^{\frac{\eta}{1-\alpha}}$$

The current value of human capital is $q_t = 1 + \zeta\frac{I_{Ht}}{H_t}$. By eliminating $K$, we also have the following law of motion of $q_t$,

$$\dot{q}_t = (r + \delta_H)q_t - \eta AK_t^{\alpha}H_t^{\eta-1} - \frac{\zeta}{2}\frac{I_{Ht}^2}{H_t^2} = (r + \delta_H)q_t - \eta A(\frac{\alpha A}{r + \delta_K})^{\frac{1}{1-\alpha}} H_t^{\frac{\alpha+\eta-1}{1-\alpha}} - \frac{\zeta}{2}(\frac{q_t - 1}{\zeta})^2$$

**d).**

$\alpha + \eta < 1$, it is saddle path stable. See Figure 2.

**e).f).g).**

To describe the behavior of variables, it is recommended to track the dynamics on the phase diagram and also plot time series for each variable of interest, starting from the steady state.

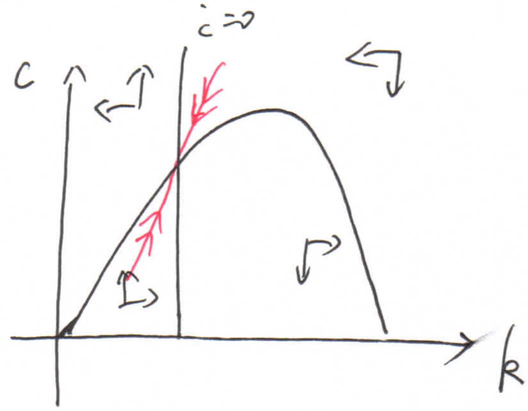


Figure 1.

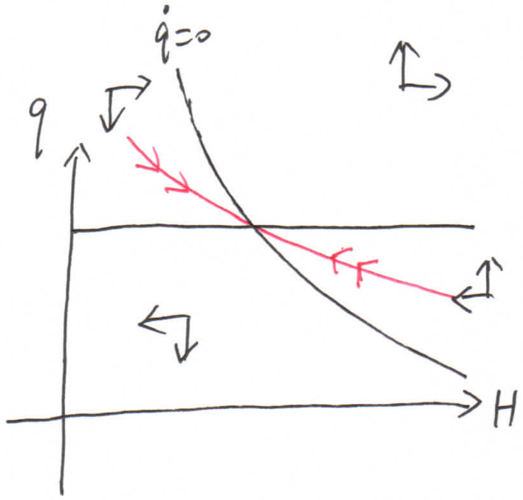


Figure 2.

# The Neoclassical Growth Model

Macroeconomic Analysis Recitation 6

Yang Jiao*

# 1 Log-linearization

The idea of log-linearization is to look at the percentage change (log-change) of a variable in response to other variables in terms of percentage change as well around a small neighborhood of some state (usually steady state).

$$y = f(x_1, ..., x_n) \tag{1}$$

Then first order expansion is

$$log(y) - log(y^*) = \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* \cdot [log(x_i) - log(x_i^*)] \tag{2}$$

Denote log-change as $\hat{x} = log(x) - log(x^*)$, the above becomes

$$\hat{y} = \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* \cdot \hat{x}_i$$

Note the coefficient $\frac{\frac{\partial f}{\partial x_i}|_{x^*}}{f(x_1^*, ..., x_n^*)} \cdot x_i^* = \frac{\partial \log f}{\partial \log x_i}|_{x^*}$ is in fact **elasticity**.

Useful formulas to remember (derived based on the equation above)

$$\widehat{a} = 0, \text{where } a \text{ is a constant.}$$

$$\widehat{ax} = \widehat{x}, \text{where } a \text{ is a constant.}$$

$$\widehat{xy} = \hat{x} + \hat{y}$$

$$\widehat{x^\alpha} = \alpha\hat{x}, \text{where } \alpha \text{ is a constant.}$$

$$\widehat{x+y} = \frac{x^*}{x^* + y^*}\hat{x} + \frac{y^*}{x^* + y^*}\hat{y}$$

$$\widehat{x-y} = \frac{x^*}{x^* - y^*}\hat{x} - \frac{y^*}{x^* - y^*}\hat{y}$$

*Please email me if you find errors or typos to yj2279@columbia.edu. All comments and suggestions are welcome and appreciated.

**Examples**

(1) Neoclassical production function $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$:

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t + (1-\alpha)\hat{N}_t$$

(2) Resource constraint $Y_t = C_t + I_t$:

$$\hat{Y}_t = \frac{C^*}{Y^*}\hat{C}_t + \frac{I^*}{Y^*}\hat{I}_t$$

(3) Capital accumulation: $K_{t+1} = (1-\delta)K_t + I_t$:

$$\hat{K}_{t+1} = \frac{(1-\delta)K^*}{K^*}\hat{K}_t + \frac{I^*}{K^*}\hat{I}_t$$

That is

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta \hat{I}_t$$

(4) Consumption Euler equation: $C_t^{-\sigma} = \beta R_t C_{t+1}^{-\sigma}$:

$$-\sigma \hat{C}_t = \hat{R}_t - \sigma \hat{C}_{t+1}$$

which implies

$$\hat{C}_{t+1} - \hat{C}_t = \frac{1}{\sigma}\hat{R}_t$$

where the left hand side is consumption growth rate. Since gross interest rate $\hat{R}_t = 1 + r_t$, then $\log R_t - \log R^* = \log(1+r_t) - \log(1+r^*) = r_t - r^*$. Therefore,

$$\hat{C}_{t+1} - \hat{C}_t = \frac{1}{\sigma}(r_t - r^*)$$

# 2 Neoclassical Growth Model

In this section, we will learn to use log-linearization in a simple neoclassical growth model without uncertainty. In contrast with what we learned in the first half semester in continuous time where labor is fixed, now we study in discrete time framework and introduce endogenous labor supply.

## 2.1 Model Setup

The problem is setup as the following social planner problem:

$$\max_{\{C_t, L_t, Y_t, I_t, K_{t+1}, N_t\}} \sum_{t=0}^{+\infty} \beta^t u(C_t, L_t)$$

s.t.

$$L_t + N_t = 1$$

$$C_t + I_t = Y_t$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

$$Y_t = F(K_t, N_t)$$

$$0 \leq L_t, N_t \leq 1; C_t > 0; K_{t+1} \geq 0$$

given $K_0$.

Utility is an increasing function of consumption $C_t$ and leisure $L_t$. The first constraint is the allocation of aggregate time (which is normalized to 1) between leisure and work. The second constraint is resource constraint. The third constraint is capital accumulation. The fourth constraint is production function which shall be neoclassical.

By substituting $y_t$, $I_t$, $C_t$ and $L_t$, the above problem can be re-written as

$$\max_{\{K_{t+1}, N_t\}} \sum_{t=0}^{+\infty} \beta^t u(F(K_t, N_t) + (1-\delta)K_t - K_{t+1}, 1 - N_t)$$

First order conditions w.r.t. $K_{t+1}$ and $N_t$ give the following **equilibrium conditions**:

$$-u_c(C_t, 1-N_t) + \beta\left(u_c(C_{t+1}, 1-N_{t+1})[F_k(K_{t+1}, N_{t+1}) + 1 - \delta]\right) = 0 \tag{3}$$

$$-u_l(C_t, 1-N_t) + u_c(C_t, 1-N_t)F_n(K_t, N_t) = 0 \tag{4}$$

where

$$C_t = F(K_t, N_t) + (1-\delta)K_t - K_{t+1} \tag{5}$$

The transversality condition is

$$\lim_{T \to +\infty} \beta^T u_c(C_T, 1-N_T)K_{T+1} = 0 \tag{6}$$

Equation (3) is similar to Euler equation in a decentralized economy except that $R_t$ is substituted by $F_k(K_{t+1}, N_{t+1}) + 1 - \delta$. It captures the intertemporal tradeoff between consuming today and invest in the capital good for consumption tomorrow. Equation (4) is a static tradeoff between consumption good or leisure, where $F_n(K_t, N_t)$ is the opportunity cost of enjoying leisure instead of produce. Equation (5) is just resource constraint. Finally, the transversality condition will guarantee the economy is on the saddle path.

## 2.2 Steady State

In steady state, $C_t, I_t, Y_t, K_{t+1}, N_t, L_t$ are all constants. Use * to denote steady state variables. From equation (3),(4) and (5), we obtain

$$F_k(K^*, N^*) = \frac{1}{\beta} - 1 + \delta$$

$$F_n(K^*, N^*) = \frac{u_l(C^*, 1-N^*)}{u_c(C^*, 1-N^*)}$$

$$C^* + \delta K^* = F(K^*, N^*)$$

Those three equations uniquely pin down $C^*, K^*, N^*$. As with other steady state variables $I^*, Y^*, L^*$, they are just functions of $C^*, K^*, N^*$.

## 2.3 Log-linearization

### 2.3.1 Equation (3)

For equation (3), rewrite it as

$$u_c(C_t, 1 - N_t) = \beta\left(u_c(C_{t+1}, 1 - N_{t+1})[F_k(K_{t+1}, N_{t+1}) + 1 - \delta]\right)$$

Then we get

$$\xi_{cc}\hat{C}_t + \xi_{cl}\widehat{1 - N_t} = \xi_{cc}\hat{C}_{t+1} + \xi_{cl}\widehat{1 - N_{t+1}} + \frac{1/\beta - 1 + \delta}{1/\beta}[\eta_{kn}\hat{N}_{t+1} + \eta_{kk}\hat{K}_{t+1}]$$

where $\xi_{ab}$ is the elasticity of marginal utility of a with respect to b, evaluated at steady state (a,b=c or l) and $\eta_{ab}$ is the elasticity of marginal product of a with respect to b, evaluated at steady state (a,b=k or n). For instance, $\eta_{kn} = \frac{\partial \log F_k(K,N)}{\partial \log N}|_{(K^*,N^*)}$. Recall why we have **elasticity** in the log-linearization.

We can derive further for the above equation to

$$\xi_{cc}\hat{C}_t - \xi_{cl}\frac{N^*}{1 - N^*}\hat{N}_t = \xi_{cc}\hat{C}_{t+1} - \xi_{cl}\frac{N^*}{1 - N^*}\hat{N}_{t+1} + [1 - \beta(1 - \delta)][\eta_{kn}\hat{N}_{t+1} + \eta_{kk}\hat{K}_{t+1}] \quad (7)$$

### 2.3.2 Equation (4)

We then rewrite equation (4) as

$$F_n(K_t, N_t) = \frac{u_l(C_t, 1 - N_t)}{u_c(C_t, 1 - N_t)}$$

which means

$$\eta_{nk}\hat{K}_t + \eta_{nn}\hat{N}_t = \xi_{lc}\hat{C}_t + \xi_{ll}\widehat{1 - N_t} - \xi_{cc}\hat{C}_t - \xi_{cl}\widehat{1 - N_t}$$

that is

$$\eta_{nk}\hat{K}_t + \eta_{nn}\hat{N}_t = \xi_{lc}\hat{C}_t - \xi_{ll}\frac{N^*}{1 - N^*}\hat{N}_t - \xi_{cc}\hat{C}_t + \xi_{cl}\frac{N^*}{1 - N^*}\hat{N}_t \quad (8)$$

Notice that the above equation is all about time $t$ variables. This result is not surprising as it is derived from a static labor supply choice problem.

### 2.3.3 Equation (5)

Equation (5) is resource constraint

$$C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, N_t)$$

Log-linearizing both sides, we obtain

$$\frac{C^*}{Y^*}\hat{C}_t + \frac{K^*}{Y^*}\hat{K}_{t+1} - \frac{(1-\delta)K^*}{Y^*}\hat{K}_t = \alpha_L \hat{K}_t + \alpha_K \hat{N}_t$$

where $\alpha_K = \frac{\partial \log F(K,N)}{\partial \log K}|_{(K^*,N^*)}$ and $\alpha_L = \frac{\partial \log F(K,N)}{\partial \log N}|_{(K^*,N^*)}$. One can show under constant returns to scale for $F(K,N)$, we have $\alpha_K + \alpha_L = 1$. Denote $\alpha = \alpha_K$. $s_c = \frac{C^*}{Y^*}$ and $s_i = \frac{I^*}{Y^*} = \frac{\delta K^*}{Y^*}$. The above equation is equivalent to the following

$$s_c\hat{C}_t + \frac{s_i}{\delta}\hat{K}_{t+1} - \frac{(1-\delta)s_i}{\delta}\hat{K}_t = (1-\alpha)\hat{K}_t + \alpha\hat{N}_t \tag{9}$$

### 2.3.4 First Order Linear System

These three equations contain three variables $\hat{C}, \hat{N}, \hat{K}$. We also have initial condition $\hat{K}_0$ and the transversality condition. Note in equation (8), there are only time $t$ variables, so we can substitute $\hat{N}_t$ (and also $\hat{N}_{t+1}$) in equation (7) and (9) by using equation (8). Then we are left with only variables $\hat{C}$ and $\hat{K}$. It is a linear system, we can write a first order system

$$\begin{pmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{pmatrix} = W \begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix} \tag{10}$$

given $\hat{K}_0$ and TVC has to be satisfied.

Do a decomposition of matrix $W$: $W = P\Lambda P^{-1}$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{11}$$

One can find usually one eigenvalue satisfies $|\lambda_1| > 1$ and the other eigenvalue satisfies $|\lambda_2| < 1$ (this is not always true, but in this model, most of the time, you will find so).

Then multiply both sides of equation (11) by $P^{-1}$:

$$P^{-1}\begin{pmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}\begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix}$$

Define

$$\begin{pmatrix} \tilde{K}_t \\ \tilde{C}_t \end{pmatrix} = P^{-1}\begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix}.$$

We arrive at

$$\begin{pmatrix} \tilde{K}_{t+1} \\ \tilde{C}_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{K}_t \\ \tilde{C}_t \end{pmatrix} = \begin{pmatrix} \lambda_1^{t+1} & 0 \\ 0 & \lambda_2^{t+1} \end{pmatrix}\begin{pmatrix} \tilde{K}_0 \\ \tilde{C}_0 \end{pmatrix}$$

Note that $|\lambda_1| > 1$, in order to guarantee that TVC is not violated. We must require $\tilde{K}_0 = 0$. From the definition of $\tilde{K}_0$, we know it is a linear combination of $\hat{K}_0$ and $\hat{C}_0$. Denote

$\tilde{K}_0 = a_1\hat{K}_0 + a_2\hat{C}_0$. Then we get the decision rule:

$$\hat{C}_0 = -\frac{a_1}{a_2}\hat{K}_0$$

This logic applies to any time $t \geq 0$, so

$$\hat{C}_t = -\frac{a_1}{a_2}\hat{K}_t$$

As with $\hat{K}_{t+1}$, since $\hat{K}_{t+1} = w_{11}\hat{K}_t + w_{12}\hat{C}_t$ and we already have $\hat{C}_t = -\frac{a_1}{a_2}\hat{K}_t$, then we get the decision rule of $\hat{K}_{t+1}$ as a function of $\hat{K}_t$. The whole transitional dynamics is solved now: it corresponds to the saddle path similar to what we learned in the first half of the semester.

Another quick way to get the solution follows *undetermined coefficients* procedure. We understand we are trying to solve a linear system so the solution has to satisfy the form $\hat{K}_{t+1} = \phi_1\hat{K}_t$ and $\hat{C}_t = \phi_2\hat{K}_2$ then we plug these two formulas to equation (10) to obtain

$$\begin{pmatrix} \phi_1 \\ \phi_2\phi_1 \end{pmatrix}\hat{K}_t = \begin{pmatrix} w_{11} + w_{12}\phi_2 \\ w_{21} + w_{22}\phi_2 \end{pmatrix}\hat{K}_t$$

Lastly, comparing coefficients and pick up the solution which satisfies TVC. In order to return to steady state in order not to violate TVC, we will in fact pick up the solution with $|\phi_1| < 1$.

# The Real Business Cycle Model

Macroeconomic Analysis Recitation 7

Yang Jiao*

## 1 Balanced Growth Preferences

Before we move to real business cycle models, we first would like to know what kind of preferences on consumption and leisure (or hours worked) are consistent with long run growth facts, despite that we aim at short run fluctuations. Recall that we proved in recitation 3 that under certain conditions, production function has to be labor-augmented form. We denote the neoclassical production function as $Y_t = H(K_t, X_t N_t)$, where $K_t$ is capital, $N_t$ is labor and $X_t$ is labor productivity. Suppose growth rate of capital stock $\gamma_K$ is equal to the growth rate of labor productivity $\gamma_X$ and $N_t$ is a constant. Then it is easy to verify that $\gamma_Y = \gamma_C = \gamma_X = \gamma_I = \gamma_K$, denoted as $\gamma$. Now we will try to derive the functional form for preference $u(C, L)$.

First, the following Euler equation should hold

$$\frac{u_c(C_t, L_t)}{u_c(C_{t+1}, L_{t+1})} = \beta(1 + r_t) \tag{1}$$

where $r_t$ is the interest rate. It satisfies the following equation

$$r_t = H_k(K_t, N_t X_t) - \delta$$

Note that $H$ is constant returns to scale, so $H_k$ and $H_n$ are both homogeneous of degree 0. As $K_t$ grows at rate $\gamma$ and $N_t X_t$ also grows at rate $\gamma$, we know that $r_t$ is a constant. Therefore, rewrite the Euler equation as

$$u_c(C_t, L) = \beta(1 + r)u_c((1 + \gamma)C_t, L) \tag{2}$$

Take derivative with respect to $C_t$, we arrive at

$$u_{cc}(C_t, L) = \beta(1 + r)u_{cc}((1 + \gamma)C_t, L)(1 + \gamma) \tag{3}$$

Divide equation (3)$*C_t$ by equation (2), we get

$$\frac{u_{cc}(C_t, L)C_t}{u_c(C_t, L)} = \frac{u_{cc}(C_{t+1}, L)C_{t+1}}{u_c(C_{t+1}, L)}.$$

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It implies that $\frac{u_{cc}(C,L)C}{u_c(C,L)}$ is a constant, that is

$$\frac{u_{cc}(C,L)C}{u_c(C,L)} = -\sigma \tag{4}$$

where $\sigma$ represents a constant number.

Secondly, wage $W_t = X_t H_n(K_t, X_t N_t)$ grows at rate $\gamma$ as well. Since $C_t$ also grows at rate $\gamma$, we conclude that $W_t = aC_t$, where $a$ is a constant number. Labor supply decision gives the following

$$W_t = \frac{u_l(C_t, L)}{u_c(C_t, L)}$$

Then

$$u_l(C_t, L) = aC_t u_c(C_t, L) \tag{5}$$

Take derivative with respect to $C_t$:

$$u_{cl}(C_t, L) = au_c(C_t, L) + C_t a u_{cc}(C_t, L) \tag{6}$$

Divide equation (6)$*C_t$ by equation (5), we get

$$\frac{u_{cl}(C_t, L)C_t}{u_l(C_t, L)} = 1 + \frac{u_{cc}(C_t, L)C_t}{u_c(C_t, L)}$$

It implies that

$$\frac{u_{cl}(C,L)C}{u_l(C,L)} = 1 - \sigma \tag{7}$$

Combining equation (4) and (7) to get the following King-Plosser-Rebelo preferences

$$u(C,L) = \begin{cases} \frac{C^{1-\sigma}}{1-\sigma} v(L) & \sigma \neq 1 \\ \log C + v(L) & \sigma = 1 \end{cases}$$

# 2 Balanced Growth Preferences and Hours Worked

Given that we have King-Plosser-Rebelo preferences, which satisfies long run growth facts, we also would like to ask whether they are also able to capture short run fluctuations. Consider a representative worker's problem

$$\max_{C_t, L_t, a_{t+1}} E_0 \sum_{t=0}^{+\infty} \beta^t u(C_t, L_t)$$

s.t. budget constraint

$$C_t + A_{t+1} = W_t(1 - L_t) + (1 + r_t)A_t$$

where $A_t$ is asset and $r_t$ is return on asset. Note that we have used total hours of a worker is normalized to 1. Denoting $\lambda_t$ as the Lagrangian multiplier of the budget constraint, one can derive the following

$$u_c(C_t, L_t) = \lambda_t$$

and

$$\frac{u_l(C_t, L_t)}{u_c(C_t, L_t)} = W_t$$

These two equations deliver

$$u_l(C_t, L_t) = \lambda_t W_t$$

For instance, if we use the following BGP preference

$$u(C, L) = \log C_t - \gamma \frac{\epsilon}{1+\epsilon}(1 - L_t)^{1+\frac{1}{\epsilon}}$$

with $\epsilon > 0$, the labor supply equation becomes

$$N_t = 1 - L_t = W_t^{\epsilon} \lambda_t^{\epsilon}$$

In a recession, usually hours worked drop a lot. The above equation shows that there are two forces that can determine hours worked. One is wage, if wage goes down, hours worked goes down (substitution effect). The other is related to consumption, if consumption goes down, that is $\lambda_t$ goes up, hours worked increases (income effect). The second force offsets part of hours worked drop, so this BGP preferences can hardly generate realistic hours worked fluctuations as in the data. GHH preferences are often used in business cycle studies to address this problem:

$$u(C_t, N_t) = U(C_t - v(N_t))$$

One can check that GHH preferences eliminate income effect so that labor supply only moves with wage. Admittedly, this is not a perfect way to capture labor market dynamics because in the data real wage doesn't move much, so even if we are only left with substitution effect of labor supply, we are not able to generate very volatile hours worked. We will move to search and match models in the future to address how to generate more volatile labor market response (by capturing unemployment dynamics as in the data, hours worked dynamics are largely driven by extensive margin instead of intensive margin.)

# 3 Balanced Growth Path

In this section, we will dive into how to deal with both long run growth and short run dynamics. In addition to labor productivity growth, as in your problem set 2, I will also include investment specific technology progress to keep the framework more general first:

$$K_{t+1} = (1 - \delta)K_t + V_t I_t \tag{8}$$

where $V_t$ represents investment specific technology. The production function is Cobb-Douglas

$$Y_t = AK_t^{1-\alpha}(X_t N_t)^{\alpha} \tag{9}$$

In a closed economy without government spending, the market clearing condition is

$$Y_t = C_t + I_t \tag{10}$$

The social planner's problem is

$$\max_{\{K_{t+1},N_t\}} \sum_{t=0}^{+\infty} \beta^t u(F(K_t,N_t) + \frac{(1-\delta)K_t - K_{t+1}}{V_t}, 1-N_t)$$

given $K_0$. First order conditions:

$$\frac{u_c(C_t,1-N_t)}{V_t} = \beta\left[u_c(C_{t+1},1-N_{t+1})\left((1-\alpha)A(\frac{K_{t+1}}{X_{t+1}N_{t+1}})^{-\alpha} + \frac{1-\delta}{V_{t+1}}\right)\right] \quad (11)$$

$$u_l(C_t,1-N_t) = u_c(C_t,1-N_t)\alpha A(\frac{K_t}{X_tN_t})^{1-\alpha} \quad (12)$$

where

$$C_t = AK_t^{1-\alpha}(X_tN_t)^\alpha + \frac{(1-\delta)K_t - K_{t+1}}{V_t} \quad (13)$$

The first question we face is how to transform the non-stationary variables into stationary variables. How to reasonable guess the growth rates? We still use $\gamma$ to denote growth rate. On the balanced growth path (short run dynamics shut down), equation (9) suggests

$$(1+\gamma_Y) = (1+\gamma_K)^{1-\alpha}(1+\gamma_X)^\alpha.$$

Equation (8) can be written as

$$\frac{K_{t+1}}{K_t} = 1-\delta + \frac{I_tV_t}{K_t}$$

which implies $I_tV_t$ has to grow at the same rate as $K_t$, so

$$1+\gamma_K = (1+\gamma_I)(1+\gamma_V)$$

and finally equation (10) tells that we can further guess

$$\gamma_Y = \gamma_C = \gamma_I.$$

Solving the relationship between growth rates, we have

$$1+\gamma_Y = 1+\gamma_C = 1+\gamma_I = (1+\gamma_X)(1+\gamma_V)^{\frac{1-\alpha}{\alpha}}$$

and

$$1+\gamma_K = (1+\gamma_X)(1+\gamma_V)^{\frac{1}{\alpha}}$$

Therefore, define $S_t = X_tV_t^{\frac{1-\alpha}{\alpha}}$ and re-define variables

$$y_t = \frac{Y_t}{S_t}, c_t = \frac{C_t}{S_t}, i_t = \frac{I_t}{S_t}, k_{t+1} = \frac{K_{t+1}}{S_{t+1}V_{t+1}}.$$

We can then verify these re-defined variables when plugged into equation (11) and (12) indeed eliminate both growing terms $X_t$ and $V_t$.

From now on, for simplicity, I will only keep $X_t$ and disregard the growth of $V_t$. Equation

(11), (12) and (13) will be transformed to

$$u_c(c_t, 1-N_t) = \beta(1+\gamma_S)^{-\sigma}(1+\gamma_V)\left[u_c(c_{t+1}, 1-N_{t+1})\left((1-\alpha)A(\frac{k_{t+1}}{N_{t+1}})^{-\alpha} + 1 - \delta\right)\right]$$

$$u_l(c_t, 1-N_t) = u_c(c_t, 1-N_t)\alpha A(\frac{k_t}{N_t})^{1-\alpha}$$

and

$$c_t = Ak_t^{1-\alpha}N_t^{\alpha} + (1-\delta)k_t - (1+\gamma_X)k_{t+1}$$

where $\beta(1+\gamma_X)^{1-\sigma} < 1$ to make sure that lifetime utility is finite. Note that we have used the fact that $u(C, L)$ is King-Plosser-Rebelo preferences.

# 4 Stochastic TFP Shocks

In previous sections, we have investigated deterministic case. In that case, in the long run, stationary variables reach their steady state. In order to explain short run business cycles, we introduce uncertainty to the model. Specifically, technology $A_t$ is no longer a constant but instead an AR(1) process

$$\log A_{t+1} - \log \bar{A} = \rho(\log A_t - \log \bar{A}) + \epsilon_{t+1}$$

Define $a_t = \log A_t - \log \bar{A}$, then

$$a_{t+1} = \rho a_t + \epsilon_{t+1} \tag{14}$$

We keep having constant growth rate of labor augmented technology

$$\frac{X_{t+1}}{X_t} = 1 + \gamma_X$$

After introducing uncertainty, the objective function of a social planner needs to include expectation operator:

$$\max_{\{K_{t+1}, N_t\}} E_0\left[\sum_{t=0}^{+\infty}\beta^t u(F(K_t, N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t)\right]$$

Accordingly, our first order conditions should also carry with expectation operators. In addition, TFP $A$ is time varying now:

$$u_c(C_t, 1-N_t) = \beta E_t\left[u_c(C_{t+1}, 1-N_{t+1})\left((1-\alpha)A_{t+1}(\frac{K_{t+1}}{X_{t+1}N_{t+1}})^{-\alpha} + 1 - \delta\right)\right] \tag{15}$$

$$u_l(C_t, 1-N_t) = u_c(C_t, 1-N_t)\alpha A_t(\frac{K_t}{X_t N_t})^{1-\alpha}X_t \tag{16}$$

and

$$C_t = A_t K_t^{1-\alpha}N_t^{\alpha} + (1-\delta)K_t - K_{t+1} \tag{17}$$

Use stationary variables $c_t = \frac{C_t}{X_t}$, $k_{t+1} = \frac{K_{t+1}}{X_{t+1}}$ to re-write the above

$$u_c(c_t, 1-N_t) = \beta(1+\gamma_S)^{-\sigma} E_t \left[ u_c(c_{t+1}, 1-N_{t+1}) \left( (1-\alpha) A_{t+1} (\frac{k_{t+1}}{N_{t+1}})^{-\alpha} + 1 - \delta \right) \right]$$

$$u_l(c_t, 1-N_t) = u_c(c_t, 1-N_t) \alpha A_t (\frac{k_t}{N_t})^{1-\alpha}$$

and

$$c_t = A_t k_t^{1-\alpha} N_t^{\alpha} + (1-\delta) k_t - (1+\gamma_X) k_{t+1}$$

Besides, remember we also need to write down exogenous TFP process:

$$a_{t+1} = \rho a_t + \epsilon_{t+1}$$

We will still do log-linearization. The linear system is

$$\xi_{cc}\hat{c}_t - \xi_{cl}\frac{N^*}{1-N^*}\hat{N}_t = E_t[\xi_{cc}\hat{c}_{t+1} - \xi_{cl}\frac{N^*}{1-N^*}\hat{N}_{t+1} + (1-\beta(1+\gamma_X)^{-\sigma}(1-\delta))(a_{t+1} + \alpha\hat{N}_{t+1} - \alpha\hat{k}_{t+1})]$$

$$a_t + (1-\alpha)\hat{k}_t - (1-\alpha)\hat{N}_t = \xi_{lc}\hat{c}_t - \xi_{ll}\frac{N^*}{1-N^*}\hat{N}_t - \xi_{cc}\hat{c}_t + \xi_{cl}\frac{N^*}{1-N^*}\hat{N}_t$$

$$s_c\hat{c}_t + \frac{s_i(1+\gamma_X)}{\delta+\gamma_X}\hat{k}_{t+1} - \frac{(1-\delta)s_i}{\delta+\gamma_X}\hat{k}_t = a_t + (1-\alpha)\hat{k}_t + \alpha\hat{N}_t$$

and

$$a_{t+1} = \rho a_t + \epsilon_{t+1}$$

Note once we have used log-linearization to get a linear system. It is enough for us to solve the model by only using expectation of the exogenous process: $E_t a_{t+1} = \rho a_t$. This will imply that the covariance matrix of stochastic shocks (in our case, just $\sigma_\epsilon$) doesn't enter policy rules (certainty equivalence).

The dynamic system is reduced to the following after we eliminate $\hat{N}_t$ by employing the second equation.

$$E_t \begin{bmatrix} \hat{k}_{t+1} \\ a_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = W \begin{bmatrix} \hat{k}_t \\ a_t \\ \hat{c}_t \end{bmatrix}$$

where

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ 0 & \rho & 0 \\ w_{31} & w_{32} & w_{33} \end{bmatrix}$$

Do a decomposition of matrix $W$ as $W = P\Lambda P^{-1}$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(Here it is for illustration purpose. It is not always true that W is diagonalizable. But one can always do ***Jordan decomposition***.) Suppose $|\lambda_1| > 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ and define

$$\begin{bmatrix} \tilde{k}_t \\ a_t \\ \tilde{c}_t \end{bmatrix} = P^{-1} \begin{bmatrix} \hat{k}_t \\ a_t \\ \hat{c}_t \end{bmatrix}.$$

We then obtain

$$E_t \begin{bmatrix} \tilde{k}_{t+1} \\ a_{t+1} \\ \tilde{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ a_t \\ \tilde{c}_t \end{bmatrix}$$

In order to satisfy $TVC$, it has to be the case that

$$\tilde{k}_t = 0$$

Recall the definition of $\tilde{k}_t$, we know $\tilde{k}_t = b_1\hat{k}_t + b_2a_t + b_3\hat{c}_t = 0$, where $[b_1, b_2, b_3]$ is the first row of matrix $P^{-1}$. The solution gives

$$\hat{c}_t = -\frac{b_1}{b_3}\hat{k}_t - \frac{b_2}{b_3}\hat{c}_t.$$

Finally, it is easy to solve all other endogenous variables.

We can also use *undetermined coefficients* method by writing the solution as $\hat{k}_{t+1} = \phi_{11}\hat{k}_t + \phi_{12}a_t$ and $\hat{c}_t = \phi_{21}\hat{k}_t + \phi_{22}a_t$ such that

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21}\phi_{11} & \phi_{21}\phi_{12} + \phi_{22}\rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ a_t \end{bmatrix} = \begin{bmatrix} w_{11} + w_{13}\phi_{21} & w_{12} + w_{13}\phi_{22} \\ w_{31} + w_{33}\phi_{21} & w_{32} + w_{33}\phi_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ a_t \end{bmatrix}$$

Compare the left and right hand side to solve the coefficients. Pick up the solution that satisfies $TVC$.

# Labor Market Search and the Real Business Cycle Model

Macroeconomic Analysis Recitation 8

Yang Jiao*

# 1 Introduction

In a standard RBC model, it is difficult to replicate large labor market volatility with reasonable Frisch elasticity that is consistent with micro-level data. Moreover, in that model, labor supply is voluntary, i.e. there is no involuntary unemployment. Labor market search model is a popular way in the literature to study unemployment dynamics. We will introduce it to the standard RBC model.

# 2 Labor Market Search Model without Capital

## 2.1 Households

The representative household's problem is

$$\max_{C_t,S_{t+1},N_t} \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t [\log C_t - \gamma N_t] \tag{1}$$

subject to the budget constraint

$$C_t + q_t S_{t+1} \leq (q_t + d_t) S_t + w_t N_t \tag{2}$$

and law of motion for employment

$$N_t = (1-\xi) N_{t-1} + \phi_{t-1}(1 - N_{t-1}) \tag{3}$$

where $C_t$ is consumption, $N_t$ labor supply.

$S_t$ is the equity share households inherit from last period and they need to pick up their demand of equity share $S_{t+1}$ in period $t$ (the total supply is fixed and normalized to 1, so in equilibrium $S_t = 1$). $d_t$ is the dividend payment per equity share in period t. $q_t$ is equity price after dividend payment ($q_t + d_t$ is equity price before dividend payment.) And $w_t$ is wage.

*Please email me if you find errors or typos to yj2279@columbia.edu. All comments and suggestions are welcome and appreciated.

$\xi$ is a constant that represents the exogenous employment exit probability. $\phi_t$ is the probability that an unemployment worker finds a job.

Denote $\lambda_t$ as the Lagrangian multiplier for the budget constraint and $V_t^n$ as the Lagrangian multiplier for the employment dynamic equation (3). We obtain the following first order conditions with respect to $C_t$, $S_{t+1}$ and $N_t$:

$$\lambda_t = \frac{1}{C_t} \tag{4}$$

$$q_t\lambda_t = \beta E_t[\lambda_{t+1}(q_{t+1} + d_{t+1})] \tag{5}$$

$$V_t^n = -\gamma + \lambda_t w_t + \beta(1 - \xi - \phi_t)E_t V_{t+1}^n \tag{6}$$

Rewrite equation (5) as

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}(d_{t+1} + q_{t+1}) \tag{7}$$

Iterate forward to arrive at

$$q_t = E_t \sum_{j=1}^{+\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} d_{t+j} \tag{8}$$

Or equivalently, stock price before dividend payment is

$$q_t + d_t = E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} d_{t+j} \tag{9}$$

That is, the equity price is the summation of discounted dividend payment. Firms' objective will be to maximize equity value of the firm.

## 2.2 Firms

Workers in the firm are divided into two departments. $N_t^y$ workers are producing good, while $N_t^r$ are recruiters.

The production function is in linear form.

$$Y_t = A_t N_t^y$$

where $A_t$ is aggregate productivity.

Recall that we already derived that stock price (before dividend payment) in period 0 is

$$q_0 + d_0 = E_0 \sum_{t=0}^{+\infty} \beta^j \frac{\lambda_t}{\lambda_0} d_t$$

Its value to equity holders in terms of utility is $J_0 = \lambda_0(q_0 + d_0)$. Therefore, the objective function of a firm is to maximize $J_0$. Note that $d_t = A_t N_t^y - w_t(N_t^y + N_t^r)$. Therefore, firms' problem is to

$$\max_{N_t^y, N_t^r} J_0 = E_0 \sum_{t=0}^{+\infty} \beta^t \lambda_t [A_t N_t^y - w_t(N_t^y + N_t^r)] \tag{10}$$

subject to

$$N_t^y + N_t^r = (1-\xi)(N_{t-1}^y + N_{t-1}^r) + \mu_{t-1} N_{t-1}^r \tag{11}$$

where $\xi$ is the aformentioned employment exit probability. $\mu_t$ is the probability that a recruiter can successfully match a new worker.

Denoting $J_t^n$ as the Lagrangian multiplier for equation (11), we have the first order conditions for $N_t^y$ and $N_t^y$ are:

$$J_t^n = \lambda_t(A_t - w_t) + \beta(1-\xi)E_t J_{t+1}^n \tag{12}$$

$$J_t^n = -\lambda_t w_t + \beta(1-\xi+\mu_t)E_t J_{t+1}^n \tag{13}$$

Combining the above two equation to yield

$$\lambda_t A_t = \mu_t \beta E_t J_{t+1}^n \tag{14}$$

The economic interpretation of this equation is that firms allocate recruiters and production workers such that the foregone production of a recruiter equals to the benefit of getting more workers and higher expected future profits.

Substitute equation (14) into equation (13),

$$J_t^n = \lambda_t \left( (1 + \frac{1-\xi}{\mu_t}) A_t - w_t \right) \tag{15}$$

## 2.3 Labor Market Search

In previouse sections, households take job finding probability $\phi_t$ as given and firms take hiring rate $\mu_t$ as given. Now we start to gauge into how these two probabilities are determined.

Define recruiter-unemployment ratio as

$$\theta_t = \frac{N_t^r}{1 - N_t}.$$

Assume that $\mu_t$ depends only on $\theta_t$, so $\mu_t = \mu(\theta_t)$. The total measure of unemployed individuals hired by firms in period t is then $N_t^r \mu(\theta_t)$. Therefore, the probability for an unemployment individual to find a job is $\frac{N_t^r}{1-N_t}\mu(\theta_t)$, i.e.

$$\phi_t = \phi(\theta_t) = \theta_t \mu(\theta_t)$$

## 2.4 Wage Determination

It remains to find a way to nail down wage. There are a range of wages which make both firms and workers obtain positive surplus. After By assuming Nash bargaining between firms and workers to divide the joint surplus from the relationship:

$$V_t^n = \zeta(J_t^n + V_t^n) \tag{16}$$

$$J_t^n = (1-\zeta)(J_t^n + V_t^n) \tag{17}$$

where $0 \leq \zeta \leq 1$ is the bargaining weight for worker. $J_t^n + V_t^n$ is the total surplus from a match.

## 2.5 Market Clearing Conditions

Labor market: $N_t = N_t^r + N_t^y$
Asset market: $S_t = 1$
Goods Market: $C_t = Y_t$

## 2.6 Competitive Equilibrium

A competitive equilibrium is defined as a set of sequences $\{Y_t, N_t, V_t^n, J_t^n, \theta_t\}$ with $0 < N_t, \theta_t < 1$ that satisfy

$$Y_t = (1+\theta_t)A_t N_t - A_t\theta_t$$

$$N_{t+1} = (1-\xi)N_t + \phi(\theta_t)(1-N_t)$$

$$J_t^n = \frac{1}{Y_t}\left((1 + \frac{1-\xi}{\phi(\theta_t)}\theta_t)A_t - w_t\right)$$

$$\frac{A_t\theta_t}{Y_t} = \phi(\theta_t)\beta E_t J_{t+1}^n$$

$$V_t^n = -\gamma + \frac{w_t}{Y_t} + \beta(1-\xi_t-\phi(\theta_t))E_t V_{t+1}^n$$

$$\zeta J_t^n = (1-\zeta)V_t^n$$

given exogenous $A_t$ and initial $N_0$.

Define $y_t = \frac{Y_t}{A_t}$ and $\tilde{w}_t = \frac{w_t}{A_t}$, and rewrite the system

$$y_t = (1+\theta_t)N_t - \theta_t$$

$$N_{t+1} = (1-\xi)N_t + \phi(\theta_t)(1-N_t)$$

$$J_t^n = \frac{1}{y_t}\left((1 + \frac{1-\xi}{\phi(\theta_t)}\theta_t) - \tilde{w}_t\right)$$

$$\frac{\theta_t}{y_t} = \phi(\theta_t)\beta E_t J_{t+1}^n$$

$$V_t^n = -\gamma + \frac{\tilde{w}_t}{y_t} + \beta(1-\xi_t-\phi(\theta_t))E_t V_{t+1}^n$$

$$\zeta J_t^n = (1-\zeta)V_t^n$$

We immediately see that the solution to labor market $N_t$ is irrelevant with technology shock $A_t$. Therefore, this model, TFP shock cannot generate labor market fluctuations at all!

Later, we first introduce capital good, in that case technology will affect the accumulation incentives for capital good and since there is no constant returns to scale to labor, labor market will respond to TFP shock. However, real wage will increase if there is a technology shock. The high wage makes firms not expand that much employment. Therefore, it is still hard to generate realistic employment fluctuations. Furthermore, real wages are highly procyclical, which is at odds with data.

In order to solve the high employment fluctuations puzzle, Hall(2005) and Hagedorn and Manovskii(2008) try to moderate real wage response so that employment responses will be large. The first paper relies on wage rigidity assumption and the second paper assigns little bargaining power to workers.

# Monetary Theory: Flexible Price Models

Macroeconomic Analysis Recitation 9

Yang Jiao*

## 1 The Perils of Taylor Rule

(Exam 2016, Q2) An **endowment** economy is populated by a representative household with **money in utility** preferences

$$U = \sum_{t=0}^{+\infty} \beta^t [\log C_t + \phi(m_t)]$$

where $0 < \beta < 1$, $\phi''(m) < 0$, $\lim_{m \to 0} \phi'(m) = +\infty$ and $\lim_{m \to +\infty} \phi'(m) = 0$. $C_t$ is consumption and $m_t = \frac{M_t}{P_t}$ is realmoney balance, where $M_t$ is nominal money balance and $P_t$ nominal price level.

The representative household's budget constraint is

$$C_t + m_t + b_t = Y + \frac{(1 + i_{t-1})b_{t-1} + m_{t-1}}{1 + \pi_t} + t_t$$

where $Y$ is the endowment, $b_t$ is real holdings of nominal bond, $i_t$ is the nominal interest rate, $t_t = \frac{T_t}{P_t}$ is real lump-sum transfer from the government to the household and $\pi_t$ is inflation rate.

The government runs a balanced budget where they collect seigniorage tax and lump-sum transfer to household:

$$m_t - \frac{m_{t-1}}{1 + \pi_t} = t_t$$

The central bank sets money supply $m_t$ in order to target the nominal interest rate according to

$$1 + i_t = \rho(1 + \pi_t)$$

where function $\rho(\cdot)$ is non-decreasing, $\rho(1) = \beta^{-1}$ and $\beta\rho'(1) > 1$.

Denoting $\lambda_t$ as the Lagrangian multiplier, we obtain the following first order conditions:

$$\frac{1}{C_t} = \lambda_t \tag{1}$$

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$$\phi'(m_t) = \lambda_t - \beta \frac{\lambda_{t+1}}{1+\pi_{t+1}} \tag{2}$$

$$\lambda_t = \beta(1+i_t)\frac{\lambda_{t+1}}{1+\pi_{t+1}} \tag{3}$$

From these conditions, the first observation is that **nominal interest rate cannot be negative.** Combining equation (2) and (3), we see that if $i_t < 0$, then $\phi'(m_t) < 0$, which is contradictory to the assumption that more real balance gives higher utility. Intuitively, if nominal interest rate is negative. The household can simply issue bond (whose gross return is smaller than 1) and store money (whose gross return is 1 and in addition, it provides utility) to arbitrage.

The market clearing condition says $C_t = Y$, therefore, by equation (1), we know $\lambda_t$ is a constant. Then equation (3) tells

$$1 + \pi_{t+1} = \beta(1 + i_t) \tag{4}$$

The interest rate rules are set by the central bank:

$$1 + i_t = \rho(1 + \pi_t) \tag{5}$$

The above two lead to

$$1 + \pi_{t+1} = \beta\rho(1 + \pi_t)$$

First, $\pi^* = 0$ is a steady state. But since $i_t \geq 0$, so $\rho(1 + \pi_t) \geq 1$, we must have another steady state, featuring deflation. See Figure 1. We can have at least **two steady states**.

Notice that the deflation steady state is stable while $\pi^* = 0$ steady state is unstable. This question illustrates by taking into account zero lower bound on nominal interest rate and doing global analysis, instead of only log-linearizing around $\pi^* = 0$ steady state, we can get quite different solutions and dynamics!

# 2 Cash In Advance with Interest Rate Target

(Exam 2015, Q2) A representative household preferences on consumption $C_t$ and hours worked $H_t$ are given by

$$U = \sum_{t=0}^{+\infty} \beta^t [\log C_t - \frac{\gamma}{1+\kappa} H_t^{1+\kappa}]$$

subject to budget constraint

$$M_t + \frac{B_t}{1+R} + \frac{\hat{B}_t}{1+\hat{R}_t} = M_{t-1} + B_{t-1} + \hat{B}_{t-1} + W_{t-1}H_{t-1} - P_{t-1}C_{t-1} + P_t T$$

$$M_0, B_0, \hat{B}_0 \text{ given.}$$

where $P_t$ is price level, $M_t$ is money, $W_t$ is the nominal wage and $T$ are lump-sum transfers by the government. $B_t$ is a one period government issued bond yielding $R > 0$. $\hat{B}_t$ is a one period privately issued bond and the interest rate is $\hat{R}_t$. Households face a **Cash-in-**

**Advance** constraint

$$P_t C_t \leq M_t$$

Competitive firms produce according to a linear technology

$$Y_t = H_t$$

Government debt evolves according to

$$\frac{B_t}{1+R} = P_t T + B_{t-1} - (M_t - M_{t-1})$$

In equilibrium, good market clears

$$C_t = Y_t$$

and private debt

$$\hat{B}_t = 0$$

Denote $m_t = \frac{M_t}{P_t}$, $b_t = \frac{B_t}{P_t}$, $w_t = \frac{W_t}{P_t}$, and $\pi_t = \frac{P_t}{P_{t-1}} - 1$. The household problem can be written as

$$\max_{C_t, m_t, H_t, b_t, \hat{b}_t} \sum_{t=0}^{+\infty} \beta^t [\log C_t - \frac{\gamma}{1+\kappa} H_t^{1+\kappa}]$$

subject to

$$m_t + \frac{b_t}{1+R} + \frac{\hat{b}_t}{1+\hat{R}_t} = \frac{m_{t-1} + b_{t-1} + \hat{b}_{t-1} + w_{t-1}H_{t-1} - C_{t-1}}{1+\pi_t} + T$$

$$C_t \leq m_t$$

Note that households solve $C_t, H_t$ for $t \geq 0$, and $m_t, b_t, \hat{b}_t$ for $t \geq 1$ as $m_0$, $b_0$ and $\hat{b}_0$ are given for individual household. Denote $\lambda_t$ ($t \geq 1$) and $\xi_t$ ($t \geq 0$) as Lagrangian multipliers of the two constraint.

For $t \geq 1$, first order conditions are

$$C_t : \beta \frac{\lambda_{t+1}}{1+\pi_{t+1}} + \xi_t = \frac{1}{C_t} \tag{6}$$

$$m_t : \lambda_t = \xi_t + \beta \frac{\lambda_{t+1}}{1+\pi_{t+1}} \tag{7}$$

$$H_t : w_t \beta \frac{\lambda_{t+1}}{1+\pi_{t+1}} = \gamma H_t^{\kappa} \tag{8}$$

$$b_t : \frac{\lambda_t}{1+R} = \beta \frac{\lambda_{t+1}}{1+\pi_{t+1}} \tag{9}$$

$$\hat{b}_t : \frac{\lambda_t}{1+\hat{R}_t} = \beta \frac{\lambda_{t+1}}{1+\pi_{t+1}} \tag{10}$$

For $t = 0$, first order conditions are

$$C_0 : \frac{1}{C_0} = \xi_0 + \beta \frac{\lambda_1}{1+\pi_1} \tag{11}$$

$$H_0 : \beta \frac{w_0}{1+\pi_1} = \gamma H_0^\kappa \tag{12}$$

Equation (7) and (9) give

$$\xi_t = \frac{R}{1+R} \lambda_t \tag{13}$$

for $t \geq 1$.

Substitute equation (9) and (13) into equation (6) to get

$$\lambda_t = \frac{1}{C_t} \tag{14}$$

for $t \geq 1$.

Equation (6) can be re-written as

$$\gamma \frac{H_t^\kappa}{w_t} + \xi_t = \frac{1}{C_t} \tag{15}$$

by substituting into equation (8).

Equation (7) can be re-written as

$$1 = \beta(1+R) \frac{C_t}{C_{t+1}} \frac{1}{1+\pi_{t+1}} \tag{16}$$

Equation (8) can be re-written as

$$w_t = (1+R)\gamma H_t^\kappa C_t \tag{17}$$

by substituting into equation (8) and (13).

Equation (10) implies

$$\hat{R}_t = R \tag{18}$$

by comparing with equation (9).

Since $\lambda_1 = \frac{1}{C_1}$ from equation (14), we re-write equation (11) and (12):

$$\frac{1}{C_0} = \xi_0 + \beta \frac{1}{(1+\pi_1)C_1} \tag{19}$$

$$\beta \frac{w_0}{1+\pi_1} \frac{1}{C_1} = \gamma H_0^\kappa \tag{20}$$

Finally, cash in advance constraint is always binding $C_t = m_t$ for $t \geq 0$.

Government debt evolves as

$$b_t = (1+R)(-m_t + T + \frac{b_{t-1} + m_{t-1}}{1+\pi_t})$$

Next we proceed to look at firms' problem whose production function is

$$Y_t = H_t.$$

Its profit maximization gives

$$w_t = 1$$

Market clearing condition tells

$$C_t = Y_t$$

Then equation (17) is simply

$$1 = (1+R)\gamma C_t^{1+\kappa} \tag{21}$$

The above gives

$$C_t = (\frac{1}{\gamma(1+R)})^{\frac{1}{1+\kappa}} = (\frac{\beta}{\gamma})^{\frac{1}{1+\kappa}}$$

for $t \geq 1$.

**Case 1.** Now suppose $T = 0$ and $R = \frac{1-\beta}{\beta}$.

Due to the cash in advance constraint, we also have

$$m_t = C_t = (\frac{\beta}{\gamma})^{\frac{1}{1+\kappa}}$$

for $t \geq 1$ and thus $\lambda_t$ is also a constant. Equation (9) then pins down

$$\pi_{t+1} = \beta(1+R) - 1 = 0$$

for $t \geq 1$.

Government debt evolution is

$$b_{t+1} = \frac{b_t}{\beta} + (\frac{1-\beta}{\beta} - R)(\frac{\beta}{\gamma})^{\frac{1}{1+\kappa}} + (1+R)T = \frac{b_t}{\beta}$$

In order to satisfy TVC, we have to pick up $b_t = 0$, for $t \geq 1$. In particular, $b_1 = 0$, which implies

$$0 = b_1 = (1+R)(-m_1 + T + \frac{b_0 + m_0}{1+\pi_1}) = (1+R)(-(\frac{\beta}{\gamma})^{\frac{1}{1+\kappa}} + 0 + \frac{B_0 + M_0}{P_0(1+\pi_1}))$$

In addition, at time 0, equation (20) gives

$$\frac{\beta}{1+\pi_1} = \gamma(\frac{M_0}{P_0})^\kappa C_1$$

The above two equations have two unknowns $P_0$ and $\pi_1$, so we are able to solve them:

$$P_0 = [\frac{\gamma}{\beta} M_0^{\kappa} (B_0 + M_0)]^{\frac{1}{1+\kappa}}$$

**Case 2.** Now suppose $R = \frac{1-\beta}{\beta}$ and

$$P_t T_t = -\rho B_{t-1}$$

with $\rho > 0$.

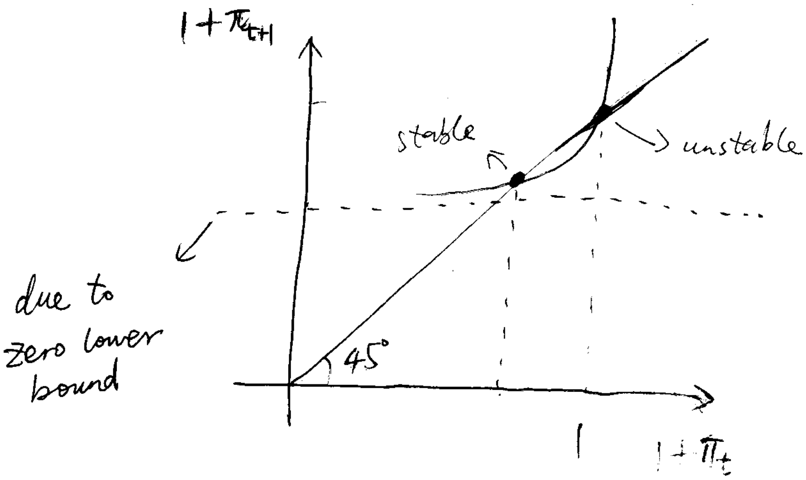
Then the evolution of government debt is

$$b_{t+1} = \frac{1-\rho}{\beta} b_t$$

We require $\frac{1-\rho}{\beta} > 1$, that is $\rho < 1 - \beta$, so that we can immediately have

$$b_t = 0$$

for all $t \geq 1$ in order to satisfy TVC. Note $b_1 = 0$ is important to pin down the unique equilibrium. Otherwise, we only have one equation for $P_0$ and $\pi_1$ (from equation (20)), which will lead to infinite solutions.




$\beta \cdot \rho'(1) > 1$

$\Rightarrow$ slope at $1+\pi_t = 1$ is larger than 1

Figure 1.

# Monetary Theory: Sticky Price Models

Macroeconomic Analysis Recitation 10

Yang Jiao*

## 1 Dixit-Stiglitz Preference

The cost minimization problem for a consumer with Dixit-Stiglitz preference is

$$\min_{C_{it}} \int_0^1 P_{it} C_{it} di$$

s.t.

$$C_t = (\int_0^1 C_{it}^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}}$$

This is in fact a static problem (and each period, households face this problem), so one can omit all the time subscript here. Denote $\psi$ as the Lagrange multiplier of the constraint.

First order condition is

$$P_i = \psi(\int_0^1 C_i^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}-1} C_i^{-\frac{1}{\theta}}$$

which will hold for any $i \in [0,1]$.

$$\to \frac{C_i}{C_j} = (\frac{P_i}{P_j})^{-\theta}$$

Rearrange the above equation to get

$$C_i^{\frac{\theta-1}{\theta}} = C_j^{\frac{\theta-1}{\theta}} P_j^{\theta-1} P_i^{1-\theta}$$

$$\to P_j^{1-\theta} C_i^{\frac{\theta-1}{\theta}} = C_j^{\frac{\theta-1}{\theta}} P_i^{1-\theta}$$

$$\to \int_0^1 P_j^{1-\theta} dj \cdot C_i^{\frac{\theta-1}{\theta}} = \int_0^1 C_j^{\frac{\theta-1}{\theta}} dj \cdot P_i^{1-\theta}$$

$$\to P^{1-\theta} \cdot C_i^{\frac{\theta-1}{\theta}} = C^{\frac{\theta-1}{\theta}} \cdot P_i^{1-\theta}$$

where **price index** $P$ is defined as $P = [\int_0^1 P_j^{1-\theta} dj]^{\frac{1}{1-\theta}}$.

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Therefore, the **demand function** for good $i$ is

$$\rightarrow C_i = C(\frac{P_i}{P})^{-\theta}$$

which leads to

$$\rightarrow P_i C_i = P_i^{1-\theta} P^{\theta} C$$

Integrate both sides from 0 to 1 to finally have the total expenditure

$$\int_0^1 P_i C_i di = PC$$

This final equation makes it clear why we call $P$ as price index. For example, when we write down the budget constraint for a household, instead of using $\int_0^1 P_{it} C_{it} di$ as the expenditure on consumption, we can simply use $P_t C_t$.

# 2 Sticky Price: the Calvo Model

Each period with probability $\lambda$, a firm can adjust its price. The firm understands that this price may stay in the future, so he takes into account today's price's effect on future periods.

Suppose the firm sets price $P_{it}$ at time $t$. With probability $(1-\lambda)$, his price is still $P_{it}$ at time $t+1$. With probability $(1-\lambda)^2$, his price is still $P_{it}$ at time $t+2$ ...With probability $(1-\lambda)^k$, his price is still $P_{it}$ at time $t+k$...That's where the following summation comes from. The firm chooses $P_{it}$ to maximize the following:

$$\max_{P_{i,t}} \sum_{k=0}^{+\infty} (1-\lambda)^k E_t \{ \frac{M_{t,t+k}}{P_{t+k}} [P_{i,t} - MC_{i,t+k}] Y_{i,t+k} \}$$

s.t.

$$Y_{i,t+k} = Y_{t+k} (P_{it}/P_{t+k})^{-\theta}$$

where $[P_{i,t} - MC_{i,t+k}]Y_{i,t+k}$ is nominal profit. $M_{t,t+k}$ is the stochastic discount factor. The constraint is the demand function for firm $i$ product.

**Remark.** This stochastic discount factor comes from households' problem. Households trade the firm's stock shares. For example, consider the following household problem

$$\max_{C_t, h_t, q_{i,t+1}} E_t \sum_{t=0}^{+\infty} \beta^t [\log C_t - \frac{\gamma\epsilon}{1+\epsilon} \int_0^1 h_{jt}^{1+1/\epsilon} dj]$$

s.t.

$$P_t C_t + B_{t+1} + P_t K_{t+1} + P_t T_t + \int_0^1 q_{i,t+1}(v_{it} - d_{it}) di \leq B_t(1+i_t) + K_t(1+r_t)P_t + \int_0^1 W_{jt} h_{jt} dj + \int_0^1 q_{it} v_{it} di$$

where $q_{i,t+1}$ is the share of stocks of firm $i$ households buy. $v_{it}$ is firm $i$'s stock price before paying dividends $d_{it}$.

Denote $\Lambda_t$ the Lagrange multiplier of the budget constraint. F.O.C. for $C_t$

$$\frac{1}{C_t} = P_t \Lambda_t$$

F.O.C. for $q_{i,t+1}$

$$\Lambda_t (v_{it} - d_{it}) = \beta E_t (\Lambda_{t+1} v_{i,t+1})$$

$\rightarrow$

$$v_{it} = d_{it} + \beta E_t (\frac{\Lambda_{t+1}}{\Lambda_t} v_{i,t+1})$$

Iterate forward to get

$$v_{it} = \sum_{k=0}^{+\infty} \beta^k E_t (\frac{\Lambda_{t+k}}{\Lambda_t} d_{i,t+k})$$

Firm stock price is the discounted dividends of the firm. Now substitute $\Lambda_t$ to have

$$v_{it} = \sum_{k=0}^{+\infty} E_t (\beta^k \frac{P_t C_t}{P_{t+k} C_{t+k}} d_{i,t+k}) = P_t \sum_{k=0}^{+\infty} E_t (\beta^k \frac{C_t}{P_{t+k} C_{t+k}} d_{i,t+k})$$

Define $M_{t,t+k} = \beta^k \frac{C_t}{C_{t+k}}$, then

$$\frac{v_{it}}{P_t} = \sum_{k=0}^{+\infty} E_t (\frac{M_{t,t+k}}{P_{t+k}} d_{i,t+k})$$

Firms should maximize its stock price (firm value for equity holders).

What is $d_{i,t+k}$ for each firm? Its nominal profits in each period. Then check the above is your firm maximiazation objective for the Calvo model (up to the consideration of changing price probability).

In equilibrium, stock shares should be $q_{it} = 1 (100\%)$, then we can see that all dividends $\int_0^1 d_{it} di$ go to households.

■

First order condition of the Calvo firm is (substitute the constraint into the objective function):

$$\sum_{k=0}^{+\infty} (1-\lambda)^k E_t \{ \frac{M_{t,t+k}}{P_{t+k}} [(1-\theta) + \frac{\theta}{P_{it}} MC_{i,t+k}] Y_{i,t+k} \} = 0$$

i.e.

$$\sum_{k=0}^{+\infty} (1-\lambda)^k E_t \{ \frac{M_{t,t+k}}{P_{t+k}} Y_{i,t+k} [P_{it} - \frac{\theta}{\theta - 1} MC_{i,t+k}] \} = 0 \quad (1)$$

$\rightarrow$

$$P_{it} = \frac{\frac{\theta}{\theta-1} \sum_{k=0}^{+\infty} (1-\lambda)^k E_t \{ \frac{M_{t,t+k}}{P_{t+k}} Y_{i,t+k} MC_{i,t+k} \}}{\sum_{k=0}^{+\infty} (1-\lambda)^k E_t \{ \frac{M_{t,t+k}}{P_{t+k}} Y_{i,t+k} \}}$$

Log-linearize the above equation to get (apply several formulas of log-linearization)

$$p_{it} = \sum_{k=0}^{+\infty} \frac{(1-\lambda)^k \beta^k \frac{MC^*}{P^*} Y^* (m_{t,t+k} + y_{i,t+k} + mc_{i,t+k} - p_{t+k})}{\sum_{s=0}^{+\infty} (1-\lambda)^s \beta^s \frac{MC^*}{P^*} Y^*} - \sum_{k=0}^{+\infty} \frac{(1-\lambda)^k \beta^k \frac{Y^*}{P^*} (m_{t,t+k} + y_{i,t+k} - p_{t+k})}{\sum_{s=0}^{+\infty} (1-\lambda)^s \beta^s \frac{Y^*}{P^*}}$$

i.e.

$$p_{it} = [1-\beta(1-\lambda)] \sum_{k=0}^{+\infty} [\beta(1-\lambda)]^k E_t mc_{i,t+k}$$

Notice under flexible price, price $p^*_{i,t+k} = mc_{i,t+k}$. Therefore,

$$p_{it} = [1-\beta(1-\lambda)] \sum_{k=0}^{+\infty} [\beta(1-\lambda)]^k E_t (p^*_{i,t+k})$$

Substitute $t$ by $t-j$, where $j = 0, 1, 2, ...$:

$$p_{i,t-j} = [1-\beta(1-\lambda)] \sum_{k=0}^{+\infty} [\beta(1-\lambda)]^k E_{t-j} (p^*_{i,t-j+k})$$

We can ignore index $i$, since each firm is subject to the same shock,

$$p_t(j) = [1-\beta(1-\lambda)] \sum_{k=0}^{+\infty} [\beta(1-\lambda)]^k E_{t-j} (p^*_{t-j+k}) \tag{2}$$

At time $t$, we know in the economy, $\lambda$ firms set price at time $t$ with price $p_t(0)$, $\lambda(1-\lambda)$ firms set price at time $t$ with price $p_t(1)$, $\lambda(1-\lambda)^2$ firms set price at time $t$ with price $p_t(2)$, etc. Hence

$$p_t = \int_0^1 p_{it} di = \lambda \sum_{j=0}^{+\infty} (1-\lambda)^j p_t(j)$$

which implies

$$p_t = \lambda p_t(0) + (1-\lambda) p_{t-1} \tag{3}$$

Equation (2) implies

$$p_t(0) = [1-\beta(1-\lambda)] p^*_t + \beta(1-\lambda) E_t[E_{t+1}(p_{t+1}(0))]$$

i.e.

$$p_t(0) = [1-\beta(1-\lambda)] p^*_t + \beta(1-\lambda) E_t(p_{t+1}(0)) \tag{4}$$

Combine equation (3) and (4) to eliminate $p_t(0)$:

$$(1-\lambda)\pi_t + \lambda p_t = \lambda[1-\beta(1-\lambda)] p^*_t + \beta(1-\lambda) E_t[(1-\lambda)\pi_{t+1} + \lambda p_{t+1}]$$

Substitute flexible price $p_t^* = p_t + \alpha y_t$ to the above equation:

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa y_t$$

where $\kappa = \frac{[1-\beta(1-\lambda)]\alpha\lambda}{1-\lambda}$.

Though Calvo model is a tractable and widespread used model in macro literature, there are many **drawbacks** of Calvo model compared with data.

- Natural Rate property

  In steady state:

  $$\pi = \beta\pi + \kappa y \rightarrow y = \frac{(1-\beta)\pi}{\kappa}$$

  It is a **non-vertical long run phillip curve**, which means you can use long run high inflation to achieve high real output. In the long run, we would like monetary policy to be neutral.

- Accelerationist Problem A simple example to illustrate why **change in inflation and real output are negatively correlated by the Calvo model**: suppose that the government announces a disinflation $\pi_{t+1} < \pi_t$(caused by contractionary monetary shocks), recall that $\beta \approx 1$ the Phillips curve is

  $$\pi_t = E_t\pi_{t+1} + \kappa y_t$$

  $\rightarrow$

  $$\pi_t - \pi_{t+1} = \kappa y_t > 0$$

  That is the Calvo model predicts that the correlation between change in inflation and output is negative.

  This is at odds with data.

  **Remark.** Acceleration phenomenon means the positive correlation between real output and change in inflation in the data, i.e. when real output is high, inflation tends to rise.

- Non Hump-Shape

  Iterate forward the Phillips Curve:

  $$\pi_t = \kappa \sum_{k=0}^{+\infty} \beta^k E_t(y_{t+k})$$

  When the economy gets hit by a negative shock to aggregate demand, and the shock dies out gradually. Real output also dies out gradually. Then the above Phillips curve would predict a **non-hump shaped** path for inflation. This result is not in line with empirical evidence, where the response of inflation to monetary shocks tends to be hump-shaped.