Exercise 12.1 Solution

We prove by contradiction. Consider that sequences c_t, d_{t+1}, and k_{t+1} satisfy optimality conditions (12.1) to (12.8) but not the transversality condition (12.9). If equation (12.9) doesn't hold and given that $d_{t+1} \leq \kappa q_t k_{t+1}$, we have

$$\lim_{t \to +\infty} \frac{\kappa q_t k_{t+1} - d_{t+1}}{(1+r)^t} = \theta > 0, \tag{1}$$

which implies that for any $\epsilon > 0$ (we choose a small enough value for ϵ so that $\theta - \epsilon > 0$, say, $\epsilon = \frac{\theta}{2}$), there exists an integer $T > 0$, such that any $t \geq T$,

$$\frac{\kappa q_t k_{t+1} - d_{t+1}}{(1+r)^t} > \theta - \epsilon > 0 \tag{2}$$

Therefore, for $t \geq T$, the collateral constraints never bind,

$$\mu_t = 0, \tag{3}$$

Choose an alternative plan (use star * to denote variables under the new plan) such that $k^*_{t+1} = k_{t+1}$ for all $t \geq 0$, and

$$c^*_t = \begin{cases} c_t, & \text{when } t < T \\ c_t + \delta, & \text{when } t = T \\ c_t, & \text{when } t > T \end{cases}$$

where $\delta > 0$. This new plan clearly provides higher welfare compared to the old plan.

We will show for some positive values of δ, the new plan is feasible. That is, d^*_{t+1} derived from the budget constraint, will not violate the collateral constraint. Since when $t < T$, we already have $d^*_{t+1} = d_{t+1} \leq \kappa q_t k_{t+1} = \kappa q^*_t k^*_{t+1}$, we only need to focus on the path of debt d^*_{t+1} for $t \geq T$.

Denote $\Delta c^*_t = c^*_t - c_t$ and $\Delta d^*_t = d^*_t - d_t$. We immediately obtain $\Delta c^*_T = \delta$ from the construction of new path for consumption. Notice that, by the budget constraint, the following equation holds

$$\Delta d^*_{t+1} = (1+r)[\Delta c^*_t + \Delta d^*_t] \tag{4}$$

Since $\Delta c^*_T = \delta$, $\Delta d^*_T = 0$ and $\Delta c^*_t = 0$ for $t > T$, we get $\Delta d^*_{T+1} = \delta(1+r)$, $\Delta d^*_{T+2} = \delta(1+r)^2$, ... , $\Delta d^*_{T+j} = \delta(1+r)^j$, ... , or equivalently, for any $t \geq T$, $\Delta d^*_{t+1} = \delta(1+r)^{t-T+1}$.

For any $t \geq T$,

$$d^*_{t+1} = \Delta d^*_{t+1} + d_{t+1} = \delta(1+r)^{t-T+1} + d_{t+1}$$

Recall that in equation (2),

$$d_{t+1} < \kappa q_t k_{t+1} - (1+r)^t(\theta - \epsilon).$$

Combine the above two formulas to deliver

$$d^*_{t+1} = \delta(1+r)^{t-T+1} + d_{t+1} < \kappa q_t k_{t+1} - (1+r)^t[\theta - \epsilon - \delta(1+r)^{-T+1}].$$

Finally, set $\theta - \epsilon - \delta(1+r)^{-T+1} > 0$, i.e., $\delta < (1+r)^{T-1}(\theta - \epsilon)$, then $d^*_{t+1} < \kappa q_t k_{t+1} = \kappa q^*_t k^*_{t+1}$ is satisfied.

In conclusion, with $0 < \delta < (1+r)^{T-1}(\theta - \epsilon)$, the new plan is feasible and associated with higher welfare. $\square$

Exercise 12.2 Solution

1. Derive the first-order conditions associated with the household's optimization problem.

Sol: The objective is

$$\max_{c_1>0, c_2>0, d_2, k_1} \ln c_1 + \ln c_2$$

s.t.

$$F(k) + d_2 - d_1 - c_1 - q(k_1 - k) = 0$$

$$F(k_1) - d_2 - c_2 = 0$$

$$d_2 \leq \kappa q k_1$$

Denoting λ, η and μ as the Lagrangian multipliers of the above 3 constraints, we write

$$\max_{c_1>0, c_2>0, d_2, k_1} \ln c_1 + \ln c_2 + \lambda[F(k) + d_2 - d_1 - c_1 - q(k_1 - k)] + \eta[F(k_1) - d_2 - c_2] + \mu[\kappa q k_1 - d_2]$$

First order conditions are then given by

$$\frac{1}{c_1} = \lambda$$

$$\frac{1}{c_2} = \eta$$

$$\lambda - \eta - \mu = 0$$

$$-\lambda q + \eta F'(k_1) + \mu \kappa q = 0$$

$$F(k) + d_2 - d_1 - c_1 - q(k_1 - k) = 0$$

$$F(k_1) - d_2 - c_2 = 0$$

$$\mu(\kappa q k_1 - d_2) = 0$$

$$\mu \geq 0$$

$$\kappa q k_1 \geq d_2$$

2. Assume that the aggregate stock of capital is fixed. Derive the complete set of equilibrium conditions.

Sol: In equilibrium:

$$k_1 = k$$

Therefore, we have the following equilibrium conditions for variables $\{c_1 > 0, c_2 > 0, d_2, \lambda, \eta, \mu, q\}$:

$$\frac{1}{c_1} = \lambda$$

$$\frac{1}{c_2} = \eta$$

$$\lambda - \eta - \mu = 0$$
$$-\lambda q + \eta F'(k) + \mu\kappa q = 0$$
$$F(k) + d_2 - d_1 - c_1 = 0$$
$$F(k) - d_2 - c_2 = 0$$
$$\mu(\kappa qk - d_2) = 0$$
$$\mu \geq 0$$
$$\kappa qk \geq d_2$$

3. Characterize the range of initial debt positions, d_1, for which the collateral constraint does not bind in equilibrium.

Sol: Let $\mu = 0$, we obtain $\lambda = \eta$ thus $c_1 = c_2$. As $c_1 = F(k) + d_2 - d_1$ and $c_2 = F(k) - d_2$, we derive $d_2 = \frac{d_1}{2}$. In order to satisfy a slack collateral constraint, it is required that

$$d_2 \leq \kappa qk$$

Substituting capital price q and d_2 into the above equation, we get

$$d_1 \leq 2\kappa F'(k)k$$

Note that consumption must be positive as well,

$$c_1 = c_2 = F(k) - \frac{d_1}{2} > 0$$

That is

$$d_1 < 2F(k)$$

Recall that $F(k)$ is concave, $0 < \kappa < 1$ and **if** $F(0) = 0$, we know that when $k > 0$,

$$\kappa F'(k)k < F'(k)k < F(k).$$

In sum, $d_1 \leq 2\kappa F'(k)k$ guarantees that an unconstrained equilibrium exists.

4. Find a sufficient condition on the initial level of debt d_1, for which the economy possesses multiple equilibria, in particular, at least one equilibrium in which the collateral constraint binds and one equilibrium in which it does not.

Sol: Now we turn to binding collateral constraint. First,

$$\mu = \lambda - \eta = \frac{1}{c_1} - \frac{1}{c_2} = \frac{1}{F(k) + d_2 - d_1} - \frac{1}{F(k) - d_2}$$

Second, by replacing λ, η and μ, the follwing equation holds,

$$q = \frac{\eta F'(k)}{\lambda - \kappa\mu} = \frac{\frac{1}{F(k)-d_2}F'(k)}{\frac{1}{F(k)+d_2-d_1} - \kappa(\frac{1}{F(k)+d_2-d_1} - \frac{1}{F(k)-d_2})}$$
$$= \frac{(F(k) + d_2 - d_1)F'(k)}{F(k) - d_2 - \kappa(d_1 - 2d_2)}$$

It is a function of d_2. When collateral constraint binds,

$$LHS = d_2 = \kappa q(d_2)k = \kappa \frac{(F(k) + d_2 - d_1)F'(k)}{F(k) - d_2 - \kappa(d_1 - 2d_2)} k = RHS$$

Take both sides as functions of d_2. The left hand side is the 45 degree line and $RHS(\frac{d_1}{2}) = \kappa F'(k)k > \frac{d_1}{2}$.

Assume the intersection of $RHS(d_2)$ with the x-axis is point $\underline{\mathrm{d}}$, since the function is continuous, we only need to impose that $\underline{\mathrm{d}} > 0$, which says

$$\underline{\mathrm{d}} = d_1 - F(k) > 0$$

or

$$d_1 > F(k)$$

It is easy to check that $\underline{\mathrm{d}} = d_1 - F(k) < \frac{d_1}{2}$, given the condition $d_1 < 2\kappa F'(k)k$ in question **3**. Therefore, the condition we need is

$$F(k) < d_1 < 2\kappa F'(k)k$$

Notice the above requires $\kappa > \frac{1}{2}$ given that $F(k)$ renders concavity.

$$RHS(d_2) = \kappa \frac{d_2 + F(k) - d_1}{(2\kappa - 1)d_2 + F(k) - \kappa d_1} k = \frac{\kappa}{2\kappa - 1} \frac{(2\kappa - 1)d_2 + (2\kappa - 1)(F(k) - d_1)}{(2\kappa - 1)d_2 + F(k) - \kappa d_1} k$$

As $(2\kappa - 1)(F(k) - d_1) - (F(k) - \kappa d_1) = (\kappa - 1)(2F(k) - d_1) < 0$, it means $RHS(d_2)$ is an increasing function of d_2. So $RHS(d_2)$ must cross the 45 degree line once and the corresponding d_2^* has to be less than $\frac{d_1}{2}$ and greater than $\underline{\mathrm{d}}$, so $c_2 = F(k) - d_2 > 0$ and $c_1 = F(k) + d_2 - d_1 = d_2 - \underline{\mathrm{d}} < 0$.

In conclusion, a sufficient condition to this question is

$$F(k) < d_1 < 2\kappa F'(k)k$$

(which implicitly requires $\kappa > \frac{1}{2}$.)

Exercise 12.3

A (bubble-free) competitive equilibrium is a set of sequences $c_t > 0.d_t+1$, $\mu_t \geq 0$ and $q_t \geq 0$ satisfying

$$d_0 = \sum_{t=0}^{+\infty} \frac{y_t - c_t}{(1+r)^t} \tag{1}$$

$$c_t + d_t = y_t + \frac{d_{t+1}}{1+r} \tag{2}$$

$$\frac{1}{c_t}[\frac{1}{\beta(1+r)} - \frac{\mu_t}{\beta}] = \frac{1}{c_{t+1}} \tag{3}$$

$$\frac{q_t}{c_t}[1 - \kappa\mu_t] = \frac{\beta}{c_{t+1}}[q_{t+1} + \alpha\frac{y_{t+1}}{k}] \tag{4}$$

$$\mu_t(\kappa q_t k - d_{t+1}) = 0 \tag{5}$$

$$d_{t+1} \leq \kappa q_t k \tag{6}$$

$$\lim_{t\to\infty}(1+r)^{-t}q_t = 0 \tag{7}$$

with d_0 given and the exogenous sequences A_t and $y_t = A_t k^\alpha$.

The Steady State

$A_t = A$ for all t. Then the output is also constant $y_t = y = Ak^\alpha$. In steady state $c_t = c^* > 0$, $d_{t+1} = d^*$, and $\mu_t = \mu^* \geq 0$ and $q_t = q^* \geq 0$. Equation (3) implies $\mu^* = \frac{1}{1+r} - \beta$, then equation (4) becomes $q_t = \frac{\beta}{1-\kappa\mu^*}(q_{t+1} + \alpha\frac{y}{k})$. In steady state capital price is

$$q^* = \frac{\beta\alpha\frac{y}{k}}{1 - \beta - \kappa\mu^*}$$

In order to guarantee the above is positive, we need to impose $1 - \beta - \kappa\mu^* > 0$, i.e. $\kappa < \frac{(1-\beta)(1+r)}{1-\beta(1+r)}$.

The sequential budget constraint equation (2) gives

$$c^* = y - \frac{r}{1+r}d^*.$$

Going back to equation (1), we obtain

$$d^* = d_0.$$

Since the debt position is constant, the current account satisfies

$$ca^* = 0.$$

On the other hand, the trade balance is

$$tb^* = y - c^* = \frac{r}{1+r} d^*.$$

The natural debt limit requires $c^* > 0$, therefore,

$$d_0 < \frac{1+r}{r} y$$

The collateral constraint has to be binding since $\mu^* > 0$, so

$$d_0 = \kappa q^* k = \kappa \frac{\beta \alpha \frac{y}{k}}{1 - \beta - \kappa \mu^*} k = \kappa \frac{\beta \alpha y}{1 - \beta - \kappa \mu^*}$$

This steady state requires fairly strong parameter choices (including an equality.)

Non-existence of the Unconstrained Equilibrium

We now turn to a case where the collateral constraint never binds, so $\mu^* = 0$. Equation (3) implies

$$c_{t+1} = \beta(1+r)c_t$$

The above relationship, combining with the lifetime budget constraint equation (1) pins down

$$c_0 = (1 - \beta)\left(\frac{1+r}{r} y - d_0\right)$$

thus

$$c_t = [\beta(1+r)]^t c_0$$

which follows a decreasing path. Capital price, derived from equation (4), is

$$q_t = \frac{1}{1+r}[q_{t+1} + \alpha \frac{y}{k}].$$

Notice that $\frac{1}{1+r} < 1$ (if $r > 0$), so the unique stationary solution is

$$q_t = q^* = \frac{\alpha y / k}{r}$$

The dynamics of debt is given by equation (2):

$$\begin{aligned} d_{t+1} &= (1+r)(c_t - y_t + d_t) \\ &= (1+r)\left([\beta(1+r)]^t c_0 - y + d_t\right) \end{aligned}$$

Divide both sides by $(1+r)^{t+1}$ to obtain

$$\frac{d_{t+1}}{(1+r)^{t+1}} = \frac{d_t}{(1+r)^t} + \beta^t c_0 - \frac{y}{(1+r)^t}$$

The solution to the above is

$$\frac{d_t}{(1+r)^t} = d_0 + \sum_{k=0}^{t}[\beta^{k-1}c_0 - \frac{y}{(1+r)^{k-1}}],$$

that is,

$$\begin{aligned} d_{t+1} &= (1+r)^{t+1}\left\{d_0 + \frac{1-\beta^{t+1}}{1-\beta}c_0 - \left[\frac{1+r}{r} - \frac{1+r}{r}\left(\frac{1}{1+r}\right)^{t+1}\right]y\right\} \\ &= \frac{1+r}{r}y - [\beta(1+r)]^{t+1}c_0. \end{aligned}$$

Debt is an increasing path. This result is intuitive: with $\beta(1+r) < 1$, the representative agent would like to consume early, accumulating external debt. Alternatively, one can get the formula of d_{t+1} by employing

$$c_t = (1-\beta)(\frac{1+r}{r}y - d_t).$$

The trade balance is

$$tb_t = y - c_t$$

which is an increasing function of time t and the current account is

$$ca_t = tb_t - r\frac{d_t}{1+r}$$

In addition, the natural debt limit demands

$$d_0 < \frac{1+r}{r}y$$

To discipline the collateral constraint so that it is not binding, the following has to hold for any $t \geq 0$,

$$d_{t+1} < \kappa q^* k.$$

In fact, we know that $\lim_{t\to\infty} d_{t+1} = \frac{1+r}{r}y$, the condition turns to

$$\frac{1+r}{r}y < \kappa q^* k.$$

which can be simplified to

$$1 + r < \kappa\alpha$$

However, when $\kappa < 1$ and $\alpha < 1$ and $r > 0$, the above condition cannot hold. We conclude that the unconstrained equilibrium doesn't exist.

Exercise 9.1

Given tradable consumption c^T, the demand schedule of nontradables is given by

$$\frac{A_2(c^T, c^N)}{A_1(c^T, c^N)} = \frac{p^N}{p^T} = p \tag{1}$$

1

We have

$$\frac{\partial p}{\partial c^N} = \frac{A_{22}(c^T, c^N)A_1(c^T, c^N) - A_{12}(c^T, c^N)A_2(c^T, c^N)}{A_1^2(c^T, c^N)}$$

Since the aggregator is assumed to be increasing and concave, we know

$$A_1, A_2 \geq 0 \tag{2}$$

$$A_{22} \leq 0 \tag{3}$$

By the linearly homogeneous property, we obtain

$$A = c^T A_1 + c^N A_2 \to A_1 = \frac{A - c^N A_2}{c^T}$$

Take derivative w.r.t c^N to get

$$A_{12} = \frac{A_2 - A_2 - c^N A_{22}}{c^T} = -\frac{c^N}{c^T} A_{22} \geq 0 \tag{4}$$

Finally, we arrive at

$$\frac{\partial p}{\partial c^N} = \frac{A_{22}A_1 - A_{12}A_2}{A_1^2} \leq 0 \tag{5}$$

which implies that the demand schedule of non-tradables is downward sloping.

2

Taking derivative w.r.t c^T in equation (1), the following has to hold

$$\frac{\partial p}{\partial c^T} = \frac{A_{21}(c^T, c^N)A_1(c^T, c^N) - A_{11}(c^T, c^N)A_2(c^T, c^N)}{A_1(c^T, c^N)^2}$$

Similar to the above question, the assumption about the consumption aggregator guarantees that

$$A_1, A_2 \geq 0$$

$$A_{11} \leq 0 \tag{6}$$

$$A_{21} \geq 0 \tag{7}$$

Therefore

$$\frac{\partial p}{\partial c^T} = \frac{A_{21}A_1 - A_{11}A_2}{A_1^2} \geq 0 \tag{8}$$

In other words, the demand schedule shifts up and to the right when c^T increases.

3

$$A_2 = \frac{1}{2}\sqrt{\frac{c^T}{c^N}}$$

$$A_1 = \frac{1}{2}\sqrt{\frac{c^N}{c^T}}$$

$$p = \frac{A_2}{A_1} = \frac{c^T}{c^N} \tag{9}$$

4

$$A_2 = \left[a\left(c^T\right)^{1-\frac{1}{\xi}} + (1-a)\left(c^N\right)^{1-\frac{1}{\xi}}\right]^{\frac{1}{1-\frac{1}{\xi}}-1}(1-a)\left(c^N\right)^{-\frac{1}{\xi}}$$

$$A_2 = \left[a\left(c^T\right)^{1-\frac{1}{\xi}} + (1-a)\left(c^N\right)^{1-\frac{1}{\xi}}\right]^{\frac{1}{1-\frac{1}{\xi}}-1}a\left(c^T\right)^{-\frac{1}{\xi}}$$

$$p = \frac{A_2}{A_1} = \frac{1-a}{a}\left(\frac{c^T}{c^N}\right)^{\frac{1}{\xi}} \tag{10}$$

which leads to

$$\frac{d\ln\left(\frac{c^T}{c^N}\right)}{d\ln p} = \xi$$

So ξ is the elasticity of subsititution between tradable goods and nontradable goods.

Exercise 9.2

Welfare when there is no interest rate change is simply given by

$$V = \frac{1}{1-\beta}\ln y^T \tag{11}$$

Suppose the interest rate r has an unanticipated decrease to $\underline{r}$ at time $t=0$ and reverts back to r afterwards. We will discuss two scenarios respectively: downward nominal wage rigidity and flexible wage.

- In the case of **downward nominal wage rigidity**, we have the following: for $t = 0$,

$$c_0^T = y^T \left(\frac{1}{1+\underline{r}} + \frac{r}{1+r} \right) \tag{12}$$

$$c_0^N = 1 \tag{13}$$

$$h_0 = 1$$

and for $t \geq 1$

$$c_t^T = y^T \left(\frac{1}{1+r} + \frac{r}{1+r} \frac{1+\underline{r}}{1+r} \right) \tag{14}$$

$$c_t^N = \left(\frac{1+\underline{r}}{1+r} \right)^{\alpha} \tag{15}$$

$$h_t = \frac{1+\underline{r}}{1+r}$$

Welfare under downward rigidity is given by

$$\begin{aligned} V_{rig} &= \sum_{t=0}^{\infty} \beta^t \left(\ln c_t^T + \ln c_t^N \right) \\ &= \ln y^T \left(\frac{1}{1+\underline{r}} + \frac{r}{1+r} \right) + \ln(1) + \frac{\beta}{1-\beta} \ln y^T \left(\frac{1}{1+r} + \frac{r}{1+r} \frac{1+\underline{r}}{1+r} \right) + \frac{\beta}{1-\beta} \ln \left(\frac{1+\underline{r}}{1+r} \right)^{\alpha} \\ &= \frac{1}{1-\beta} \ln y^T + \ln \left(\frac{1}{1+\underline{r}} + \frac{r}{1+r} \right) + \frac{\beta}{1-\beta} \ln \left(\frac{1}{1+r} + \frac{r}{1+r} \frac{1+\underline{r}}{1+r} \right) + \frac{\alpha\beta}{1-\beta} \ln \left(\frac{1+\underline{r}}{1+r} \right) \end{aligned}$$

The welfare change compared to the no interest rate shock result is

$$\Delta_{rig} = V_{rig} - V = \ln \left(\frac{1}{1+\underline{r}} + \frac{r}{1+r} \right) + \frac{\beta}{1-\beta} \ln \left(\frac{1}{1+r} + \frac{r}{1+r} \frac{1+\underline{r}}{1+r} \right) + \frac{\alpha\beta}{1-\beta} \ln \left(\frac{1+\underline{r}}{1+r} \right) \tag{16}$$

Take derivative w.r.t to $\underline{r}$

$$\begin{aligned} \frac{\partial \Delta_{rig}}{\partial \underline{r}} &= \left(1 - \frac{1+r}{1+\underline{r}} \right) \frac{1}{1+r+r(1+\underline{r})} + \frac{\alpha}{r} \frac{1}{1+\underline{r}} \\ &= \frac{1}{1+r+r(1+\underline{r})} \left[\frac{r(\underline{r}-r) + \alpha(1+r+r(1+\underline{r}))}{r(1+\underline{r})} \right] \end{aligned} \tag{17}$$

The condition for it to be positive is that

$$\alpha > \frac{r\left(r-\underline{r}\right)}{1+r+r\left(1+\underline{r}\right)} \tag{18}$$

Note that for $\underline{r}$ around a small neighborhood of r, the above can be easily satisfied. Therefore, it is going to be welfare decreeasing with the interest rate shock under downward nominal wage rigidity.

- While in the case of **flexible wage**. For time $t=0$,

$$c_0^T = y^T\left(\frac{1}{1+\underline{r}} + \frac{r}{1+r}\right) \tag{19}$$

$$c_0^N = 1 \tag{20}$$

$$h_0 = 1$$

and for $t \geq 1$

$$c_t^T = y^T\left(\frac{1}{1+r} + \frac{r}{1+r}\frac{1+\underline{r}}{1+r}\right) \tag{21}$$

$$c_t^N = 1 \tag{22}$$

$$h_t = 1$$

Welfare under flexible wage is given by

$$\begin{aligned} V_{fle} &= \sum_{t=0}^{\infty}\beta^t\left(\ln c_t^T + \ln c_t^N\right) \\ &= \ln y^T\left(\frac{1}{1+\underline{r}} + \frac{r}{1+r}\right) + \ln\left(1\right) + \frac{\beta}{1-\beta}\ln y^T\left(\frac{1}{1+r} + \frac{r}{1+r}\frac{1+\underline{r}}{1+r}\right) + \frac{\beta}{1-\beta}\ln 1 \\ &= \frac{1}{1-\beta}\ln y^T + \ln\left(\frac{1}{1+\underline{r}} + \frac{r}{1+r}\right) + \frac{\beta}{1-\beta}\ln\left(\frac{1}{1+r} + \frac{r}{1+r}\frac{1+\underline{r}}{1+r}\right) \end{aligned} \tag{23}$$

The welfare change compared to no interest rate shock result is

$$\Delta_{fle} = V_{fle} - V = \ln\left(\frac{1}{1+\underline{r}} + \frac{r}{1+r}\right) + \frac{\beta}{1-\beta}\ln\left(\frac{1}{1+r} + \frac{r}{1+r}\frac{1+\underline{r}}{1+r}\right) \tag{24}$$

Take derivative w.r.t to $\underline{r}$

$$
\begin{aligned}
\frac{\partial \Delta_{fle}}{\partial \underline{r}} &= \left(\frac{\beta r}{1-\beta} - \frac{1+r}{1+\underline{r}}\right)\frac{1}{1+r+r\left(1+\underline{r}\right)} \\
&= \left(1-\frac{1+r}{1+\underline{r}}\right)\frac{1}{1+r+r\left(1+\underline{r}\right)} \\
&< 0
\end{aligned} \tag{25}
$$

We conclude that when $\underline{r} < r$, the flexible wage setting is always welfare-improving.

- The intuition behind the above results is as follows. With downward nominal wage rigidity, when interest rate drops, individuals expand consumption in tradables and thus demand for non-tradables. This pushes up nominal wage, but when interest rate reverts back, they decrease their demand for non-tradables, but firms find that they cannot cut down nominal wages due to the downward nominal wage rigidity, and have to cut employment. The involuntary unemployment leads to a drop in production (so is consumption) of non-tradables. There are pecuniary externalities (or demand externalities) here. Individuals don't take into account the fact that their expansion in consumption upon good shock is putting the economy in a high nominal wage, which will later cause involuntary unemployment.

Exercise 9.3

Now assume the nominal wage is rigid in both directions. The equilibrium is characterized by the following conditions:

$$\frac{c_t^T}{c_t^N} = p_t$$

$$c_t^T(1+\beta)r_t = c_{t+1}^T$$

$$c_t^T + d_t = \frac{d_{t+1}}{1+r_t} + y_t^T$$

$$c_t^N = h_t^\alpha$$

$$p_t = \frac{w_t}{\alpha(h_t^d)^{\alpha-1}}$$

$$h_t \leq \bar{h} = 1$$

$$h_t = \min\{h_t^d, 1\}$$

Under full wage rigidity, for any t

$$w_t = \alpha y^T$$

Thanks to the property of separability of tradables and non-tradables, the path for tradables' consumption will not change: for $t = 0$

$$c_0^T = y^T \left(\frac{1}{1+\underline{r}} + \frac{r}{1+r} \right) \tag{26}$$

and for $t \geq 1$

$$c_t^T = y^T \left(\frac{1}{1+r} + \frac{r}{1+r} \frac{1+\underline{r}}{1+r} \right) \tag{27}$$

Therefore, for $t = 0$

$$h_0 = 1 \tag{28}$$

and for $t \geq 1$

$$h_t = \frac{1}{1+r} + \frac{r}{1+r} \frac{1+\underline{r}}{1+r} \tag{29}$$

Note that when $t \geq 1$ full wage rigidity case gives higher employment than the downward nominal wage case thus higher non-tradables consumption. Therefore, full nominal wage rigidity delivers higher welfare. The intuition is that downward wage rigidity makes the system adjustable when there's positive movement, while not fully adjustable in response to negative movements (since wage can't decrease as needed). Full wage rigidity helps restrict temporary wage increase upon a good shock, so when interest rate reverts back it won't be as bad as in the case when there was wage increase in previous period in terms of employment (thus consumption in nontradables).

Exercise 9.12

1

With downward nominal wage rigidity, the family of optimal exchange rate policy is

$$\epsilon_t \geq \gamma \frac{w_{t-1}}{\omega(c_t^T)}$$

where

$$\omega(c_t^T) = \frac{A_2(c_t^T, F_t(\bar{h}))}{A_1(c_t^T, F_t(\bar{h}))} F_t'(\bar{h})$$

It will deliver full employment.

With economy starting from the full-employment steady state, the family of full employment exchange rate policy under downward wage rigidity is written as

$$\epsilon_t \geq \gamma \frac{A_2(c_{t-1}^T, F_{t-1}(\bar{h}))}{A_1(c_{t-1}^T, F_{t-1}(\bar{h}))} \left(\frac{A_2(c_t^T, F_t(\bar{h}))}{A_1(c_t^T, F_t(\bar{h}))} \right)^{-1} \frac{F_{t-1}'(\bar{h})}{F_t'(\bar{h})} \tag{1}$$

Similar to downward nominal wage rigidity, under downward price rigidity, the family of optimal exchange rate policy is

$$\epsilon_t \geq \gamma_p \frac{p_{t-1}}{\rho(c_t^T)}$$

where

$$\rho(c_t^T) = \frac{A_2(c_t^T, F(\bar{h}))}{A_1(c_t^T, F(\bar{h}))}$$

With economy starting from the full-employment steady state, the family of full employment exchange rate policy under downward wage rigidity is written as

$$\epsilon_t \geq \gamma_p \frac{A_2(c_{t-1}^T, F_{t-1}(\bar{h}))}{A_1(c_{t-1}^T, F_{t-1}(\bar{h}))} \left(\frac{A_2(c_t^T, F_t(\bar{h}))}{A_1(c_t^T, F_t(\bar{h}))} \right)^{-1} \tag{2}$$

When $F_t(h) = F_{t-1}(h)$ and $\gamma = \gamma_p$, the two families of optimal exchange rate policies are the same.

2

Exchange rate which stabilizes nominal wage is

$$\begin{aligned}
\epsilon_t &= \frac{W_t A_1(c_t^T, F_t(\bar{h}))}{F_t'(\bar{h}) A_2(c_t^T, F_t(\bar{h}))} \left(\frac{W_{t-1} A_1(c_{t-1}^T, F_{t-1}(\bar{h}))}{F_{t-1}'(\bar{h}) A_2(c_{t-1}^T, F_{t-1}(\bar{h}))} \right)^{-1} \\
&= \frac{W_{-1} A_1(c_t^T, F_t(\bar{h}))}{F_t'(\bar{h}) A_2(c_t^T, F_t(\bar{h}))} \left(\frac{W_{-1} A_1(c_{t-1}^T, F_{t-1}(\bar{h}))}{F_{t-1}'(\bar{h}) A_2(c_{t-1}^T, F_{t-1}(\bar{h}))} \right)^{-1} \\
&= \frac{A_1(c_t^T, F_t(\bar{h}))}{A_2(c_t^T, F_t(\bar{h}))} \frac{A_2(c_{t-1}^T, F_{t-1}(\bar{h}))}{A_1(c_{t-1}^T, F_{t-1}(\bar{h}))} \frac{F_{t-1}'(\bar{h})}{F_t'(\bar{h})}
\end{aligned} \tag{3}$$

It is included in the family of optimal exchange rate policies.

Similarly, exchange rate that stabilizes nominal price is

$$\begin{aligned}
\epsilon_t &= \frac{P_t A_1(c_t^T, F_t(\bar{h}))}{A_2(c_t^T, F_t(\bar{h}))} \left(\frac{P_{t-1} A_1(c_{t-1}^T, F_{t-1}(\bar{h}))}{A_2(c_{t-1}^T, F_{t-1}(\bar{h}))} \right)^{-1} \\
&= \frac{P_{-1} A_1(c_t^T, F_t(\bar{h}))}{A_2(c_t^T, F_t(\bar{h}))} \left(\frac{P_{-1} A_1(c_{t-1}^T, F_{t-1}(\bar{h}))}{A_2(c_{t-1}^T, F_{t-1}(\bar{h}))} \right)^{-1} \\
&= \frac{A_1(c_t^T, F_t(\bar{h}))}{A_2(c_t^T, F_t(\bar{h}))} \frac{A_2(c_{t-1}^T, F_{t-1}(\bar{h}))}{A_1(c_{t-1}^T, F_{t-1}(\bar{h}))} \qquad (4)
\end{aligned}$$

It is included in the family of optimal exchange rate policies.

When $F_t(h) = F_{t-1}(h)$, stabilizing nominal wage is equivalent to stabilizing nominal price.

3

Suppose there is productivity shock, then $F_t(h) \neq F_{t-1}(h)$, from the above, we see that the two families of optimal exchange rate policies are not the same and stabilizing nominal wage is not equivalent to stabilizing nominal price.

Exercise 9.13

From the question, we obtain:

exogenouse tradable goods: $y_1^T = 10$ and $y_2^T = 13.2$
exogenouse domestic interest rate: $r = 0.1$
fixed exchange rate: $\xi_1 = \xi_2 = 1$
foreign price of tradable goods: $P_1^{T*} = P_2^{T*} = 1$
initial wage: $W_0 = 8.25$
labor supply: $\bar{h} = 1$
production function: $y_t^N = h_t^\alpha \quad \alpha = 0.75$

The optimization problem of households can be written as

$$\max_{C_1^T, C_1^N, C_2^T, C_2^N, d_2, h_1, h_2} \ln C_1^T + \ln C_1^N + \ln C_2^T + \ln C_2^N$$

subject to the budget constraints

$$\begin{aligned}
\xi_1 C_1^T + P_1^N C_1^N &= \xi_1 y_1^T + W_1 h_1 + \Phi_1 + \xi_1 \frac{d_2}{1+r} \\
\xi_2 C_2^T + P_2^N C_2^N + \xi_2 d_2 &= \xi_2 y_2^T + W_2 h_2 + \Phi_2
\end{aligned}$$

Denoting λ_1 and λ_2 the Lagrangian multiplier of the above two constraints, we get the first-order conditions:

$$\begin{aligned}
\frac{1}{C_1^T} &= \lambda_1 \\
\frac{1}{C_1^N} &= \lambda_1 p_1 \\
\frac{1}{C_2^T} &= \lambda_2 \\
\frac{1}{C_2^N} &= \lambda_2 p_2 \\
\frac{\lambda_1}{1+r} &= \lambda_2 \\
C_1^T + p_1 C_1^N &= y_1^T + w_1 h_1 + \Phi_1 + \frac{d_2}{1+r} \\
C_2^T + p_2 C_2^N + d_2 &= y_2^T + w_2 h_2 + \Phi_2
\end{aligned}$$

The optimization problem of firm can be written as

$$\max_{h_t^d} \Phi_t = P_t^N \left(h_t^d\right)^\alpha - W_t h_t^d$$

which gives us

$$W_t = P_t^N \alpha \left(h_t^d\right)^{\alpha-1}$$

Equivalently,

$$\begin{aligned}
w_1 &= p_1 \alpha \left(h_1^d\right)^{\alpha-1} \\
w_2 &= p_2 \alpha \left(h_2^d\right)^{\alpha-1}
\end{aligned}$$

Finally, market clearing conditions of nontradables:

$$C_t^N = (h_t)^\alpha$$

Equilibrium characterization

$$
\begin{aligned}
\frac{1}{C_1^T} &= \lambda_1 && (5)\\
\frac{1}{C_2^T} &= \lambda_2 && (6)\\
\lambda_1 &= (1+r)\lambda_2 && (7)\\
C_1^T &= y_1^T + \frac{d_2}{1+r} && (8)\\
C_2^T + d_2 &= y_2^T && (9)\\
\frac{C_1^T}{C_1^N} &= p_1 && (10)\\
\frac{C_2^T}{C_2^N} &= p_2 && (11)\\
C_1^N &= h_1^\alpha && (12)\\
C_2^N &= h_2^\alpha && (13)\\
w_1 &= p_1\alpha\left(h_1^d\right)^{\alpha-1} && (14)\\
w_2 &= p_2\alpha\left(h_2^d\right)^{\alpha-1} && (15)\\
h_1 &\leq \bar{h} = 1 && (16)\\
h_2 &\leq \bar{h} = 1 && (17)\\
h_1 &= \min\{h_1^d, 1\} && (18)\\
h_2 &= \min\{h_2^d, 1\} && (19)\\
w_1 &\geq w_0 && (20)\\
w_2 &\geq w_1 && (21)\\
(w_t - w_{t-1})\left(h_t - \bar{h}\right) &= 0 \quad for\ t = 1,2 && (22)
\end{aligned}
$$

(1)

Equilibrium levels of consumption of tradables can be solved using the first five conditions, or equivalently

$$
\begin{aligned}
C_2^T &= (1+r)C_1^T\\
C_1^T &= y_1^T + \frac{d_2}{1+r}\\
C_2^T + d_2 &= y_2^T
\end{aligned}
$$

which gives

$$
\begin{aligned}
d_2 &= 1.1\\
C_1^T &= 11 && (23)\\
C_2^T &= 12.1 && (24)
\end{aligned}
$$

and trade balance

$$
\begin{aligned}
tb_1 &= y_1 - C_1^T = -1 && (25)\\
tb_2 &= y_2 - C_2^T = 1.1 && (26)
\end{aligned}
$$

(2)

Consider the rest of F.O.C.s and we can derive

$$
\begin{aligned}
\frac{C_1^T}{h_1^\alpha} &= p_1\\
\frac{C_2^T}{h_2^\alpha} &= p_2\\
w_1 &= p_1 \alpha \left(h_1^d\right)^{\alpha-1}\\
w_2 &= p_2 \alpha \left(h_2^d\right)^{\alpha-1}\\
h_1 &\leq \bar{h} = 1\\
h_2 &\leq \bar{h} = 1\\
h_1 &= \min\{h_1^d, 1\}\\
h_2 &= \min\{h_2^d, 1\}\\
w_1 &\geq w_0\\
w_2 &\geq w_1\\
(w_t - w_{t-1})\left(h_t - \bar{h}\right) &= 0 \quad for\ t = 1, 2
\end{aligned}
$$

For period 1, assume $h_1 = h_1^d < 1 = \bar{h}$, then

$$
\begin{aligned}
w_1 &= \frac{C_1^T}{h_1^\alpha} \alpha \left(h_1\right)^{\alpha-1}\\
&= \frac{C_1^T}{h_1} \alpha\\
&> \alpha C_1^T = 8.25 = w_0
\end{aligned}
$$

which leads to a contradition against the last slackness condition.

Hence it must be true that

$$
\begin{aligned}
h_1 &= 1 && (27)\\
C_1^N &= 1 && (28)\\
p_1 &= 11 && (29)\\
w_1 &= 8.25 && (30)
\end{aligned}
$$

For period 2, assume $h_2 = h_2^d < 1$, then

$$
\begin{aligned}
w_2 &= \frac{C_2^T}{h_2^\alpha}\alpha\left(h_2\right)^{\alpha-1} \\
&= \frac{C_2^T}{h_2}\alpha \\
&> \alpha C_2^T = 9.075 > w_0
\end{aligned}
$$

which aso leads to a contradition against the slackness condition.

Therefore, it must be true that

$$
\begin{aligned}
h_2 &= 1 && (31) \\
C_2^N &= 1 && (32) \\
p_2 &= 12.1 && (33) \\
w_2 &= 9.075 && (34)
\end{aligned}
$$

(3)

Now study the case that the country interest rate increases to $r = 0.32$.

First, tradable goods consumption and trade balance in two periods are given by

$$
\begin{aligned}
d_2 &= 0 \\
C_1^T &= 10 && (35) \\
C_2^T &= 13.2 && (36) \\
tb_1 &= 0 && (37) \\
tb_2 &= 0 && (38)
\end{aligned}
$$

It can be easily verified that the economy cannot be at full employment in period 1, i.e. $h_1^d = h_1 < 1$. Then we have

$$
\begin{aligned}
w_1 &= w_0 = 8.25 && (39) \\
h_1 &= \alpha\frac{C_1^T}{w_1} = \frac{10}{11} && (40) \\
C_1^N &= \left(\frac{10}{11}\right)^{0.75} \approx 0.93 && (41) \\
p_1 &= \frac{C_1^T}{h_1^\alpha} = 10\left(\frac{10}{11}\right)^{-0.75} \approx 10.74 && (42)
\end{aligned}
$$

It can also be easily verified that the economy should be at full employment in period 2, i.e. $h_2^d = h_2 = 1$. Then we have

$$h_2 = 1 \tag{43}$$
$$C_2^N = 1 \tag{44}$$
$$p_2 = 13.2 \tag{45}$$
$$w_2 = 9.9 \tag{46}$$

In summary, downward wage rigidity is binding in period 1 but it is slack in period 2.

The intuition is as follows. Thanks to the separability of utility function, we can easily show that the dynamics of tradable goods become steeper after an increase in interest rate. In other words, households have more incentives to save by reducing tradables consumption in period 1. The less tradables consumption means a less demand for nontradables in period one, which forces firm to cut employmnet, simply because they cannot cut nominal wage as restricted by downward nominal wage rigidity. But in period 2, due to the increase in tradable good consumption, the demand for non-tradables increases, which drives up market equilibrium wage and makes the downward wage rigidity not binding any more. Thus, the economy reaches full employment in period 2.

(4)

Consider a devaluation to achieve full employment in both periods. Rewrite all the relevant conditions below after imposing $h_1 = h_2 = 1$

$$\begin{aligned}
C_1^T &= p_1 \\
C_2^T &= p_2 \\
w_1 &= p_1\alpha \\
w_2 &= p_2\alpha \\
W_2 &\geq W_1 \geq W_0
\end{aligned}$$

Therefore, we have the solution that

$$p_1 = 10 \tag{47}$$
$$p_2 = 13.2 \tag{48}$$
$$w_1 = 7.5 \tag{49}$$
$$w_2 = 9.9 \tag{50}$$

Note that $w = \frac{W}{\xi}$ by definition, hence we have

$$\frac{\xi_1}{\xi_0} = \frac{w_0}{w_1}\frac{W_1}{W_0} \geq \frac{w_0}{w_1} = 1.1 \tag{51}$$
$$\frac{\xi_2}{\xi_1} = \frac{w_1}{w_2}\frac{W_2}{W_1} \geq \frac{w_1}{w_2} \approx 0.76 \tag{52}$$

In sum, the minimum gross devaluation rate in period 1 is 1.1, while the minimum gross devaluation in period 2 is about 0.76.

(5)

The less constrained Ramsey problem is written as follows

$$\max \ln C_1^T + \ln h_1^\alpha + \ln C_2^T + \ln h_2^\alpha$$

subject to

$$
\begin{aligned}
C_1^T &= y_1^T + \frac{d_2}{1+r} && (53)\\
C_2^T + d_2 &= y_2^T && (54)\\
w_1 &= \alpha\frac{C_1^T}{h_1} && (55)\\
w_2 &= \alpha\frac{C_2^T}{h_2} && (56)\\
W_2 \geq W_1 &\geq W_0 && (57)\\
h_1, h_2 &\leq \bar{h} && (58)
\end{aligned}
$$

The solution to the less constrained problem is given by (check several possibilities and compare: 1). $W_1 = W_0$, $W_2 = W_1$; 2). $W_1 = W_0$, $W_2 > W_1$; 3). $W_1 > W_0$, $W_2 = W_1$; 4). $W_1 > W_0$, $W_2 > W_1$)

$$C_1^T = C_2^T = 11.38$$

$$h_1 = h_2 = 1$$

$$W_1 = W_2 = 8.535 > W_0$$

Now we go back to Ramsey problem indexed by τ_1. Using the following condition we can solve for optimal τ_1

$$\frac{1-\tau_1}{1+r}\frac{1}{C_1^T} = \frac{1}{C_2^T}$$

$$\rightarrow \tau_1 = -0.32 \qquad (59)$$

It is left to check that all the conditions stated in the original Ramsey problem are satisfied by the solution above, which is trivial. In summary, if $\tau_1 = -0.32$, which is a subsidy, the economy is Ramsey optimal and

$$\begin{aligned} h_1 = h_2 &= 1 & (60)\\ C_1^N = C_2^N &= 1 & (61)\\ W_1 = W_2 &= 8.535 & (62)\\ C_1^T = C_2^T &= 11.38 & (63) \end{aligned}$$

Excercise 9.14

1

The optimality problem of the representative household is

$$\max_{c^T, c^N, d_{+1}} E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, c_t^N))$$

subject to the budget constraint

$$c_t^T + p_t c_t^N + d_t = y_t^T + w_t h_t + \phi_t + \frac{d_{t+1}}{1+r_t}$$

The first-order conditions are shown below

$$p_t = \frac{A_2(c_t^T, c_t^N)}{A_1(c_t^T, c_t^N)} \rightarrow p_t = \frac{1-a}{a} \left(\frac{c_t^N}{c_t^T} \right)^{-\frac{1}{\xi}}$$

$$\lambda_t = U'(A_t) A_1(c_t^T, c_t^N) \rightarrow \lambda_t = a A_t^{-\sigma} A_t^{\frac{1}{\xi}} (c_t^T)^{-\frac{1}{\xi}}$$

$$\frac{\lambda_t}{1+r_t} = \beta E_t \lambda_{t+1}$$

Given that $\xi = \frac{1}{\sigma}$ and $\beta(1+r) = 1$ we have

$$\begin{aligned} \lambda_t &= a (c_t^T)^{-\frac{1}{\xi}} \\ \lambda_t &= E_t \lambda_{t+1} \end{aligned}$$

Therefore, the equilibrium process of tradables is characterized by

$$\left(c_t^T\right)^{-\frac{1}{\xi}} = E_t \left(c_{t+1}^T\right)^{-\frac{1}{\xi}} \tag{64}$$

Since $\xi = \frac{1}{\sigma}$, the above problem is in fact separable and because there is no uncertainty in tradable output and interest rate, one can drop the expectation operator. In addition, we use the assumption that $d_0 = 0$, the tradable consumption can then be shown to be

$$c_t^T = y^T \tag{65}$$

2

$$\begin{aligned}
\epsilon_t = \frac{\xi_t}{\xi_{t-1}} &\geq \gamma \left[\frac{A_2(c_{t-1}^T, \bar{h})}{A_1(c_{t-1}^T, \bar{h})} F'_{t-1}(\bar{h}) \right] \left[\frac{A_2(c_t^T, \bar{h})}{A_1(c_t^T, \bar{h})} F'_{t-1}(\bar{h}) \right]^{-1} \\
&= \gamma \left(e^{z_{t-1} - z_t} \right)^{1-\frac{1}{\xi}} \qquad (66)
\end{aligned}$$

3

Demand curve:

$$p_t = \frac{1-a}{a} \left(\frac{c_t^N}{c_t^T} \right)^{-\frac{1}{\xi}} = \frac{1-a}{a} \left(\frac{y^T}{e^{z_t} h_t^\alpha} \right)^{\frac{1}{\xi}} \qquad (67)$$

Supply curve:

$$p_t = \frac{w_t}{\alpha e^{z_t}} h_t^{1-\alpha} = \frac{W_t/\xi_t}{\alpha e^{z_t}} h_t^{1-\alpha} \qquad (68)$$

Suppose the economy is at full employment at $t = -1$ and an increase in productivity occurs, i.e. $z_0 > z_{-1}$, demand curve and supply curve both shift downwards. Note that $(e^{z_t})^{-\frac{1}{\xi}} < (e^{z_t})^{-1}$ as $\xi < 1$. Under peg, the shift in suply curve is smaller than the shift in demand curve, leading to involuntary unemployment. Under the optimal exchange rate policy, a devaluation should decrease the domestic real wage such that the full employment is obtained again.

4

Following the result in part 2,

$$\begin{aligned}
\epsilon_t &= \gamma \left(e^{z_{t-1} - z_t} \right)^{1-\frac{1}{\xi}} \\
\ln \epsilon_t &= \left(1 - \frac{1}{\xi} \right) (z_{t-1} - z_t) + \ln \gamma \qquad (69)
\end{aligned}$$

The correlation between devaluation rate and productivity growth rate is

$$corr = \frac{E[(\ln \epsilon_t - E \ln \epsilon)(z_t - z_{t-1})]}{\sigma_{\ln \epsilon_t} \sigma_{z_t - z_{t-1}}} = 1 > 0 \qquad (70)$$

As analyzed in part 3, when technology gets increased, household tends to supply less labor while firm wants to hire more labor. The net effect is a decline in labor in equilibrium. In order to induce full employment again, the economy needs devaluation and further reduction of real wage so as to encourage firm's hiring of labor. Therefore, the correlation is positive.

5

If $\xi > 1$, then $\frac{1}{\xi} - 1 < 0$. In this case, the downward shift in the supply curve exceeds that of demand curve, so that there will be more labor demand from production side. Recall that it has been assumed that the domestic currency will remain as strong as possible. Therefore, we could implement an appreciation to increase the real wage and still achieve full employment. In other words, the correlation is negative.

$$corr = \frac{E[(\ln \epsilon_t - E \ln \epsilon)(z_t - z_{t-1})]}{\sigma_{\ln \epsilon_t} \sigma_{z_t - z_{t-1}}} = -1 < 0 \tag{71}$$

Exercise 10.1

Household Problem:

$$\max_{c_t^T, c_t^N, d_{t+1}, h_t^v, l_t^v} \sum_{t=0}^{\infty} \beta^t U(C_t, l_t) = \sum_{t=0}^{\infty} \beta^t \left[\frac{\left[A(c_t^T, c_t^N)\right]^{1-\sigma} - 1}{1-\sigma} + \psi \frac{(l_t^v)^{1-\theta} - 1}{1-\theta} \right] \tag{1}$$

subject to

$$\begin{aligned} c_t^T + p_t c_t^N + d_t &= (1-\tau_t)(y_t^T + w_t h_t^v + \phi_t) + \frac{d_{t+1}}{1+r_t} && (2) \\ h_t^v + l_t^v &= \bar{h} && (3) \\ d_{t+1} &\leq \bar{d} && (4) \end{aligned}$$

where $A(c^T, c^N)$ is the consumption bundle and τ_t is income tax rate.

Note that given the property of utility function, it is always true that voluntary leisure is positive and so

$$h_t^v < \bar{h} \tag{5}$$

Firm Problem:

$$\max_{h_t} \phi_t = p_t F(h_t) - (1-s_t) w_t h_t \tag{6}$$

Labor Market:

Suppose firms cannot force workers to work, i.e.

$$h_t \leq h_t^v \tag{7}$$

and nominal wage is downward rigid, i.e.

$$W_t \geq \gamma W_{t-1} \rightarrow w_t \geq \gamma w_{t-1} \tag{8}$$

$$(h_t^v - h_t)(w_t - \gamma w_{t-1}) = 0 \tag{9}$$

Equation (8) says if firm's demand for labor is strictly lower than the supply, the wage rigidity is binding.

Nontradables Market Clearing:

$$F(h_t) = c_t^N \tag{10}$$

Government Budget:

$$s_t w_t h_t = \tau_t (y_t^T + w_t h_t^v + \phi_t) \tag{11}$$

Competitive Equilibrium:

We are now ready to write all the optimality conditions below

$$
\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (12)\\
p_t &= \frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} && (13)\\
p_t &= (1-s_t)\frac{w_t}{F'(h_t)} && (14)\\
\psi\left(\bar{h} - h_t^v\right)^{-\theta} &= \lambda_t(1-\tau_t)w_t && (15)\\
h_t &\leq h_t^v && (16)\\
w_t &\geq \gamma w_{t-1} && (17)\\
d_{t+1} &\leq \bar{d} && (18)\\
\lambda_t &= U_1\left(A(c_t^T, F(h_t)\right) A_1\left(c_t^T, F(h_t)\right) && (19)\\
\frac{\lambda_t}{1+r_t} &= \beta E_t \lambda_{t+1} + \mu_t && (20)\\
\mu_t &\geq 0 && (21)\\
\mu_t(d_{t+1} - \bar{d}) &= 0 && (22)\\
(h_t^v - h_t)(w_t - \gamma w_{t-1}) &= 0 && (23)\\
s_t w_t h_t &= \tau_t(y_t^T + w_t h_t + \phi_t) && (24)
\end{aligned}
$$

Pareto optimal allocation

$$
\max_{c_t^T, l_t, h_t, d_{t+1}} \sum_{t=0}^{\infty} \beta^t U(C_t, l_t) = \sum_{t=0}^{\infty} \beta^t \left[\frac{\left[A(c_t^T, F(h_t))\right]^{1-\sigma} - 1}{1-\sigma} + \psi \frac{(l_t)^{1-\theta} - 1}{1-\theta} \right] \tag{25}
$$

subject to

$$
\begin{aligned}
l_t + h_t &= \bar{h} && (26)\\
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (27)\\
d_{t+1} &\leq \bar{d} && (28)
\end{aligned}
$$

The optimality condition to planner's problem is given by

$$
\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (29)\\
\psi\left(\bar{h} - h_t\right)^{-\theta} &= U_1\left(A(c_t^T, F(h_t)\right) A_2\left(c_t^T, F(h_t)\right) F'(h_t) && (30)\\
d_{t+1} &\leq \bar{d} && (31)\\
\lambda_t &= U_1\left(A(c_t^T, F(h_t)\right) A_1\left(c_t^T, F(h_t)\right) && (32)\\
\frac{\lambda_t}{1+r_t} &= \beta E_t \lambda_{t+1} + \mu_t && (33)\\
\mu_t &\geq 0 && (34)\\
\mu_t(d_{t+1} - \bar{d}) &= 0 && (35)
\end{aligned}
$$

Consider a policy maker who wishes to set the income tax and wage subsidy to achieve Pareto optimality. The optimization problem faced by this policy maker is to maximize (1) subject to (12) - (24). To see that the Ramsey-optimal policy supports the Pareto optimal allocation, we transform the optimality conditions under competitive equilibrium into the following:

$$
\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (36)\\
p_t &= \frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} && (37)\\
p_t &= (1-s_t)\frac{w_t}{F'(h_t)} && (38)\\
\psi\left(\bar{h} - h_t^v\right)^{-\theta} &= U_1\left(A(c_t^T, F(h_t)\right) A_2\left(c_t^T, F(h_t)\right) \frac{1-\tau_t}{1-s_t} F'(h_t) && (39)\\
h_t &\leq h_t^v && (40)\\
w_t &\geq \gamma w_{t-1} && (41)\\
d_{t+1} &\leq \bar{d} && (42)\\
\lambda_t &= U_1\left(A(c_t^T, F(h_t)\right) A_1\left(c_t^T, F(h_t)\right) && (43)\\
\frac{\lambda_t}{1+r_t} &= \beta E_t \lambda_{t+1} + \mu_t && (44)\\
\mu_t &\geq 0 && (45)\\
\mu_t(d_{t+1} - \bar{d}) &= 0 && (46)\\
(h_t^v - h_t)(w_t - \gamma w_{t-1}) &= 0 && (47)\\
s_t w_t h_t &= \tau_t(y_t^T + w_t h_t + \phi_t) && (48)
\end{aligned}
$$

Equations (36), (42)-(46) are identical to (29), (31)-(35). Now set p_t to satisfy (37), $w_t = \gamma w_{t-1}$, and then s_t to satisfy (38), τ_t to satisfy (48). Note $\tau_t < s_t$. Therefore, from (39), we know that $h_t^v > h_t$, which satisfies (4). The above procedures replicate Pareto optimal allocation.

Exercise 10.2

The policy tool is the combination of lump-sum taxes and external debt. Exchange rate is fixed.
Household Problem:

$$\max \sum_{t=0}^{\infty} \beta^t U\left(A(c_t^T, c_t^N)\right) \tag{49}$$

subject to budget constraint

$$c_t^T + p_t c_t^N \quad = \quad y_t^T + w_t h_t + \phi_t + T_t \tag{50}$$

where T_t is the transfer denominated in foreign currency.

Since household is restricted from borrowing, its problem is equivalent to maximize utility period by period.
Firm Problem:

$$\max_{h_t} \phi_t = p_t F(h_t) - w_t h_t \tag{51}$$

Labor Market:

$$h_t \leq \bar{h} \tag{52}$$

$$w_t \geq \gamma w_{t-1} \tag{53}$$

$$\left(\bar{h} - h_t\right)(w_t - \gamma w_{t-1}) = 0 \tag{54}$$

Nontradables Market:

$$F(h_t) = c_t^N \tag{55}$$

Government Budget:

$$\frac{d_{t+1}}{1+r_t} \quad = \quad d_t + T_t \tag{56}$$

$$d_{t+1} \quad \leq \quad \bar{d} \tag{57}$$

where d_{t+1} is the external debt position demoninated in foreign currency.
Competitive Equilibrium

We are now ready to write all the optimality conditions below

$$
\begin{aligned}
c_t^T &= y_t^T + T_t && (58)\\
p_t &= \frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} && (59)\\
p_t &= \frac{w_t}{F'(h_t)} && (60)\\
w_t &\geq \gamma w_{t-1} && (61)\\
h_t &\leq \bar{h} && (62)\\
\mu &\geq 0 && (63)\\
d_{t+1} &\leq \bar{d} && (64)\\
\mu_t\left(d_{t+1} - \bar{d}\right) &= 0 && (65)\\
\left(\bar{h} - h_t\right)(w_t - \gamma w_{t-1}) &= 0 && (66)\\
\frac{d_{t+1}}{1 + r_t} &= d_t + T_t && (67)
\end{aligned}
$$

Consider a policy maker who wishes to set transfer and external debt to obtain Pareto optimality. After simple rearrangement, this policy maker is to maximize household utility subject to (58)-(67).

Eqm(1):

$$
\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1 + r_t} && (68)\\
\frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} F'(h_t) &= w_t && (69)\\
w_t &\geq \gamma w_{t-1} && (70)\\
h_t &\leq \bar{h} && (71)\\
\mu &\geq 0 && (72)\\
d_{t+1} &\leq \bar{d} && (73)\\
\mu_t\left(d_{t+1} - \bar{d}\right) &= 0 && (74)\\
\left(\bar{h} - h_t\right)(w_t - \gamma w_{t-1}) &= 0 && (75)
\end{aligned}
$$

We already know that Ramsey optimal capital control problem described in section 10.3 is characterized by the following conditions.

Eqm(2):

$$
\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (76)\\
\frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} F'(h_t) &= w_t && (77)\\
w_t &\geq \gamma w_{t-1} && (78)\\
h_t &\leq \bar{h} && (79)\\
d_{t+1} &\leq \bar{d} && (80)\\
\left(\bar{h} - h_t\right)(w_t - \gamma w_{t-1}) &= 0 && (81)\\
\lambda_t &= U'\left(A(c_t^T, F(h_t))\right) A_1(c_t^T, F(h_t)) && (82)\\
\frac{\lambda_t(1-\tau_t^d)}{1+r_t} &= \beta E_t \lambda_{t+1} + \mu_t && (83)\\
\mu_t &\geq 0 && (84)\\
\mu_t \left(d_{t+1} - \bar{d}\right) &= 0 && (85)
\end{aligned}
$$

The solutions to Eqm(1) and Eqm(2) are also the solution to the less-constrained problem of Ramsey capital control policy studied in section 10.3 (note we can drop the complementary slackness condition by contradiction). Thus, the equilibrium real allocation in this setting is identical to the one obtained in section 10.3 under Ramsey optimal control policy.

Excercise 10.3

Consider pre-determined consumption tax as a policy tool. Again exchange rate is fixed.

Household Problem:

$$
\max_{c_t^T, c_t^N, d_{t+1}} \sum_{t=0}^{\infty} \beta^t U\left(A(c_t^T, c_t^N)\right) \tag{86}
$$

subject to budget constraint

$$
\begin{aligned}
(1+\tau_t^c)\left(c_t^T + p_t c_t^N\right) + d_t &= y_t^T + w_t h_t + \phi_t + \frac{d_{t+1}}{1+r_t} + T_t && (87)\\
d_{t+1} &\leq \bar{d} && (88)
\end{aligned}
$$

Firm Problem:

$$
\max_{h_t} \phi_t = p_t F(h_t) - w_t h_t \tag{89}
$$

Labor Market: Downward wage rigidity must be satisfied and labor supply cannot exceed the upper bound

$$h_t \leq \bar{h} \tag{90}$$

$$w_t \geq \gamma w_{t-1} \tag{91}$$

$$\left(\bar{h} - h_t\right)\left(w_t - \gamma w_{t-1}\right) = 0 \tag{92}$$

Nontradables Market:

$$F(h_t) = c_t^N \tag{93}$$

Government Budget:

$$T_t \quad = \quad \tau_t^c \left(c_t^T + p_t c_t^N\right) \tag{94}$$

where T_t is the transfer to household denominated in foreign currency.
Competitive Equilibrium:
We are now ready to write and simplify all the optimality conditions below. It is worth noticing that Ramsey allocation calls for $\mu_t = 0$ and $d_{t+1} < \bar{d}$, hence we can impose this fact directly. We also write the multiplier λ_t as a function nontradables and labor.

Eqm(1):

$$\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (95)\\
\frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} F'(h_t) &= w_t && (96)\\
h_t &\leq \bar{h} && (97)\\
w_t &\geq \gamma w_{t-1} && (98)\\
d_{t+1} &\leq \bar{d} && (99)\\
\lambda_t &= U'(A(c_t^T, F(h_t)))A_1(c_t^T, F(h_t)) && (100)\\
\frac{1}{1+r_t}\frac{1+\tau_{t+1}^c}{1+\tau_t^c}\lambda_t &= E_t \beta \lambda_{t+1} + \mu_t && (101)\\
\mu_t &\geq 0 && (102)\\
\mu_t(d_{t+1} - \bar{d}) &= 0 && (103)\\
\left(\bar{h} - h_t\right)\left(w_t - \gamma w_{t-1}\right) &= 0 && (104)\\
& && (105)
\end{aligned}$$

We already know that Ramsey optimal capital control problem described in section 10.3 is characterized by the following conditions

Eqm(2):

$$
\begin{aligned}
c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t} && (106)\\
\frac{A_2\left(c_t^T, F(h_t)\right)}{A_1\left(c_t^T, F(h_t)\right)} F'(h_t) &= w_t && (107)\\
h_t &\leq \bar{h} && (108)\\
w_t &\geq \gamma w_{t-1} && (109)\\
d_{t+1} &\leq \bar{d} && (110)\\
\lambda_t &= U'(A(c_t^T, F(h_t)))A_1(c_t^T, F(h_t)) && (111)\\
\frac{1}{1+r_t}(1-\tau_t^d)\lambda_t &= E_t \beta \lambda_{t+1} + \mu_t && (112)\\
\mu_t &\geq 0 && (113)\\
\mu_t(d_{t+1} - \bar{d}) &= 0 && (114)\\
\left(\bar{h} - h_t\right)(w_t - \gamma w_{t-1}) &= 0 && (115)\\
& && (116)
\end{aligned}
$$

Comparing the optimal conditions under two equilibria, it is obvious that we can set

$$
\begin{aligned}
1 - \tau_t^d &= \frac{1+\tau_{t+1}^c}{1+\tau_t^c} \rightarrow \\
\tau_{t+1}^c &= \left(1 - \tau_t^d\right)(1 + \tau_t^c) - 1 && (117)
\end{aligned}
$$

By comparing the two sets of equilibrium conditons, the Ramsey consumption tax above is able to replicate solution to optimal capital control problem.

Exercise 13.3

Now the maximization problem is to

$$\max_{\{c(\epsilon),d(\epsilon)\}} \int_{\epsilon_L}^{\epsilon_H} u(c(\epsilon))\pi(\epsilon)d\epsilon$$

subject to

$$c(\epsilon) = \bar{y} + \epsilon - d(\epsilon)$$

$$\int d(\epsilon)\pi(\epsilon)d\epsilon = 0$$

$$d(\epsilon) \leq min\{d(\epsilon), k\}$$

The last constraint is equivalent to

$$d(\epsilon) \leq k$$

Since the problem is exactly the same as in section 13.3.3, the analysis there goes through.

Exercise 13.4

We consider the case that $0 < \alpha(\bar{y} + \epsilon^H) < \epsilon^H$, which means $\frac{\alpha}{1-\alpha}\bar{y} < \epsilon_H$.

The optimal contract is to

$$\max_{\{c(\epsilon),d(\epsilon)\}} \int_{\epsilon_L}^{\epsilon_H} u(c(\epsilon))\pi(\epsilon)d\epsilon$$

subject to

$$c(\epsilon) = \bar{y} + \epsilon - d(\epsilon)$$

$$\int d(\epsilon)\pi(\epsilon)d\epsilon = 0$$

$$d(\epsilon) \leq \alpha(\bar{y} + \epsilon)$$

The Lagrangian associated with the above problem can be written as:

$$L = \int_{\epsilon^L}^{\epsilon^H} \{u(\bar{y} + \epsilon - d(\epsilon)) + \lambda d(\epsilon) + \gamma(\epsilon)[\alpha(\bar{y} + \epsilon) - d(\epsilon)]\}\pi(\epsilon)d\epsilon$$

when the incentive-compatibility constraint does not bind, the argument still holds as before, we would have $\gamma(\epsilon)$ vanishes in the first order condition and that $d(\epsilon)$ takes the form of $d(\epsilon) = \bar{d} + \epsilon$ and consumption is a constant $c(\epsilon) = \bar{y} - \bar{d}$.

Then we can prove that if the IC constraint is binding for some ϵ', then it is binding for all $\epsilon'' > \epsilon'$. We prove by contradiction. Let $\epsilon'' \in (\epsilon', \epsilon^H]$. Suppose, the IC constraint is binding for ϵ', so that $d(\epsilon') = \alpha(\bar{y} + \epsilon')$, but is not binding for ϵ'', so that $d(\epsilon'') < \alpha(\bar{y} + \epsilon'')$. Then

$$c(\epsilon'') = \bar{y} + \epsilon'' - d(\epsilon'') > (1 - \alpha)(\bar{y} + \epsilon'') > (1 - \alpha)(\bar{y} + \epsilon') = c(\epsilon')$$

We obtain $u'(c(\epsilon'')) < u'(c(\epsilon'))$ and thus $\gamma(\epsilon'') > \gamma(\epsilon') \geq 0$. But since IC constraint is not binding for ϵ'', we must have $\gamma(\epsilon'') = 0$, which is a contradiction. So there is a cutoff $\bar{\epsilon}$ such that

$$d(\epsilon) = \begin{cases} \bar{d} + \epsilon, \epsilon < \bar{\epsilon} \\ \alpha(\bar{y} + \epsilon), \epsilon > \bar{\epsilon} \end{cases}$$

We will further show that

1. $\bar{d} > 0$
2. $d(\bar{\epsilon}) = \bar{d} + \bar{\epsilon} = \alpha(\bar{y} + \bar{\epsilon})$

If the contract is indeed continuous, which means the second condition is satisfied, then

$$\begin{aligned} 0 &= \int_{\epsilon^L}^{\bar{\epsilon}} (\bar{d} + \epsilon)\pi(\epsilon) d\epsilon + \int_{\bar{\epsilon}}^{\epsilon^H} \alpha(\bar{y} + \epsilon)\pi(\epsilon) d\epsilon \\ &= \int_{\epsilon^L}^{\bar{\epsilon}} (\bar{d} + \epsilon)\pi(\epsilon) d\epsilon + \int_{\bar{\epsilon}}^{\epsilon^H} \alpha(\bar{y} + \bar{\epsilon})\pi(\epsilon) d\epsilon + \int_{\bar{\epsilon}}^{\epsilon^H} \alpha(\epsilon - \bar{\epsilon})\pi(\epsilon) d\epsilon \\ &= \bar{d} - (1 - \alpha) \int_{\bar{\epsilon}}^{\epsilon^H} (\epsilon - \bar{\epsilon})\pi(\epsilon) d\epsilon \end{aligned}$$

This gives $\bar{d} > 0$, then we move on to show continuity.

The optimal contract sets $\bar{\epsilon}$ and $\bar{d}$ to maximize

$$\int_{\epsilon^L}^{\bar{\epsilon}} u(\bar{y} - \bar{d})\pi(\epsilon) d\epsilon + \int_{\bar{\epsilon}}^{\epsilon^H} u((1 - \alpha)(\bar{y} + \epsilon))\pi(\epsilon) d\epsilon$$

subject to

$$\int_{\epsilon^L}^{\bar{\epsilon}} (\bar{d} + \epsilon)\pi(\epsilon) d\epsilon + \int_{\bar{\epsilon}}^{\epsilon^H} \alpha(\bar{y} + \epsilon)\pi(\epsilon) d\epsilon = 0$$

Differentiate the objective function with respect to $\bar{\epsilon}$ and $\bar{d}$ and set the result equal to 0 to get

$$-u'(\bar{y} - \bar{d})F(\bar{\epsilon})d\bar{d} + u(\bar{y} - \bar{d})\pi(\bar{\epsilon})d\bar{\epsilon} - u((1 - \alpha)(\bar{y} + \bar{\epsilon}))\pi(\bar{\epsilon})d\bar{\epsilon} = 0$$

Differentiate the constraint to arrive at

$$F(\bar{\epsilon})d\bar{d} + (\bar{d} + \bar{\epsilon})\pi(\bar{\epsilon})d\bar{\epsilon} - \alpha(\bar{y} + \bar{\epsilon})\pi(\bar{\epsilon})d\bar{\epsilon} = 0$$

Combining the two gives

$$u'(\bar{y} - \bar{d})(\bar{d} + \bar{\epsilon} - \alpha(\bar{y} + \bar{\epsilon})) + [u(\bar{y} - \bar{d}) - u((1 - \alpha)(\bar{y} + \bar{\epsilon}))] = 0$$

Apparently $\bar{d} + \bar{\epsilon} = \alpha(\bar{y} + \bar{\epsilon})$ satisfies the above condition.

Finally, we check the participation constraint by replacing $\bar{d}$ by $\alpha(\bar{y}+\bar{\epsilon})-\bar{\epsilon}$

$$0=\alpha(\bar{y}+\bar{\epsilon})-\bar{\epsilon}-(1-\alpha)\int_{\bar{\epsilon}}^{\epsilon^H}(\epsilon-\bar{\epsilon})\pi(\epsilon)d\epsilon$$

$$\frac{\alpha}{1-\alpha}\bar{y}=\int_{\epsilon^L}^{\bar{\epsilon}}(\bar{\epsilon}-\epsilon)\pi(\epsilon)d\epsilon$$

The left hand side ranges from 0 to ϵ^H when $\bar{\epsilon}$ goes from ϵ^L to ϵ^H. Note we have restricted $\frac{\alpha}{1-\alpha}\bar{y}<\epsilon^H$. By continuity, there is an $\bar{\epsilon}$ that equates left hand side with right hand side.

Exercise 13.5

The incentive compatibility constraint is

$$u(\bar{y}+\epsilon)-m\leq u(\bar{y}+\epsilon-d(\epsilon))$$

The maximization problem is to

$$\max_{c(\epsilon),d(\epsilon)}\int_{\epsilon_L}^{\epsilon_H}u(c)\pi(\epsilon)d\epsilon$$

subject to

$$c(\epsilon)=\bar{y}+\epsilon-d(\epsilon)$$

$$\int d(\epsilon)\pi(\epsilon)d\epsilon=0$$

$$u(\bar{y}+\epsilon)-m\leq u(\bar{y}+\epsilon-d(\epsilon))$$

The Lagrangian associated with the above problem can be written as:

$$L=\int_{\epsilon^L}^{\epsilon^H}\{u(\bar{y}+\epsilon-d(\epsilon))+\lambda d(\epsilon)+\gamma(\epsilon)[u(\bar{y}+\epsilon-d(\epsilon))+m-u(\bar{y}+\epsilon)]\}\pi(\epsilon)d\epsilon$$

Still, it is easy to check when incentive compatibility constraint does not bind, we would have $\gamma(\epsilon)$ vanishes in the first order condition and that $d(\epsilon)$ takes the form of $d(\epsilon)=\bar{d}+\epsilon$ and consumption is a constant $c(\epsilon)=\bar{y}-\bar{d}$.

We go on to prove that when IC constraint is not binding for some ϵ', then for all $\epsilon''<\epsilon'$, the IC constraint does not bind, either. This is also proved by contradiction. Suppose we have ϵ' does not bind, but $\epsilon''\in[\epsilon_L,\epsilon')$ and the IC constraint is binding

$$u(\bar{y}+\epsilon')-m<u(\bar{y}+\epsilon'-d(\epsilon'))$$

$$u(\bar{y}+\epsilon'')-m=u(\bar{y}+\epsilon''-d(\epsilon''))$$

$$u(\bar{y}+\epsilon'')-u(\bar{y}+\epsilon''-d(\epsilon''))>u(\bar{y}+\epsilon')-u(\bar{y}+\epsilon'-d(\epsilon'))$$

We then have that

$$u(\bar{y}+\epsilon''-d(\epsilon''))-u(\bar{y}+\epsilon'-d(\epsilon'))<u(\bar{y}+\epsilon'')-u(\bar{y}+\epsilon')<0$$

That is,

$$u(c(\epsilon''))<u(c(\epsilon')),$$

and then

$$u'(c(\epsilon''))>u'(c(\epsilon')),$$

thus

$$\gamma(\epsilon'')<\gamma(\epsilon')=0$$

which is a contradiction since $\gamma(\epsilon'')\geq 0$.

Therefore, the optimal contract takes the form of a cutoff strategy

$$d(\epsilon)=\begin{cases}\bar{d}+\epsilon, \epsilon<\bar{\epsilon}\\ \bar{y}+\epsilon-u^{-1}(u(\bar{y}+\epsilon)-m), \epsilon>\bar{\epsilon}\end{cases}$$

We will also show that

1. $\bar{d}>0$
2. $d(\bar{\epsilon})=\bar{d}+\bar{\epsilon}=\bar{y}+\bar{\epsilon}-u^{-1}(u(\bar{y}+\bar{\epsilon})-m)$

If the contract is indeed continuous as stated in the second condition, then

$$\begin{aligned}0&=\int_{\epsilon^L}^{\bar{\epsilon}}(\bar{d}+\epsilon)\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}(\bar{y}+\epsilon-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon\\ &=\int_{\epsilon^L}^{\bar{\epsilon}}(\bar{d}+\epsilon)\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}(\bar{d}+\epsilon+u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d(\epsilon)\\ &=\bar{d}+\int_{\bar{\epsilon}}^{\epsilon^H}(u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon\end{aligned}$$

This proves that $\bar{d}>0$ since $u(\cdot)$ is an increasing function. Then we proceed to show continuity.

The optimal contract sets $\bar{\epsilon}$ and $\bar{d}$ to maximize

$$\int_{\epsilon^L}^{\bar{\epsilon}}u(\bar{y}-\bar{d})\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}(u(\bar{y}+\epsilon)-m)\pi(\epsilon)d\epsilon$$

subject to

$$0=\int_{\epsilon^L}^{\bar{\epsilon}}(\bar{d}+\epsilon)\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}(\bar{y}+\epsilon-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon$$

Differentiating the objective function and the constraint shows

$$-u'(\bar{y}-\bar{d})F(\bar{\epsilon})d\bar{d}+u(\bar{y}-\bar{d})\pi(\bar{\epsilon})d\bar{\epsilon}-(u(\bar{y}+\bar{\epsilon})-m)\pi(\bar{\epsilon})d\bar{\epsilon}=0$$

$$(\bar{d}+\bar{\epsilon})\pi(\bar{\epsilon})d\bar{\epsilon}-(\bar{y}+\bar{\epsilon}-u^{-1}(u(\bar{y}+\bar{\epsilon})-m))\pi(\bar{\epsilon})d\bar{\epsilon}+F(\bar{\epsilon})d\bar{d}=0$$

Combining the above two equations, we would have

$$u'(\bar{y}-\bar{d})[(\bar{d}+\bar{\epsilon})-(\bar{y}+\bar{\epsilon}-u^{-1}(u(\bar{y}+\bar{\epsilon})-m))]+u(\bar{y}-\bar{d})-(u(\bar{y}+\bar{\epsilon})-m)=0$$

Apparently, $\bar{d}+\bar{\epsilon}=\bar{y}+\bar{\epsilon}-u^{-1}(u(\bar{y}+\bar{\epsilon})-m)$ satisfies the above equation.

Finally, we go to check the IC constraint:

$$\begin{aligned}
0 &= \int_{\epsilon^L}^{\bar{\epsilon}}(\bar{d}+\epsilon)\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}(\bar{y}+\epsilon-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon \\
&= \int_{\epsilon^L}^{\bar{\epsilon}}(\bar{d}+\epsilon)\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}(\bar{d}+\epsilon+u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d(\epsilon) \\
&= \bar{d}+\int_{\bar{\epsilon}}^{\epsilon^H}(u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon \\
&= \bar{y}-u^{-1}(u(\bar{y}+\bar{\epsilon})-m)+\int_{\bar{\epsilon}}^{\epsilon^H}(u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon
\end{aligned}$$

Or equivalently,

$$\begin{aligned}
\bar{y} &= u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-\int_{\bar{\epsilon}}^{\epsilon^H}(u^{-1}(u(\bar{y}+\bar{\epsilon})-m)-u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon \\
&= \int_{\epsilon^L}^{\bar{\epsilon}}(u^{-1}(u(\bar{y}+\bar{\epsilon})-m)\pi(\epsilon)d\epsilon+\int_{\bar{\epsilon}}^{\epsilon^H}u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon
\end{aligned}$$

The right hand side is increasing in $\bar{\epsilon}$. When $\bar{\epsilon}=\epsilon_L$ the right hand side is $\int_{\epsilon^L}^{\epsilon^H}u^{-1}(u(\bar{y}+\epsilon)-m))\pi(\epsilon)d\epsilon<\int_{\epsilon^L}^{\epsilon^H}u^{-1}(u(\bar{y}+\epsilon-d(\epsilon))\pi(\epsilon)d\epsilon=\int_{\epsilon^L}^{\epsilon^H}(\bar{y}+\epsilon-d(\epsilon)\pi(\epsilon)d\epsilon=\bar{y}$. When $\bar{\epsilon}=\epsilon_H$ the right hand side is $\int_{\epsilon^L}^{\epsilon^H}u^{-1}(u(\bar{y}+\epsilon^H)-m))\pi(\epsilon)d\epsilon=\int_{\epsilon^L}^{\epsilon^H}u^{-1}(u(\bar{y}+\epsilon^H-d(\epsilon^H))\pi(\epsilon)d\epsilon\geq\int_{\epsilon^L}^{\epsilon^H}u^{-1}(u(\bar{y}+\epsilon-d(\epsilon))\pi(\epsilon)d\epsilon=\bar{y}$. We have proved that the RHS monotonically increases from some value below $\bar{y}$ to some value above $\bar{y}$. By continuity, there is a level of $\bar{\epsilon}$ that will equate both sides.