When it Rains, it Pours:
Capital Flows with Twin External Crises

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Abstract

This paper examines the joint dynamics of private and public external debt for countries. We develop a model with the co-occurrence of banking crisis and sovereign debt crisis in open economies, formalizing Reinhart and Rogoff (2011) findings “from financial crash to debt crisis”. External interest rate spikes or sudden stop shocks force banks to cut down debt position and fire-sale capital. The existence of frictions in bank equity market creates incentives for the government to initiate a bailout. The government bails out banks by increasing external borrowing and implementing fiscal austerity to undo inefficiencies in the private sector. Under optimal bailout scheme, the model generates diverging external debt dynamics for the private sector and the government during a crisis, as we document in the European data. Finally, we investigate two rationales for ex-ante macro-prudential regulations on private external debt: fire-sale externalities between banks and moral hazard by banks.

**JEL classification**: F34 F41 G01 G21 G28

**Key words**: external shocks, banking crises, sovereign debt crises, bailout policies, fiscal policies, prudential policies

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1 Introduction

The past several decades have been characterized by substantial increase in cross-border debt. In particular, the fraction of private sector debt stock now are unprecedentedly high, see Figure 1. The size of private gross external debt stock is almost on par with public gross external debt stock now. Nevertheless, international macroeconomics literature on cross-border debt flows has largely focused on representative private agents’ borrowing or only government borrowing. It is not innocuous if analyzing only aggregate external debt masks interesting and important interactions between private debt and public debt that affect the boom-bust cycles and implications for policy interventions. This paper examines the joint dynamics of both private and public external debt to fill this void in the literature.

[Figure 1 here]

The recent European financial integration and the ensuing debt crisis is among the notable examples to suggest why studying private and public external debt jointly might be of particular interest and of importance. Figure 2 breaks a panel of EU peripheral countries’ net external debt into government external debt and private external debt. It shows that during the recent debt crisis, by and large, government debt GDP ratio shoots up while private debt GDP ratio plummets. In other words, we observe divergent government and private debt dynamics during the crisis. On the other hand, before the crises, in most countries we have witnessed massive net private capital inflows but little increase or even slightly decline in government debt/GDP.

[Figure 2 here]

We notice that an important feature in the recent European debt crises is that government usually spends considerable resources on bailout. For instance, Ireland spends more than 40% GDP on bank bailout, Slovenia casts more than 10% of its annual GDP. These large scale government interventions pose additional burden on the government’s balance sheet, pushing up sovereign debt. This narrative is far from unique. For example, in 1982, Chile was hit by an international debt crisis, resulting in deterioration in its borrowing terms and terms of trade. In order to mitigate the banking crisis, the Chilean government assumed external debt of several private banks. It contributed to the

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1We don’t have data on banking sector’s net external debt position, the gross external debt of banking sectors, however, follow similar pattern, see Figure 3.


3See e.g., http://www.politico.eu/article/slovenia-turns-itself-around-greece-bailout-bank-crisis, where the article also explicitly mentions Slovenia funds bailout by government bond issuance and fiscal tightening, as we exactly model in this paper.
huge deficit on the government’s balance sheet. In a recent illuminating empirical work by Reinhart and Rogoff (2011), they show that banking crisis and sovereign debt crisis usually go hand in hand. More importantly, they find that most often, banking crisis precedes sovereign debt crisis. In this paper, we explore a model to produce these European countries’ private and public debt dynamics during crises, in a framework where banking external debt crisis ignites the hike of sovereign debt via optimal bailout policies.

The severity of the European debt crises also naturally raises the question of whether ex-ante prudential regulations on bank external debt are desirable. Private and public debt are not isolated in our study. If these prudential policies have been implemented before the crisis, what are the implications for the severity of crisis and private and public debt dynamics during bad times?

This paper constructs a small open economy model nesting both private and public external debt flows. We first build up a banking crisis model. The model features that in bad times, banks have to fire-sale assets and capital misallocation becomes inevitable. We then explicitly incorporate government balance sheet with sovereign external debt. The above-mentioned inefficiencies call for the government to bail out banks. This is the intuitive reason government increases their debt even when borrowing also becomes more costly for the government in bad times, in contrast with private debt’s dynamics. Finally, we propose two reasons why macro-prudential policies could improve upon the decentralized economy. The first one lies in fire-sale externalities between banks and the second one is collective moral hazard problem by banks.

In the banking crisis model, domestic banks issue debt to foreign investors and thus they are exposed to external interest rate shocks or borrowing constraint shocks. Banks also face equity market frictions, which prevent them from downsizing dividends freely (or raising equity freely) when facing adverse shocks. Therefore, upon bad shocks, banks have to sell part of its capital stock, creating mis-allocation of capital within the economy and it is translated into shriveling in aggregate output. Relied on the banking crisis model, we then insert a benevolent government who collects proportional output tax and funds public good to households. The inefficiencies associated with financial frictions in the private sector generate scope of the government’s bailout to banks. When the optimal bailout package is large enough, the benevolent government finds it reluctant to reduce current public good provision too much so the government increases external debt to smooth public good across time. Public external debt dynamics are jointly driven by interest rate and the amount of bailout to banks. The second force could possibly revert the drop in public debt (in the absence of bailout) to increase instead, offering an explanation to the observed diverging debt dynamics for private and public sector during crises.

Lastly, we show two reasons why macro-prudential policies are needed in good times. The first
one is fire-sale externalities between banks. In a decentralized economy, banks take asset prices as given when they fire-sale assets. However, ex-ante choice of high debt leads to more sell of assets in bad times, pushing down asset prices. The dropping price tightens other banks’ financial frictions in bad times. Nevertheless, these banks fail to internalize this pecuniary externalities when they borrow in good times. The second reason for prudential policies arises from moral hazard by the banking sector. Even though the bailout is designed for the problem of the whole banking sector’s balance sheet, instead of an individual bank’s balance sheet, banks know that in a systemic banking crisis ex-post, the government has no choice but to bailout. This makes banks more bold in borrowing ex-ante because they don’t internalize the fiscal cost, i.e., reduction in public good provision to households when bailout is implemented.

Related Literature

This paper is connected to several strands of literature. It falls into the research on the relationship between sovereign debt and domestic banking sector. Sosa-Padilla (2012) and Bocola (2014) study how sovereign default affects banking sector’s balance sheet and the associated output cost. Mengus (2014) and Perez (2015) further explore how this cost affects government’s incentives to default. Different from their focus on sovereign default’s disturbances on banks’ balance sheet as banks hold domestic sovereign bond, our paper is on banks and the government’s external debt dynamics. We abstract away from banks’ holdings of sovereign bonds but instead focus on the link between banks and the sovereign through optimal bailout policies.

With the key role of bailout to banks during crises, our paper also relates to ex-post government intervention and bailout policy literature in macroeconomics. Gertler, Kiyotaki et al. (2010) evaluate how various credit market interventions might mitigate the severity of crises in a closed economy. In their paper, credit policies are given by some exogenous rules instead of optimally derived. Bianchi (2012) study efficient bailout with distortionary tax instruments and government runs a balanced budget in each period. More importantly, this paper departs from this literature by studying government interventions’ implications on external government debt dynamics.

Another contribution this paper adds value to is to the macro-prudential policies literature. Korinek (2010) and Bianchi (2011) emphasize fire-sale externalities related to collateral constraints. Schmitt-Grohé and Uribe (2015) pinpoint that in the presence of downward wage rigidity, firms don’t internalize that during booms that they raise wages too high which are hard to adjust downward when adverse shocks arrive, leading to employment loss. This paper highlights the role of domestic financial constraints as opposed to external collateral constraint to induce fire-sale externalities. Secondly, we show that moral hazard is also a channel through which prudential policies on banks could be necessary. We are not the first to propose the second channel, e.g., Gertler, Kiyotaki and
Queralto (2012) and Chari and Kehoe (2015) point out how the anticipation of ex-post government interventions or bailout can distort risk taking incentives. However, we additionally illustrate its implications on government external debt dynamics and severity of sovereign debt crises.

In a closely related paper, Acharya, Drechsler and Schnabl (2014) also study how banking problem will affect government debt via bank bailout in a three period model. As their paper is in the context of a closed economy, that paper is silent on external debt, which is the concentration of the current article. In their framework, banks are always passive in managing its debt, so how a banking crisis is triggered and how would drivers of the crises affect both private and public debt are not presented. Our paper also provides two rationales for the necessity of ex-ante prudential policies. Furthermore, in their model, the problem in the financial sector is a debt overhang problem which distorts bankers’ effort supply, while our model delivers asset fire-sales and consequent misallocation thus inefficient output drop. The importance of misallocation channel during crises has been highlighted in recent empirical literature. Lastly, the cost of bailout in their model is future distortionary tax while we emphasize fiscal austerity in the form of government spending cut.

Layout

The remainder of the paper is organized as follows. In Section 2, we present a banking crisis model in a small open economy. In Section 3, we incorporate government balance sheet to produce the divergent private and public debt dynamics in crises. In Section 4, we extend the model by making government debt from default-free debt to defaultable debt. In Section 5, we study two rationales for macro-prudential policies and its consequences. Finally, we conclude in Section 6.

2 Banking Crisis in Open Economy

In this section, we formulate an infinite horizon model of banking crisis in a small open economy. In the private sector, only banks have access to international debt market and a banking crisis is featured by bank capital fire-sales and inefficient aggregate output drop in the economy.

There is fixed capital stock (or land) $K$ within the economy, which is not tradable internationally. Capital stock is allocated between productive household sector and banking sector $^4$:

$$K = K_{ht} + K_{bt},$$

where $K_{ht}$ denotes aggregate capital stock in the household sector and $K_{bt}$ aggregate capital stock in the banking sector.

$^4$The household sector in the model can be viewed as combination of households and banks who have little exposure to international debt market. What we attempt to capture here is that banks differ in their exposure to international debt market and those banks who borrow from abroad excessively will be more affected by external shocks.
2.1 Households

Representative household’s preferences are defined over an infinite stream of consumption:

\[ E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(C_t) \right], \]

where \( \beta \in (0, 1) \) is the subjective discount factor, \( C_t \) is consumption in period \( t \) and \( u(\cdot) \) is increasing and strictly concave.

Households trade equity of banks, so in equilibrium they receive dividend flows \( \text{div}_t \) from banks. Each household can produce with capital good \( k_{ht} \) with technology \( Z_t H(k_{ht}) \), where \( Z_t \) is exogenous aggregate productivity shock that will also appear in the banking sector’s production technology later. \( H(\cdot) \) is increasing, concave and \( \lim_{x \to 0} H'(x) = +\infty \). Besides, households engage in a competitive market where they can trade capital good with banks. Therefore, the budget constraint of a representative household is

\[ C_t + x_{t+1}(e_t - \text{div}_t) = x_t e_t + Z_t H(k_{ht}) + q_t (k_{ht} - k_{h,t+1}), \]

where \( q_t \) is the price of capital good, \( e_t \) is the equity price of a bank, and \( x_t \) is the share of equity of banks.\(^5\)

Denoting \( \Lambda_t \) as the Lagrangian multiplier of the budget constraint, we obtain the following first order conditions, with respect to \( C_t, k_{h,t+1} \) and \( x_{t+1} \):

\[ u'(C_t) = \Lambda_t, \quad (1) \]

\[ q_t = \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} (Z_{t+1} H'(k_{h,t+1}) + q_{t+1}) \right], \quad (2) \]

\[ e_t = \text{div}_t + \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} e_{t+1} \right]. \quad (3) \]

The second equation illustrates that capital good price is the summation of future discounted value of marginal product of capital. Iterate forward the last equation and rule out bubbles in the equity price to arrive at

\[ e_t = E_t \left[ \sum_{j=0}^{\infty} \beta^j \frac{\Lambda_{t+j}}{\Lambda_t} \text{div}_{t+j} \right]. \]

\(^5\)By writing the budget constraint without household external borrowing, we have implicitly assumed that households don’t have access to international debt market and we provide discussions on this assumption in the appendix.
Finally, the market clearing condition of the bank equity market is

\[ x_t = 1, \]

so that

\[ C_t = \text{div}_t + Z_t H(k_{ht}) + q_t(k_{ht} - k_{h,t+1}). \] (4)

2.2 Banks

A representative bank’s objective is to maximize its equity price:

\[ e_0 = E_0 \left[ \sum_{t=0}^{\infty} \beta^t \Lambda_t \text{div}_t \right] \] (5)

Banks have access to foreign external borrowing and possess production technology \( Z_t F(k_{bt}) \). Here \( Z_t \), as mentioned above, is aggregate productivity shock and \( k_{bt} \) is capital stock owned by banks. \( F(\cdot) \) is increasing, concave and \( \lim_{x \to 0} F'(x) = +\infty \). Banks can trade capital good with households in a competitive market. Therefore, the budget constraint of a bank is:

\[ \text{div}_t = Z_t F(k_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(k_{bt} - k_{b,t+1}), \] (6)

where \( R_t \) is gross interest rate shock and \( b_{t+1} \) is bank’s new borrowing that matures next period.

We then introduce the key financial friction in this model. We assume that when paying dividends, banks must pay at least a certain fraction of its revenue,

\[ \text{div}_t \geq d Z_t F(k_{bt}), \] (7)

or equivalently,

\[ (1 - d) Z_t F(k_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(k_{bt} - k_{b,t+1}) \geq 0, \] (8)

where \( d \) measures the extent of financial friction. The financial friction is an equity market friction, possibly originating from some agency or informational frictions between equity holders and managers. Empirically, Brav et al. (2005) find managers’ particularly strong desire to avoid dividend cuts. A special setting \( d = 0 \) means banks cannot raise new funds from equity owners, which is a restriction widespread imposed in the existing literature, e.g., Brunnermeier and Sannikov (2014). When \( d < 0 \), the friction restricts the amount of fund banks can raise from equity market to certain

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6Banks are endowed directly with a production technology, so we abstract away frictions between banks and firms as in Gertler and Kiyotaki (2015).
extent.

Banks pick up $b_{t+1}$, $k_{b,t+1}$ and $div_t$ to maximize equity value as formalized in equation (5), subject to budget constraint (6) and dividend constraint (8). Substituting equation (6) into equation (5) to replace $div_t$ and denoting $\mu_t$ as the Lagrangian multiplier on the dividend constraint, we have the following first order conditions, with respect to $b_{t+1}$ and $k_{b,t+1}$:

$$
(1 + \mu_t) \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} (1 + \mu_{t+1}) \right),
$$

(9)

$$
q_t (1 + \mu_t) = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} \left[ (Z_{t+1} F'(k_{b,t+1}) + q_{t+1}) (1 + \mu_{t+1}) - d \mu_{t+1} Z_{t+1} F'(k_{b,t+1}) \right] \right).
$$

(10)

Equation (9) is a revised Euler equation, taking into account the dividend constraints. The left hand side is the marginal value of one more unit of external borrowing, which could relax the dividend constraint, while the right hand side is the marginal cost of one more unit of external borrowing, which makes next period’s dividend constraint possibly tighter. The Lagrangian multipliers show up in equation (10) as well. They also capture that by selling one unit of capital good, banks relax their current dividend constraint but also risk changing next period’s tightness of dividend constraint.

Finally, the standard complementary slackness conditions are simply written as:

$$
\mu_t \left( (1 - d) Z_t F(k_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t (k_{bt} - k_{b,t+1}) \right) = 0, \mu_t \geq 0,
$$

(11)

$$
(1 - d) Z_t F(k_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t (k_{bt} - k_{b,t+1}) \geq 0.
$$

(12)

2.3 Equilibrium Conditions

Now we start to nail down the system of equilibrium conditions. Since we have representative households and banks, so in aggregation $K_{bt} = k_{bt}$ and $K_{ht} = k_{ht}$. Thereafter, we will only use capital letter $K$ in equilibrium conditions. Furthermore, substitute equation (6) into equation (4) to get

$$
C_t = Z_t H(K_{ht}) + Z_t F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t},
$$

(13)

which can be directly obtained as a market clearing condition as well. It is also clear that as aggregate capital stock is fixed in the economy, we will replace $K_{ht}$ by $K - K_{bt}$ whenever possible.

Rewrite the above equation (13) and keep equations (1), (2), (9), (10), (11) and (12). A
competitive equilibrium is defined as a set of sequences \( \{C_t, \Lambda_t, b_{t+1}, K_{b,t+1}, q_t, \mu_t \geq 0\} \) satisfying

\[
C_t = Z_t F(K_{bt}) + Z_t H(K - K_{bt}) - b_t + \frac{b_{t+1}}{R_t},
\]

(14)

\[
\Lambda_t = u'(C_t),
\]

(15)

\[
q_t = \beta E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t} (Z_{t+1} H'(K - K_{b,t+1}) + q_{t+1})\right],
\]

(16)

\[
(1 + \mu_t) \frac{1}{R_t} = \beta E_t\left(\frac{\Lambda_{t+1}}{\Lambda_t} (1 + \mu_{t+1})\right),
\]

(17)

\[
q_t (1 + \mu_t) = \beta E_t\left(\frac{\Lambda_{t+1}}{\Lambda_t} [(Z_{t+1} F'(K_{b,t+1}) + q_{t+1})(1 + \mu_{t+1}) - \frac{d}{dt} \mu_{t+1} Z_{t+1} F'(K_{b,t+1})]\right),
\]

(18)

\[
\mu_t \left( (1 - d) Z_t F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t (K_{bt} - K_{b,t+1}) \right) = 0,
\]

(19)

\[
(1 - d) Z_t F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t (K_{bt} - K_{b,t+1}) \geq 0,
\]

(20)
given initial \( K_{b0}, b_0 \) and exogenous \( \{Z_t, R_t\} \).

2.4 First Best Economy

Before gauging into deeper analysis of the competitive equilibrium, we first characterize the first-best allocation as a benchmark. The objective of a social planner is:

\[
\max_{b_{t+1}, K_{b,t+1}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(C_t) \right],
\]

subject to the resource constraint:

\[
C_t = Z_t F(K_{bt}) + Z_t H(K - K_{bt}) - b_t + \frac{b_{t+1}}{R_t}.
\]

Denoting \( \Lambda_t \) as the Lagrangian multiplier, we obtain the following first order conditions with respect to \( C_t, b_{t+1} \) and \( K_{b,t+1} \):

\[
\Lambda_t = u'(C_t),
\]

\[
\frac{1}{R_t} = \beta E_t\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right),
\]

\[
F'(K_{b,t+1}) = H'(K - K_{b,t+1}).
\]
In this economy, capital good allocation will always be efficient: the marginal products of capital in household sector and banking sector always coincide.

**Proposition 1** When \( d = -\infty \), the decentralized competitive economy is equivalent to first-best economy.

**Proof.** If \( d = -\infty \), then dividend constraint never binds, thus we can set \( \mu_t = 0, \forall t \), in the decentralized economy. It is easy to verify that it fully replicates the first-best allocation by comparing equilibrium conditions. 

This result is not surprising as the equity market friction is the key financial friction we add to a frictionless economy. Once we get rid of it, the economy returns to first-best efficiency.

### 2.5 Steady State

We further characterize the steady state of the competitive equilibrium, where \( \{C_t, \Lambda_t, b_{t+1}, K_{b,t+1}, q_t, \mu_t \geq 0\} \) are constants. Assume that \( R_t = \frac{1}{\beta} \) always holds and \( Z_t = 1 \). We let the economy stay at a steady state where the dividend constraint is not binding so that \( \mu_t \) is nil. The steady state of the competitive equilibrium is described by:

\[
C = F(K_b) + H(K - K_b) - b + \frac{b}{R},
\]

\[
\Lambda = u'(C),
\]

\[
q = \beta \frac{H'(K - K_b)}{1 - \beta},
\]

\[
q = \beta \frac{F'(K_b)}{1 - \beta},
\]

\[
(1 - d)F(K_b) - b + \frac{b}{R} \geq 0.
\]

Therefore, \( H'(K - K_b) = F'(K_b) \) so that capital stock allocation is always efficient and \( q^* = \frac{F'(K_b)}{1 - \beta} \).

As consumption is constant, next periods’ bank debt also duplicates previous periods’. We will pick up the initial bank debt \( b_0 \) so that the dividend constraint doesn’t bind. Absent from any shock, the competitive equilibrium shall stay in the steady state forever.

### 2.6 One Time Interest Rate Shock

To inspect the model’s mechanism, we first do the following experimentation. Suppose interest rate \( R_t \) is always at its steady state \( R_t = R = \frac{1}{\beta} \), \( \forall t < 0 \). At \( t = 0 \), the economy unexpectedly gets hit
by an interest rate spike $\hat{R} > \frac{1}{\beta}$. But the interest rate $R_t$ will immediately revert back to its steady state value $R_t = \frac{1}{\beta}$, $\forall t \geq 1$.

We have in mind that the shock is not too small so that the financial constraint at least binds at $t = 0$. Meanwhile, the shock is not so large such that the financial constraint will not bind more than once. We guess and verify numerically that this is true for some values of $\hat{R} > \frac{1}{\beta}$. The procedures to compute the dynamic responses to the shock is provided in the Appendix.

2.6.1 Parameterization

We are now ready to move to the numerical work. We specify the following functional forms:

$$u(C) = \frac{C^{1-\sigma} - 1}{1 - \sigma}$$

$$F(x) = A_b x^\alpha$$

$$H(x) = A_h x^\alpha$$

where $0 < \alpha < 1$.

We take one period as, say, 5 years to mimic an on average 5 years’ crisis, which implies a relatively small $\beta = 0.75 = 0.944^5$. Capital share is set to 0.33. We set bank productivity twice as large as the household sector. For simplicity, we employ log utility. The minimum fraction of revenue that has to be paid to equity owners is assumed to be 0.5. Set initial debt $b_0 = 1.04$ so that debt GDP ratio is roughly 55%. The interest rate shock is an increase of $0.8/5 = 16\%$ in annualized rate in crises. The parameter names and values are summarized in Table 1.

[Table 1 here]

2.6.2 Dynamic Responses

In order to solve the dynamic responses of key variables, we proceed as follows. We first guess that the dividend constraint binds only at $t = 0$ and numerically solve the model. Then we check that dividend constraint is indeed slack $\forall t \geq 1$ and Lagrangian multiplier $\mu_0 > 0$. The dynamic responses are shown in Figure 4.

[Figure 4 here]

Solid lines represent the dynamic responses with dividend constraint $d = 0.5$. Upon the interest rate shock, as banks’ borrowing cost is too high, they retreat debt position. It is well known that there are both wealth effect and substitution effect so the direction of debt depends on which force is stronger. In our simulation, substitution effect dominates wealth effect.

\[7\] It is well known that there are both wealth effect and substitution effect so the direction of debt depends on which force is stronger. In our simulation, substitution effect dominates wealth effect.
constraint, despite that they would like to cut dividends, they find that they cannot do that so they have to sell part of its capital stock to the household sector. The appearance of the Lagrangian multiplier $\mu_0 > 0$ creates mis-allocation thus output drop in the economy in period 1, which means that the marginal products of capital are not equalized in the two sectors. In fact, from equilibrium conditions, we can derive

$$1 + \mu_0 = \frac{F'(K_{b1}) + q^*}{H'(K - K_{b1}) + q^*}$$

so it is clear that the larger $\mu_0$ is, the more severe the misallocation is. Consumption drops in period 0 because the interest rate is high so they choose to save instead of consume that much.

In comparison, in the case without dividend constraint, illustrated in dashed lines, there is no misallocation of capital and aggregate output is always at its maximum level. Banks also cut debt more, because with dividend constraint, even if borrowing cost soars, they don’t decrease debt that much in order to pay the minimum dividend. As with consumption dynamics, the minimum dividend requirement not only distorts capital allocation, but also forces households to receive more dividends to be consumed given that we don’t allow households to save.

In sum, an external interest rate shock brings about a banking crisis to the small open economy. Banks who are exposed to external borrowing have to reduce debt and sell its capital upon the shock. The consequent misallocation and intertemporal distortion of consumption highlight the inefficiencies caused by the financial frictions.

2.7 Simple Policy Analysis

We have seen that dividend constraints distort the economy to deviate from the first-best. We next deliver some simple policy analysis to have an idea of how policies can kick in and its interplay with the financial frictions.

2.7.1 Household-Bank Transfer

We first allow lump-sum transfer $T_i$ directly from households to banks by the government.

**Proposition 2** With lump-sum transfer from households to banks, the economy can achieve first-best equilibrium.

**Proof.** See Appendix. ■

Proposition 2 is intuitive in the sense that the financial friction in the model is that banks have to pay shareholders, i.e., households, a minimum amount, then the transfer from households to banks effectively lowers the dividend requirement, until making it irrelevant. If the government has
access to implementing the direct transfer from households to banks, there is no role of government external debt at all as the economy can already restore first-best.

2.7.2 Government-Bank Transfer

We then analyze another case that the government cannot do the lump-sum transfer from households at all. The transfer is only intertemporal between banks and government. Denote \( B_{gt} \) as government external debt and \( T_t \) as lump-sum transfer to banks. The budget constraint of the government is:

\[
\frac{B_{g,t+1}}{R_t} = B_{gt} + T_t
\]

Proposition 3 With lump-sum transfer between banks and the government, the economy is equivalent to the competitive equilibrium.

Proof. See Appendix. ■

This result is a repeat of the famous Ricardian Equivalence. Banks understand that any transfer to them today will have to be paid by themselves in the future. Therefore, there is no improvement beyond the decentralized equilibrium.

So far we have considered two extreme cases. One is that we can do the direct transfer from households to banks, then the government external debt doesn’t need to appear and we obtain first-best allocation. The other is that the government cannot touch the household and has to use external debt to do finance a transfer, then we cannot outperform the decentralized equilibrium. In the following Section, we shall introduce explicit government budget constraint to make sense the tradeoff of bank bailout policy.

3 Banking Crisis, Bailout and Government Debt

In this section, we introduce the above banking crisis into a model with explicit government balance sheet, where the government employs a fixed tax rate to output to pay for public good provision. To shed light on the role of financial frictions on the incentives to bailout, we deliberately set up the model so that without financial frictions (i.e., first-best), the balance sheets of private and public sectors are “parallel” thus no bailout is needed. But upon bad shocks, with financial frictions (i.e. decentralized equilibrium), the benevolent government find itself incentivized to do bailout transfer to banks. The optimal bailout scheme has implications for external debt dynamics in both private and public sectors.
We first let each agent in the economy take bailout policy $\omega_t$ as given to list the equilibrium conditions. Then the government selects the optimal bailout in period 0 to maximize a representative household’s total welfare.

3.1 Households

We revise representative households’ preferences to incorporate public good provision $G_t$, supplied by the benevolent government:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t [u(C_t) + v(G_t)] \right],$$

where $v(\cdot)$ is increasing and concave. Households take public good $G_t$ as given. The government finances public good by taxing output. So households’ budget constraint is changed to take into account a fixed tax rate $\tau$ to output:

$$C_t + x_{t+1}(e_t - \text{div}_t) = x_te_t + Z_t(1 - \tau)H(k_{ht}) + q_t(k_{ht} - k_{ht+1}).$$

3.2 Banks

Representative banks objective is always to maximize their equity price:

$$e_0 = E_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \text{div}_t \right],$$

subject to budget constraint

$$\text{div}_t = Z_t(1 - \tau)F(k_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(k_{bt} - k_{bt+1}) + \omega_t,$$

and dividend constraint

$$(1 - \delta)(1 - \tau)F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(K_{bt} - K_{bt+1}) + \omega_t \geq 0,$$

where $\omega_t \geq 0$ is bailout transfer to banks by the government and in the constraints, tax rate $\tau$ also appears. When we shut down the bailout transfer, $\omega_t = 0, \forall t$.

3.3 Government

Given bailout transfer, the benevolent government’s problem is

$$\max_{B_0, \omega, t+1, G_t} \sum_{t=0}^{\infty} \beta^t \left[ u(C_t) + v(G_t) \right],$$

where $\omega_t \geq 0$ is bailout transfer to banks by the government and in the constraints, tax rate $\tau$ also appears. When we shut down the bailout transfer, $\omega_t = 0, \forall t$. 

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subject to budget constraint

\[ G_t + B_{gt} = \frac{B_{g,t+1}}{R_t} + \tau (H(K - K_{bt}) + F(K_{bt})) - \omega_t, \]  

(26)

where private consumption \( C_t \) is not directly selected by the government but by the households.

Denoting \( \Lambda_{gt} \) as the Lagrangian multiplier of the budget constraint, we have the following first order conditions with respect to \( G_t \) and \( B_{g,t+1} \):

\[ \Lambda_{gt} = v'(G_t), \]  

(27)

\[ \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{g,t+1}}{\Lambda_{gt}} \right). \]  

(28)

### 3.4 Equilibrium Conditions

A competitive equilibrium with public good is defined as a set of sequences \( \{C_t, \Lambda_t, G_t, \Lambda_{gt}, b_{t+1}, B_{g,t+1}, K_{b,t+1}, q_t, \mu_t \geq 0\} \) satisfying

\[ C_t = Z_t(1 - \tau)F(K_{bt}) + Z_t(1 - \tau)H(K - K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + \omega_t, \]  

(29)

\[ \Lambda_t = u'(C_t), \]  

(30)

\[ q_t = \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} (Z_{t+1}(1 - \tau)H'(K - K_{b,t+1}) + q_{t+1}) \right], \]  

(31)

\[ (1 + \mu_t) \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} (1 + \mu_{t+1}) \right), \]  

(32)

\[ q_t(1 + \mu_t) = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} [(Z_{t+1}(1 - \tau)F'(K_{b,t+1}) + q_{t+1})(1 + \mu_{t+1}) - d\mu_{t+1} Z_{t+1}(1 - \tau)F'(K_{b,t+1})] \right), \]  

(33)

\[ \mu_t \left( (1 - d) Z_t(1 - \tau)F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(K_{bt} - K_{b,t+1}) + \omega_t \right) = 0, \]  

(34)

\[ (1 - d) Z_t(1 - \tau)F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(K_{bt} - K_{b,t+1}) + \omega_t \geq 0, \]  

(35)

\[ G_t = \tau F(K_{bt}) + \tau H(K - K_{bt}) - b_{gt} + \frac{b_{g,t+1}}{R_t} - \omega_t, \]  

(36)

\[ \Lambda_{gt} = v'(G_t), \]  

(37)

\[ \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{g,t+1}}{\Lambda_{gt}} \right). \]  

(38)
3.5 Steady State

We assume that in steady state there is no bailout and confirm it is indeed optimal given the relationship between initial government debt and bank debt. The private sector steady state is very similar to the case in Section 2.5 thus omitted. The public sector steady state is:

\[ G = \tau F(K^*_b) + \tau H(K - K^*_b) - B_g + \frac{B_g}{R}, \]  

(39)

\[ \Lambda_g = v'(G). \]  

(40)

3.6 Initial State

We further assume that the economy starts with \( B_{g0} = \frac{\tau}{1-\tau} b_0 \) and \( u(X) = X^{1-\sigma} - 1 \) and \( v(X) = \left( \frac{\tau}{1-\tau} \right)^\sigma u(X) \). Then in the steady state, there is no incentive to do any bailout transfer because in any time \( t \), \( u'(C) = v'(G) \). Note it is also true that if we exclude the financial frictions (no banking crisis), even we have stochastic \( \{Z_t, R_t\} \), there is no incentive for the government to do any bailout transfer either, because \( u'(C_t) = v'(G_t) \) is always true, \( \forall t \). Now what we are interested in is that if there is an interest rate shock at time 0 in the frictional decentralized equilibrium and we allow the government to do a bailout transfer \( \omega_0 \geq 0 \) in period 0, will the government optimally step in to recapitalize banks and what are the consequences for debt dynamics.

3.7 One Time Interest Rate Shock

Let the economy experience an interest rate shock \( \hat{R} \) in period 0. We first solve the equilibrium given a one-time bailout transfer to banks \( \omega_0 \geq 0 \). We keep assuming that the shock will let banks’ dividend constraint bind only in period 0 and then we verify numerically. The detailed equilibrium conditions in private and public sector with given bailout \( \omega_0 \geq 0 \) is in the Appendix.

3.7.1 Bailout Transfer \( \omega_0 \)

After obtaining the equilibrium with a given \( \omega_0 \), we next proceed to pick up the optimal \( \omega_0 \). The problem of the benevolent government is

\[
\max_{\omega_0} \frac{C^1_{0}^{1-\sigma} - 1}{1 - \sigma} + \frac{\beta}{1 - \beta} \frac{C^1_{0}^{1-\sigma} - 1}{1 - \sigma} + \left( \frac{\tau}{1 - \tau} \right)^\sigma \left[ \frac{G^1_{0}^{1-\sigma} - 1}{1 - \sigma} + \frac{\beta}{1 - \beta} \frac{G^1_{1}^{1-\sigma} - 1}{1 - \sigma} \right].
\]

Note that we have used the fact that \( C_t = C_{t-1} \) and \( G_t = G_{t-1}, \forall t \geq 2 \). We will select the optimal bailout \( \omega_0 \) numerically.
3.7.2 Parameterization

Since we have introduced tax rate to output, we will adjust total factor productivity $A_b = 1$ and $A_h = 0.5$ to $A_b = \frac{1}{1-\tau}$ and $A_h = \frac{0.5}{1-\tau}$ so that without bailout transfer, the private sector’s debt $b_t$ and capital stock $K_{bt}$ remain the same dynamics as in Section 2.6.2. We will keep everything else the same in Table 1. New parameter values are summarized in Table 2.

[Table 2 here]

3.7.3 Dynamic Responses

We first consider the case where bailout option is not available for the government, i.e. $\omega_0 = 0$. In Figure 5 dashed lines, we can confirm that bank capital and bank debt are exactly the same as in Figure 4. In addition, increasing borrowing cost culminates in government debt drop.

In order to highlight the role of financial frictions, we also consider the optimal bailout transfer without dividend constraints in the private sector. In Figure 6, we can see that the optimal bailout transfer is indeed 0. This result is by design. We set a specific relationship between utility functions of private and public good, and also between the initial debts of them carefully, to make sure that without financial frictions, marginal utilities of public and private good are always equalized.

After we consider optimal bailout transfer, exhibited in Figure 5 solid lines, we can see that optimal bailout transfer is positive and as a consequence, government debt goes up instead of drops. Due to the bailout transfer, bank’ external debt also reduces more. Bank capital drops less and mis-allocation is alleviated. Figure 7 does show that welfare function is hump shaped in bailout transfer in the region $\omega_0 \geq 0$. The intuition is that at $\omega_0 = 0$, the presence of binding financial friction in the private sector makes one more unit transfer to banks more valuable than spending it in public good from the benevolent government’s standpoint. But as the transfer increases, the marginal value of transfer will decrease and also marginal cost will increase because after all the transfer has to be born by the reduction of public good today and in the future. When the optimal bailout transfer package is large enough, government debt has to increase instead of decrease.

**Proposition 4** In the above decentralized economy, the marginal value of positive bailout transfer at point $\omega_0 = 0$ is strictly positive as long as the Lagrangian multiplier $\mu_0$ is a decreasing function of the amount of bailout at point $\omega_0 = 0$.

**Proof.** See Appendix. ■

Intuitively, less tightening dividend constraint will first raise aggregate output in period 1 and also mute the intertemporal distortions of consumption. These two forces are the values of bank bailout and are in fact presented in the details of the proof.
Proposition 5 A sufficiently small transfer $\omega = \epsilon > 0$ strictly decreases $\mu_0$.

Proof. We prove by contradiction. Suppose the reverse is true, $\mu_0(\epsilon) \geq \mu_0(0)$.

Combining equations (52) and (53), and combining (53) and (55) show that when $\mu_0$ goes up, $K_{bi} < K_b^*$ goes down and then $q_0$ goes up. Therefore, equations (54) and (56) gives that $C_0$ drops. According to the lifetime budget constraint for the private sector, a transfer to them must either increase $C_0$ or $C_1$ or both. Given that $C_0$ declines, it must be that $C_1$ increases.

However, if the right hand side of equation (55) must drop, then $\mu_0$ has to drop as well. It constitutes a contradiction. Therefore, $\mu_0(\epsilon) < \mu_0(0)$.

It makes sense that effectively with less debt on burden, the dividend constraint will be less binding, as confirmed in the numerical exercises. On the other hand, the transfer should have an upper bound as when we reduce public sector’s available resources further, the public welfare diminishes more sharply in the end due to the limiting property of the utility function of public good provision.

3.8 Differential Shocks to Private and Public Sector

3.8.1 Debt Inelastic Interest Rate

To highlight the role of financial frictions for banks in impelling the government’s bailout, we have assumed that banks and the government face the same interest rate. Nevertheless, in reality, the government usually borrows in cheaper terms than the private sector. The interest rate gap between public and private debt is particularly high in bad times. See e.g., Acharya, Drechsler and Schnabl (2014) evidence on Ireland bank and sovereign spread. If we set the interest rate shock to the government in period 0 only equal to, say, $R + 0.5$, which is smaller than the shock $R + 0.8$ which banks have to pay. Figure 8 shows the results. Not surprisingly, less borrowing cost for the government transfers into more incentives to bailout thus higher increase in government debt and lower bank external debt. Correspondingly, bank capital cut is lower and inefficient output loss becomes smaller.
3.8.2 (Internal) Debt Elastic Interest Rate

We have considered interest rate shock to the government debt, where borrowing cost is insensitive to the quantity she borrows. In sovereign default literature, when representative agent increases debt, debt price drops dramatically especially during crisis time. In order to capture this elastic price response, in stead of setting interest rate in period $0$ as $R + 0.5$, I add an debt elastic term $p(b_1 - b_0)$, where $p(\cdot)$ is an increasingly convex function and $p(0) = 0$. We parameterize following Schmitt-Grohé and Uribe (2003)

$$p(x) = \psi_1(e^x - 1),$$

(41)

The larger $\psi_1$ is, the higher elasticity of interest rate in response to debt increase. When $\psi_1 = 0$, we return to the debt inelastic interest rate. we will set $\psi_1 = 0.5$ to make comparisons.

Figure 9 illustrate the dynamics of key variables under no bailout and optimal bailout. The optimal bailout transfer here is less than the debt inelastic interest rate case, as we have put the debt elastic part as additional borrowing cost, disincentiving the government to borrow from abroad. In consequence, government debt increases less, and the private sector’s misallocation is more severe.

3.9 Debt Limit Shocks

We have imposed only equity market frictions so far. The literature has largely constrained agents to debt market frictions by contrast, typically debt ceiling constraints, e.g., Uribe (2006) and Bianchi, Hatchondo and Martinez (2012). We will also consider those frictions. To be precise, in addition to the previous equity market frictions, banks are subject to the following frictions:

$$b_{t+1} \leq \bar{b}_{t+1},$$

where $\bar{b}_{t+1}$ is exogenously given. We will introduce shocks to debt limit. The definition of competitive equilibrium with debt limit shocks is given in Appendix.

3.10 One Time Interest Rate and Debt Limit Shock

We will still do a one-time shock experimentation. The economy starts with steady state without binding dividend constraints and binding debt limit constraints. In period $0$, the economy is hit by an interest rate shock as before, plus a debt limit shock. To make the debt limit shock relevant, we set the debt limit in period $0$ low enough to let the debt ceiling constraint be binding. The dual shocks immediately fade away after period $0$. The description of equilibrium conditions are in Appendix.
3.10.1 Dynamic Responses

Set debt limit $\bar{b}_1 = 0.9$. As the debt limit constraint is going to be binding one period, we shall see $b_1 = \bar{b}_1$. In comparison with section 3.7.5., where banks can adjust both through external debt and capital, here banks are also debt constrained, so the only way they adjust is to get rid of capital good more upon the shock, creating larger output drop. That brings more incentives for the government to revive the banking sector, i.e. optimal bailout is larger, as shown in Figure 10. In this case, government debt jumps more to finance the desired bailout package. As with bank external debt, the long run equilibrium exhibits less debt burden for the banking sector. The bailout also prevents capital good price from plunging and enlarges the wedge between private good consumption and public good consumption. In the following Sections, we shall keep both interest rate shock and debt limit shock. We take a position that during crises not only banks face higher interest rate but they can hardly borrow to the amount they would like to given any interest rate.

[Figure 8 here]

[Figure 9 here]

[Figure 10 here]

4 Sovereign Debt Default

In the previous sections, we have assumed government external debt is default-free. We now extend our model to include sovereign default risk. We introduce productivity shocks. We assume that in addition to $R_0$ and $\bar{b}_1$ shock unexpectedly arrives in period 0 to the private sector, there is also aggregate productivity shock $Z_0$ and all agents understand that there is aggregate productivity shock $Z$ realized in period 1, which lasts to all future periods. The government can choose to default in period 1 with punishment of exclusion from international debt market forever\(^8\) and default incurs a permanent cost to productivity such that productivity drops to $\phi < 1$ fraction of the original productivity.

Denote the cumulative distribution function of $Z$ as $\theta(Z)$ and default threshold as $Z^*$. Risk neutral pricing implies that government borrowing cost in period 0 is

\[
R^g_0 = \frac{R}{1 - \theta(Z^*)}. \tag{42}
\]

\(^8\)This assumption is not essential as if the government starts with 0 debt and $\beta R = 1$, government debt level will stay at 0 afterwards.
In period 1, given $B_{g1}$ and $K_{b1}$ and realized shock $Z$, the government chooses whether to default to maximize households’ lifetime utility from public good consumption from period 1:

$$V^g(B_{g1}, K_{b1}, Z) = \max \{ V^{gd}(B_{g1}, K_{b1}, Z), V^{gc}(B_{g1}, K_{b1}, Z) \},$$

where $d$ means default and $c$ means no default (continue).

Write $y_1 = \tau Z[F(K_{b1}) + H(K - K_{b1})]$, $y^* = \tau Z[F(K^*_b) + H(K - K^*_b)]$. Under no default choice, government spending is

$$G_c^t = \frac{R - 1}{R} (y_1 - B_{g1}) + \frac{y^*}{R}, t \geq 1.$$ 

While with default, government spending is

$$G_d^t = \frac{R - 1}{R} y_1 \phi + \frac{y^*}{R} \phi, t \geq 1.$$ 

Therefore, when

$$B_{g1} > (1 - \phi)(y_1 + \frac{y^*}{R - 1}),$$
the government will default. This equation illustrates that government defaults in bad times, as in Arellano (2008): high debt and low income. The cutoff $Z^*$ is given by

$$Z^* = \frac{B_{g1}}{1 - \phi} \frac{1}{\tau[Z\{F(K_{b1}) + H(K - K_{b1})\} + \tau\{F(K^*_b) + H(K - K^*_b)\}] / (R - 1)}. \quad (43)$$

In period 0, government’s problem is to pick up the size of bailout $\omega_0$ and government debt $B_{g0}$ to maximize representative household’s welfare

$$C_0^{1-\sigma} \frac{1}{1 - \sigma} + \beta E_0 \left[ C_1^{1-\sigma} \frac{1}{1 - \sigma} \right] + \frac{\beta}{1 - \beta} E_0 \left[ G_0^{1-\sigma} \frac{1}{1 - \sigma} \right] + E_0[V^g(B_{g1}, K_{b1}, Z)].$$

We notice that the government is subject to debt pricing equation given by equation (42) and (43). The amount of debt the government can raise in period 0 is

$$\frac{B_{g1}}{R_0} = \frac{B_{g1}}{R} [1 - \theta(Z^*)],$$

where $Z^*$ is given by equation (43).

In Figure 11, we conduct a numerical experiment to illustrate that it is possible that despite the increased default risk and thus borrowing cost, the government still would like to increase government debt due to the need to bail out banks. Furthermore, Figure 12 shows the government
debt Laffer curve given the bailout size is fixed at the above optimal level. The logic of the existence of a Laffer curve is also similar to Arellano (2008). When government debt issuance is very high, foreign lenders price in the high default risk, making the debt price very low.

[Figure 11 here]

[Figure 12 here]

5 Ex-ante Prudential Policies

Analyzing the economy’s response’s to an unanticipated shock is a study of ex-post policy intervention. A natural question is whether there is also scope for ex-ante policy intervention. To mimic the boom-bust cycles in European countries, we introduce pre-crisis favorable interest rate shock and the economy understands that bad shocks (high interest rate and tightening borrowing constraint) can come next period. We assume that in period $-1$, there is a good interest rate shock $R_{-1} < \frac{1}{\beta}$. At the same time, all agents understand that with probability $p_B > 0$, the interest rate and debt limit shock will arrive in period 0, while with $p_G = 1 - p_B$, the economy will revert back to steady state interest rate $R_t = \frac{1}{\beta}$, $\forall t \geq 1$. We set $p_B = 0.2$. To simplify analysis, we will still assume that government debt is risk-free.

5.1 Fire-sale Externalities

In face of interest rate or tighter borrowing constraint shocks, when the dividend constraint begins to bind, banks need to sell its capital stock to the household sector, which depresses capital good price $q_t$. The decline in capital good price in turn tightens each bank’s dividend constraint because dividend is given by

$$
div_t = (1 - \tau)F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(k_{bt} - k_{b,t+1}).$$

If $q_t$ drops, each bank has to sell more of its capital stock to satisfy the dividend constraint. The extent to which $q_0$ drops in bad times depends on debt $b_0$. However, each individual bank fails to internalize this pecuniary externality on other banks when choosing $b_0$. We shut down bailout so that $\omega_t = 0$ so as to isolate the role of fire-sale externalities.

In the decentralized case, in period $-1$, we have Euler equation

$$\Lambda_{-1} = \beta R_{-1} E_{-1}[\Lambda_0]$$
where $E_{-1}[\Lambda_0] = p_B \Lambda_0^B + p_G \Lambda_0^G$ and $\Lambda_t = u'(C_t)$. Consumption in period $-1$ is given by

$$C_{-1} = (1 - \tau)Z_{-1} (F(K_b^*) + H(K - K_b^*)) - b_{-1} + \frac{b_0}{R_{-1}}$$

We guess and verify when the bad state happens, dividend constraint binds, while when good state happens, dividend constraint will not bind. Then we solve the model by letting a planner pick up $b_0$ for banks.

The planner’s objective is to maximize private welfare. She solves the following problem

$$\max_{b_0} u(C_{-1}) + \beta E_{-1}[W(b_0)]$$

s.t.

$$C_{-1} = (1 - \tau)Z_{-1} (F(K_b^*) + H(K - K_b^*)) - b_{-1} + \frac{b_0}{R_{-1}}$$

where $W(b_0)$ denotes the welfare of the representative household, given initial $b_0$ in the decentralized economy. We label this economy as "second best".

We expect the planner selects a lower $b_0$ so that when the bad shock arrives, fire-sale of capital will be less. Figure 13 shows that this is indeed the case. The social planner’s choice also makes the economy less volatile as in crises, with less debt on burden, banks fire-sale less capital. Output loss is alleviated.

[Figure 13 here]

5.2 Moral Hazard

We also investigate the case with government bailout ex-post. To simplify the analysis, we assume that the government only has option to bailout when bad shock happens. It could be rationalized by a non-monetary cost to implement bank bailout. Therefore, banks have the expectation that when bad shock arrives, the government will optimally choose to implement a bailout transfer.

In the decentralized case, in period $-1$, we still have Euler equation

$$\Lambda_{-1} = \beta R_{-1} E_{-1}[\Lambda_0]$$

where $E_{-1}[\Lambda_0] = p_B \Lambda_0^B + p_G \Lambda_0^G$ and $\Lambda_t = u'(C_t)$. Consumption in period $-1$ is given by

$$C_{-1} = (1 - \tau)Z_{-1} (F(K_b^*) + H(K - K_b^*)) - b_{-1} + \frac{b_0}{R_{-1}}$$
At the same time, for the public good, we obtain similar expressions

\[ \Lambda_{g,-1} = \beta R_{-1} E_{-1} [\Lambda_{g0}] \]

where \( E_{-1} [\Lambda_{g0}] = p_B A_{g0}^B + p_G A_{g0}^G \) and \( \Lambda_{gt} = v'(G_t) \). Public good in period \(-1\) is given by

\[ G_{-1} = \tau Z_{-1} (F(K_b^*) + H(K - K_b^*)) - B_{g,-1} + \frac{B_{g0}}{R_{-1}} \]

Ex-post, when the bad shock happens, government optimally selects \( \omega_0 \geq 0 \) to maximize total welfare of the representative households. The equilibrium choice of \( b_0 \) and \( B_{g0} \) is a fixed point problem.

Then we solve the model with social planner helps pick up \( b_0 \) and \( B_{g0} \) to maximize total welfare starting from period \(-1\). The social planner select \( b_0 \) and \( B_{g0} \) to

\[ \max_{b_0, B_{g0}} u(C_{-1}) + v(G_{-1}) + E_{-1} [W(b_0, B_{g0})] \]

where \( W(b_0, B_{g0}) \) denotes the total welfare starting from period 0 with bank debt \( b_0 \) and \( B_{g0} \), taking into account that when bad shock arrives, the government optimally intervenes to improve banks’ balance sheet. We label this economy as 'second best'.

Figure 14 shows dynamic responses of the decentralized economy and the second-best economy. First, comparing with Figure 13, we see that in decentralized economy, banks become more aggressive in choosing \( b_0 \) in the boom phase when anticipate future bailout. Besides, in the planner’s comparisons, the government prefers to select a more conservative debt \( b_0 \) as they want to reduce the bailout cost, which is borne by households’ reduction in welfare and taken care by the government. As expected, with social planner’s choice of \( b_0 \), the severity of crises is reduced. As a result, there is little need for bailout. What’s more impressive is that the government debt doesn’t need to increase during crises. In a word, if the social planner has directly picked up bank debt in the boom phase, a severe twin external crises might have been avoided.

[Figure 14 here]

6 Conclusion

This paper explores the external debt dynamics of both private and public sector. We start with some data patterns. In the recent European experience, during the crises, private debt and public debt move in opposite directions. Before the crises, most countries experience private debt boom
while public debt GDP ratio remains stable or slightly declines. One salient fact during crises is that the government usually steps in to bailout its banking system. Motivated by the bailout connection between bank balance sheet and government balance sheet, we build up a small open economy model with both bank external debt and sovereign external borrowing.

In the private sector, banks face equity market frictions despite that they can borrow from abroad. When adverse shocks arrive, like interest rate shock or borrowing constraint shock, banks have to fire-sale capital, creating misallocation and output drop. The government optimally intervenes by trading off the benefit of relaxing banks' financial frictions and the cost of cut in (current and future) public good spending. When the optimal bailout package is large enough, in order to smooth public good provision, the government increases intertemporal debt. Banks’ debt decrease is driven jointly by foreign shocks and the bailout transfer.

Finally, we do ex-ante policy analysis. We gauge into two rationales why prudential policies are desirable ex-ante. The first one is fire-sale externalities between banks. When banks borrow ex-ante, high debt level shall affect the amount of asset they need to sell during crises, driving down asset prices. Lower asset prices tighten other banks’ financial frictions. However, banks don’t internalize this price effect and instead take asset price as given. The second one is moral hazard problem by banks. Anticipation of bailout encourages ex-ante aggressive debt position of banks. Banks don’t internalize that bailout comes at the cost of public good spending cut for households. If ex-ante prudential policies, say bank leverage restrictions had been implemented, the severity of crises will be alleviated and even possibly change the direction of government debt from increase to decrease.

While the current paper is intended to qualitatively assess the joint dynamics of public and private external debt, a potential exploration is to quantitatively study a sovereign debt crisis ignited by bailout policies. We also have not allowed private borrowing to be defaultable and simplified the role of domestic financial frictions on banks to be only creating inefficient misallocation. Deeper understanding and richer modeling of the role of financial frictions and the financial sector are possibly fruitful research topics in open economies.
References


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Figures

Figure 1: External Debt Stock in Private Sectors of Developing Countries

Notes: this figure displays private sector external debt stock’s fraction of total external debt stock (long term) in developing countries. Data source: International Debt Statistics
Figure 2: Private and Public Net External Debt/GDP

Notes: this figure displays private and public net external debt/GDP in year 2002-2011. Blue lines with circles are private sector net external debt/GDP and red lines with diamonds are government debt/GDP. Vertical axis on the left is private net external debt/GDP and vertical axis on the right is government debt/GDP. Notice Ireland is home to many international financial institutions (including hedge funds) due to tax reasons, so its total external debt/GDP (thus the constructed private external debt/GDP) should not be taken at its face value.
Figure 2 (continued): Private and Public Net External Debt/GDP

Notes: this figure displays private and public net external debt/GDP in year 2002-2011. Blue lines with circles are private sector net external debt/GDP and red lines with diamonds are government debt/GDP. Vertical axis on the left is private net external debt/GDP and vertical axis on the right is government debt/GDP.
Figure 3: Banking Sector Gross External Debt/GDP

Notes: this figure displays banking sector gross external debt/GDP in year 2002-2011.
Figure 3 (continued): Banking Sector Gross External Debt/GDP

Notes: this figure displays banking sector gross external debt/GDP in year 2002-2011.
Figure 4: Role of Domestic Financial Frictions

Notes: this figure shows the dynamic responses under a one-time shock in period 0. Solid blue lines with circles are with domestic equity frictions; dashed red lines with diamonds are without domestic equity frictions.
Figure 5: Dynamics under Optimal Bailout and No Bailout

Notes: this figure shows the dynamic responses under a one-time shock in period 0. Solid blue lines with circles are with optimal bailout; dashed red lines with diamonds are without bailout.
Figure 6: Optimality of No Bailout without Domestic Financial Frictions

Figure 7: Optimality of Bailout with Domestic Financial Frictions
Figure 8: Dynamics under Optimal Bailout and No Bailout with Differential Interest Rate (1)

Notes: this figure shows the dynamic responses under a one-time shock in period 0. The interest rate shocks to the private sector and public sector are set to be different. Solid blue lines with circles are with optimal bailout; dashed red lines with diamonds are without bailout.
Figure 9: Dynamics under Optimal Bailout and No Bailout with Differential Interest Rate (2)

Notes: this figure shows the dynamic responses under a one-time shock in period 0. The interest rate shocks to the private sector and public sector are set to be different and government faces debt elastic interest rate. Solid blue lines with circles are with optimal bailout; dashed red lines with diamonds are without bailout.
Figure 10: Dynamics under Optimal Bailout and No Bailout with Borrowing Constraint Shock

Notes: this figure shows the dynamic responses under a one-time shock in period 0. There is also borrowing constraint shock to the private sector. Solid blue lines with circles are with optimal bailout; dashed red lines with diamonds are without bailout.
Figure 11: Dynamics under Optimal Bailout with Government Default

Notes: this figure shows the dynamic response with government debt default. In this simulation, parameters $Z_0 = 0.8$, $\phi = 0.72$, and $1/Z$ follows Pareto distribution with support [1,5] and pareto parameter is 14. The realized $Z$ is 0.75 and the government defaults under optimal policies.
Figure 12: Government Debt Laffer Curve

Notes: this figure shows government debt Laffer curve. The x-axis is debt issuance which is the amount government needs to repay next period. The y-axis is the debt issuance multiplied by debt price.

Figure 13: Pecuniary Externalities

Notes: this figure shows dynamics under planner’s choice of ex-ante bank debt and decentralized choice of ex-ante bank debt (both under no bailout). Solid blue lines with circles are planner’s choice; dashed red lines with diamonds are decentralized economy.
Figure 14: Moral Hazard

Notes: this figure shows dynamics under planner’s choice of ex-ante bank debt and decentralized choice of ex-ante bank debt (both under bailout). Solid blue lines with circles are planner’s choice; dashed red lines with diamonds are decentralized economy.
Tables

Table 1: Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Note</th>
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<tr>
<td>$\beta$</td>
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<td>Minimum Dividend Requirement</td>
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<tr>
<td>$R$</td>
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<td>Steady State Interest Rate</td>
</tr>
<tr>
<td>$\hat{R}$</td>
<td>$\frac{1}{\beta} + 0.8$</td>
<td>Interest Rate Shock</td>
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Table 2: Parameters (continued)

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<tr>
<td>$B_{g0}$</td>
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<tr>
<td>$\hat{R}$</td>
<td>$\frac{1}{\beta}$</td>
<td>Steady State Interest Rate</td>
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7 Appendix

7.1 Data Sources

Figure 1 takes data from International Debt Statistics available in WDI. Public external debt refers to “public and publicly guaranteed external debt, including the national government, political subdivisions (or an agency of either), and autonomous public bodies, and external obligations of private debtors that are guaranteed for repayment by a public entity.” Private external debt means “external obligations of private debtors that are not guaranteed for repayment by a public
entity."

Figure 2 utilizes data from Lane and Milesi-Ferretti (2007), Merler, Pisani-Ferry et al. (2012) and WDI. Lane and Milesi-Ferretti (2007) provides data on debt asset, debt liability\textsuperscript{9}, GDP of nations in current US dollars, while Merler, Pisani-Ferry et al. (2012) contains information on foreign residents’ holdings of a nation’s government debt in terms of current Euros, which we take as public sector’s (net) external debt. These holdings are then converted to US dollars by exchange rate data between US dollars and Euros from WDI. A nation’s aggregate net external debt deducted by the government’s external debt is regarded as private sector’s net external debt. We look into sixteen EU27 countries. Germany, France, Italy and UK are excluded since they are big countries in the Europe. We also exclude advanced countries Belgium, Denmark, Finland, Netherland, Sweden and Luxembourg. Cyprus is further dropped as she is sometimes labeled as a “tax haven” which could mask its debt data. We keep Ireland (despite she is also sometimes labeled as a “tax haven”) as she is explored in the literature that bailout ignites her sovereign debt crisis. However, one should not read its constructed private sector’s net external debt in Figure 2 at its face value, either.

Figure 3 data are downloaded from each central bank’s website.

7.2 Discussions on Households’ Access to External Saving

In the paper, we have forbid households to participate in the international debt market. Now we allow households to be able to save in international debt market with \( R_s \), we derive the Euler equation for households:

\[
\frac{1}{R_s^t} = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} \right) \tag{44}
\]

Denote \( b_{ht} \) as households’ holding of foreign bonds. The system of equations for the competitive equilibrium includes

\[
C_t = Z_tF(K_{bt}) + Z_tH(K - K_{bt}) - b_t - b_{ht} + \frac{b_{t+1}}{R_t} + \frac{b_{h,t+1}}{R_s^t} \tag{45}
\]

\[
\Lambda_t = u'(C_t) \tag{46}
\]

\[
q_t = \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} (Z_{t+1}H'(K - K_{b,t+1}) + q_{t+1}) \right] \tag{47}
\]

\[
(1 + \mu_t) \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} (1 + \mu_{t+1}) \right) \tag{48}
\]

\textsuperscript{9}Here we don’t use IIP (international investment position) asset and liability, as other international investment positions, for example, equity is contingent liability and FDI may involve technology transfer. However, as robustness check, if we use IIP instead, similar patterns in general preserve.
\[ q_t(1 + \mu_t) = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} \left[ (Z_{t+1}F'(K_{b,t+1}) + q_{t+1})(1 + \mu_{t+1}) - d\mu_{t+1}Z_{t+1}F'(K_{b,t+1}) \right] \right) \]  \hspace{1cm} (49)

\[ \mu_t \left( (1 - d)Z_tF(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(K_{bt} - K_{b,t+1}) \right) = 0 \]  \hspace{1cm} (50)

\[ (1 - d)Z_tF(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(K_{bt} - K_{b,t+1}) \geq 0 \]  \hspace{1cm} (51)

given initial \( K_{b0}, b_0, b_{h0} = 0 \) and exogenous \{Z_t, R_t\}.

**Proposition 6** If \( R_t^s = R_t \), the decentralized competitive equilibrium can replicate first-best economy.

**Proof.** Set \( b_{t+1} \geq -R_t ((1 - d)Z_tF(K_{bt}) - b_t) \) and \( b_{h,t+1} = b_{t+1}^F - b_{t+1} \), then the financial frictions don’t bind and all the equilibrium conditions are satisfied as the first-best economy, taking \( b_t + b_{ht} \) as the corresponding equivalence of \( b_t \) in the model without household access to foreign bond. ■

The intuition behind the above proposition is that once households can save as much as possible with banks’ borrowing cost, the marginal value of consumption becomes very high. The high marginal value of consumption translates into banks’ incentives to pay more dividends, making the dividend constraints irrelevant. The key is to allow households have a saving technology with interest rate \( R_t \). In other words, we just need household debt \( b_{ht} \) to be unbounded below, the above proposition still applies.

The interest rate shock \( R_t \) to banks in this paper could be understood as a spread shock during the crisis. Admittedly, I didn’t introduce an endogenous default of banks or intermediation cost of foreign investors so as to produce an endogenous spread. Therefore, my banking crisis model’s interest rate shock is a short-cut representation of spread hike. Suppose \( R_t^s < R_t \), say, Ireland households can save in risk free German bonds with interest rate \( R_t^s \) and German bonds pay smaller interest rate than Ireland bonds. In this scenario, the next proposition establishes that dividend constraints would still matter when households in addition face borrowing constraint. As a matter of fact, we return to the decentralized competitive equilibrium without households’ access to external saving.

**Proposition 7** In the one-time shock case, if \( R_t^s < R_t \) and sufficiently small, and households are not allowed to borrow so that \( b_{ht} \geq 0 \), then households external debt would always be 0.

**Proof.** Denote the Lagrangian multiplier of households’ no borrowing constraint as \( \nu_t \). The Euler equation for households saving is

\[ \Lambda_t = \beta R_t^s E_t \Lambda_{t+1} + \nu_t \]
In steady state, $R^s < R$, we get that $\nu_t > 0$, so $b_{ht} = 0$. While in period $0$,

$$(1 + \mu_0)\Lambda_0 = \beta \hat{R}\Lambda_1$$

Therefore,

$$\nu_0 = \beta \Lambda_1 (\frac{\hat{R}}{1 + \mu_0} - \hat{R}^s)$$

It is also true that $\mu_0 > 0$ as long as $R^s < \frac{\hat{R}}{1 + \mu_0}$. In my numerical example, $\frac{\hat{R}}{1 + \mu_0} > \frac{1}{\beta} > R^s$, so as long as $R^s < \frac{1}{\beta}$, it is enough to guarantee that $b_{ht} = 0$, $\forall t$. ■

From the small open economy’s perspective, since households external saving pays less than banks’ borrowing cost, when the economy is indebted, she doesn’t want households’ to save at all. One may wonder that not allowing households to access borrowing at all is too strict a precondition. However, if households are able to borrow but at a higher interest rate than banks, we can show that still $b_{ht} = 0$ $\forall t$ holds.

Proposition 8 In the one-time shock case, if $R^s_t < R_t$ and sufficiently small, and households’ borrowing cost is $R^b_t > R_t$, then household saving/borrowing would always be 0.

Proof. We only need to tackle with excluding the borrowing possibility now. We first add a pseudo constraint $b_{ht} \geq 0$. Denote the Lagrangian multiplier on this constraint as $\nu_{bt}$. The Euler equation is

$$\Lambda_t = \beta R^b_t E_t \Lambda_{t+1} - \nu_{bt}$$

In steady state, $R^b > R$, we get that $\nu_{bt} > 0$, so that $b_{ht} = 0$. While in period $0$, recall that the Euler equation for banks’ debt is

$$(1 + \mu_0)\Lambda_0 = \beta \hat{R}\Lambda_1$$

Therefore,

$$\nu_{00} = \beta \Lambda_1 (\hat{R}^b - \frac{\hat{R}}{1 + \mu_0})$$

As long as $\hat{R}^b > \hat{R}$, we conclude that $\nu_{00} > 0$, so households will not borrow. ■

When households need to pay higher borrowing cost above banks’, the economy would rather let banks borrow from abroad instead of households. In reality, this higher external borrowing cost could be a result of additional intermediation cost for households or more severe agency frictions of households beyond banks.
7.3 Solution Procedures in Section 2.6

Bank capital starts at $K_{b0} = K_b^*$. Upon that the interest rate shock arrives in period 0, financial constraint binds. Banks choose debt $b_1$ and capital $K_{b1}$. From period 1 on, the interest rate returns to $\frac{1}{\beta}$, the financial constraint no longer binds (as mentioned above, we guess and verify under some values of $\hat{R}$, this is true.) In other words, given $b_1$ and $K_{b1}$, we first solve the no-binding equilibrium for $t \geq 1$. Consumption in period 1 is

$$C_1 = F(K_{b1}) + H(K - K_{b1}) - b_1 + \frac{b_2}{\hat{R}}$$

Since financial constraints will not bind for $t \geq 1$, we have $K_{bt} = K_b^*$, for $t \geq 2$, consumption in period 2 is

$$C_2 = F(K_b^*) + H(K - K_b^*) - b_2 + \frac{b_2}{\hat{R}}$$

Recall that the Euler equation shows that

$$\Lambda_1 = \Lambda_2$$

So we obtain $C_1 = C_2$ thus

$$b_2 = b_1 + F(K_b^*) + H(K - K_b^*) - F(K_{b1}) - H(K - K_{b1})$$

We conclude that $C_1$ is a function of $b_1$ and $K_{b1}$ in the following form:

$$C_1 = F(K_{b1}) + H(K - K_{b1}) - b_1 + \frac{b_1 + F(K_b^*) + H(K - K_b^*) - F(K_{b1}) - H(K - K_{b1})}{\hat{R}}$$

It is also straightforward to see that capital price $q_t = q^*$ when $t \geq 1$.

Next we go back to period 0, the following equations hold:

$$C_0 = F(K_b^*) + H(K - K_b^*) - b_0 + \frac{b_1}{\hat{R}}$$

$$q_0 = \beta \frac{\Lambda_1}{\Lambda_0} (H'(K - K_{b1}) + q^*)$$

$$q_0(1 + \mu_0) = \beta \frac{\Lambda_1}{\Lambda_0} (F'(K_{b1}) + q^*)$$

$$(1 + \mu_0) \frac{1}{\hat{R}} = \beta \frac{\Lambda_1}{\Lambda_0}$$
\[(1 - d)F(K_b^*) - b_0 + \frac{b_1}{R} + q_0(K_b^* - K_{b1}) = 0\]

\[\Lambda_0 = u'(C_0)\]

\[\Lambda_1 = u'(C_1)\]

Remember that \(C_1\) is in fact a function of \(b_1\) and \(K_{b1}\), so the last equation can be substituted by

\[\Lambda_1 = u' \left( F(K_{b1}) + H(K - K_{b1}) - b_1 + \frac{b_1 + F(K_b^*) + H(K - K_{b1}) - F(K_{b1}) - H(K - K_{b1})}{R} \right)\]

We have in total 7 equations and 7 unknowns \(\{q_0, \mu_0, \Lambda_0, \Lambda_1, C_0, b_1, K_{b1}\}\) given \(b_0\) and \(\hat{R}\), which will be solved numerically.

### 7.4 Equilibrium Conditions in Section 3.7

#### 7.4.1 Private Sector

In order to avoid repeating the tedious procedures in Section 2.6, we directly write down the equilibrium conditions in the private sector:

\[q_0 = \beta \frac{\Lambda_1}{\Lambda_0} \left( (1 - \tau)H'(K - K_{b1}) + q^* \right)\]  \(\text{(52)}\)

\[q_0(1 + \mu_0) = \beta \frac{\Lambda_1}{\Lambda_0} \left( (1 - \tau)F'(K_{b1}) + q^* \right)\]  \(\text{(53)}\)

\[C_0 = (1 - \tau)F(K_b^*) + (1 - \tau)H(K - K_b^*) - b_0 + \frac{b_1}{R} + \omega_0\]  \(\text{(54)}\)

\[(1 + \mu_0) \frac{1}{R} = \beta \frac{\Lambda_1}{\Lambda_0}\]  \(\text{(55)}\)

\[(1 - d)(1 - \tau)F(K_b^*) - b_0 + \frac{b_1}{R} + q_0(K_b^* - K_{b1}) + \omega_0 = 0\]  \(\text{(56)}\)

\[\Lambda_0 = u'(C_0)\]  \(\text{(57)}\)

\[\Lambda_1 = u' \left( (1 - \tau)F(K_{b1}) + (1 - \tau)H(K - K_{b1}) - b_1 + \frac{b_1 + (1 - \tau)F(K_b^*) + (1 - \tau)H(K - K_b^*) - (1 - \tau)F(K_{b1}) - (1 - \tau)H(K - K_{b1})}{R} \right)\]  \(\text{(58)}\)

We have in total 7 equations and 7 unknowns \(\{q_0, \mu_0, \Lambda_0, \Lambda_1, C_0, b_1, K_{b1}\}\).
7.4.2 Public Sector

At time $0$, the government implements a transfer $\omega_0 \geq 0$ to banks.

$$G_0 = \tau F(K^*_b) + \tau H(K - K^*_b) - B_{g0} + \frac{B_{g1}}{R} - \omega_0$$

$$G_1 = \tau F(K_{b1}) + \tau H(K - K_{b1}) - B_{g1} + \frac{B_{g2}}{R}$$

and

$$\frac{1}{R} = \beta \frac{\Lambda_{g1}}{\Lambda_{g0}}$$

$$\Lambda_{g1} = v'(G_1)$$

$$\Lambda_{g0} = v'(G_0)$$

where we should notice that when interest goes back to steady state value, government debt $B_{g2} = B_{g1}$. So we have 6 equations and 6 unknowns $\{\Lambda_{g0}, \Lambda_{g1}, G_0, G_1, B_{g1}, B_{g2}\}$.

7.5 Competitive Equilibrium in Section 3.9

Denoting $\nu_t$ as the Lagrangian multiplier on the debt limit constraint. We again omit the long derivation procedures. Given bailout $\omega_t$ to banks, a competitive equilibrium with public good is defined as a set of sequences $\{C_t, \Lambda_t, G_t, \Lambda_{gt}, b_{t+1}, B_{g,t+1}, K_{b,t+1}, q_t, \mu_t \geq 0, \nu_t \geq 0\}$ satisfying

$$C_t = Z_t(1 - \tau)F(K_{bt}) + Z_t(1 - \tau)H(K - K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + \omega_t$$

(59)

$$\Lambda_t = u'(C_t)$$

(60)

$$q_t = \beta E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}(Z_{t+1}(1 - \tau)H'(K - K_{b,t+1}) + q_{t+1})\right]$$

(61)

$$(1 + \mu_t)\frac{1}{R_t} - \nu_t = \beta E_t \left(\frac{\Lambda_{t+1}}{\Lambda_t}(1 + \mu_{t+1})\right)$$

(62)

$$q_t(1 + \mu_t) = \beta E_t \left(\frac{\Lambda_{t+1}}{\Lambda_t}[(Z_{t+1}(1 - \tau)F'(K_{b,t+1}) + q_{t+1})(1 + \mu_{t+1}) - d\mu_{t+1}Z_{t+1}(1 - \tau)F'(K_{b,t+1})]\right)$$

(63)

$$\mu_t \left[(1 - d)Z_t(1 - \tau)F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t(K_{bt} - K_{b,t+1}) + \omega_t\right] = 0$$

(64)

$$q_t(1 + \mu_t) = \beta E_t \left(\frac{\Lambda_{t+1}}{\Lambda_t}[(Z_{t+1}(1 - \tau)F'(K_{b,t+1}) + q_{t+1})(1 + \mu_{t+1}) - d\mu_{t+1}Z_{t+1}(1 - \tau)F'(K_{b,t+1})]\right)$$

(65)

$$\nu_t(\bar{b}_{t+1} - b_{t+1}) = 0$$

(66)
\[ b_{t+1} \leq b_{t+1} \]  
\[ G_t = \tau F(K_{bt}) + \tau H(K - K_{bt}) - b_{gt} + \frac{b_{gt+1}}{R_t} - \omega_t \]  
\[ \Lambda_{gt} = v'(G_t) \]  
\[ \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{gt+1}}{\Lambda_{gt}} \right) \]  

### 7.6 Equilibrium Conditions in Section 3.10

#### 7.6.1 Private Sector

The private sector equilibrium conditions carry a debt constraint with multiplier \( \nu_0 \) as well:

\[ q_0 = \beta \frac{A_1}{A_0} \left( [(1 - \tau)H'(K - K_{b1}) + q^*] \right) \]  
\[ q_0(1 + \mu_0) = \beta \frac{A_1}{A_0} \left( [(1 - \tau)F'(K_{b1}) + q^*] \right) \]  
\[ C_0 = (1 - \tau)F(K_{b1}^*) + (1 - \tau)H(K - K_{b1}^*) - b_0 + \frac{b_1}{R} + \omega_0 \]  
\[ (1 + \mu_0) \frac{1}{R} - \nu_0 = \beta \frac{A_1}{A_0} \]  
\[ (1 - \mu)(1 - \tau)F(K_{b1}^*) - b_0 + \frac{b_1}{R} + q_0(K_{b1}^* - K_{b1}) + \omega_0 = 0 \]  
\[ b_1 = \bar{b}_1, \nu_0 > 0 \]  
\[ \Lambda_0 = u'(C_0) \]  
\[ \Lambda_1 = u'((1 - \tau)F(K_{b1}) + (1 - \tau)H(K - K_{b1}) - b_1 + \frac{b_1 + (1 - \tau)F(K_{b1}^*) + (1 - \tau)H(K - K_{b1}^*) - (1 - \tau)F(K_{b1}) - (1 - \tau)H(K - K_{b1})}{R}) \]  

#### 7.6.2 Public Sector

The public sector equilibrium conditions don’t change, see Section 7.3.2.
7.7 Proofs

7.7.1 Proof of Proposition 2

With lump-sum transfer from household to banks, the equilibrium conditions are as follows:

\[ C_t = Z_t F(K_{bt}) + Z_t H(K - K_{bt}) - b_t + \frac{b_{t+1}}{R_t} \]  \tag{79}

\[ \Lambda_t = u'(C_t) \]  \tag{80}

\[ q_t = \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} (Z_{t+1} H'(K - K_{b,t+1}) + q_{t+1}) \right] \]  \tag{81}

\[ (1 + \mu_t) \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} (1 + \mu_{t+1}) \right) \]  \tag{82}

\[ q_t (1 + \mu_t) = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} [(Z_{t+1} F'(K_{b,t+1}) + q_{t+1}) (1 + \mu_{t+1}) - d \mu_{t+1} Z_{t+1} F'(K_{b,t+1})] \right) \]  \tag{83}

\[ \mu_t \left( (1 - d) Z_t F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t (K_{bt} - K_{b,t+1}) + T_t \right) = 0 \]  \tag{84}

\[ (1 - d) Z_t F(K_{bt}) - b_t + \frac{b_{t+1}}{R_t} + q_t (K_{bt} - K_{b,t+1}) + T_t \geq 0 \]  \tag{85}

Set \( T_t \geq - \left( (1 - d) Z_t F(K^*_b) - \frac{b_{F}^t}{R_t} + \frac{b_{F}^{t+1}}{R_t} \right) \), we can see that the first-best allocations satisfy all the equilibrium conditions with \( \mu_t = 0, \forall t \).

7.7.2 Proof of Proposition 3

With the balance sheet of the government, the equilibrium conditions are

\[ \frac{B_{g,t+1}}{R_t} = B_{gt} + T_t \]  \tag{86}

\[ C_t = Z_t F(K_{bt}) + Z_t H(K - K_{bt}) - b_t + \frac{b_{t+1}}{R_t} - B_{gt} + \frac{B_{g,t+1}}{R_t} \]  \tag{87}

\[ \Lambda_t = u'(C_t) \]  \tag{88}

\[ q_t = \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} (Z_{t+1} H'(K - K_{b,t+1}) + q_{t+1}) \right] \]  \tag{89}

\[ (1 + \mu_t) \frac{1}{R_t} = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} (1 + \mu_{t+1}) \right) \]  \tag{90}

\[ q_t (1 + \mu_t) = \beta E_t \left( \frac{\Lambda_{t+1}}{\Lambda_t} [(Z_{t+1} F'(K_{b,t+1}) + q_{t+1}) (1 + \mu_{t+1}) - d \mu_{t+1} Z_{t+1} F'(K_{b,t+1})] \right) \]  \tag{91}
\[ \mu_t \left( (1 - d)Z_t F(K_{bt}) - bt + \frac{b_{t+1}}{R_t} + qt(K_{bt} - K_{b,t+1}) + T_t \right) = 0 \]  \hspace{1cm} (92)

\[ (1 - d)Z_t F(K_{bt}) - bt + \frac{b_{t+1}}{R_t} + qt(K_{bt} - K_{b,t+1}) + T_t \geq 0 \]  \hspace{1cm} (93)

By substituting \( T_t \) using the government’s budget constraint, the last two equations can be re-written as

\[ \mu_t \left( (1 - d)Z_t F(K_{bt}) - bt + \frac{b_{t+1}}{R_t} + qt(K_{bt} - K_{b,t+1}) - B_{gt} + \frac{B_{g,t+1}}{R_t} \right) = 0 \]  \hspace{1cm} (94)

\[ (1 - d)Z_t F(K_{bt}) - bt + \frac{b_{t+1}}{R_t} + qt(K_{bt} - K_{b,t+1}) - B_{gt} + \frac{B_{g,t+1}}{R_t} \geq 0 \]  \hspace{1cm} (95)

Set \( b_{t+1} + B_{g,t+1} = b_{DE}^{t+1}, K_{bt} = K_{bt}^{DE} \), all the equilibrium conditions are satisfied, where the superscript \( DE \) represents the decentralized competitive equilibrium in Section 2.3 with initial bank debt equals to \( b_0 + B_{g0} \).

### 7.7.3 Proof of Proposition 4

To simplify notations, without loss of generality, let’s proceed with \( \tau = \frac{1}{2} \) so that the private and the public sector have the same utility function, the same revenue flows and the same initial debt. Write \( y^* = \frac{1}{2} (H(K - K^*_b) + F(K^*_b)) \) and \( y_1 = \frac{1}{2} (H(K - K_{b1}) + F(K_{b1})) \). With \( \mu_0 > 0 \), we must have \( y_1 < y^* \). I will keep using log utility but the conclusion should be applied to other CRRA utility functions as well.

For the public sector, the lifetime budget constraint is:

\[ G_0 + \frac{1}{R} \left( \sum_{j=1}^{+\infty} \frac{G_j}{R^{j-1}} \right) = y^* + \frac{1}{R} \left( y_1 + \sum_{j=2}^{+\infty} \frac{y^*}{R^{j-1}} - b_0 \right) \]

By Euler equation, we also know that \( G_j = G_{j-1}, \forall j \geq 2 \), therefore, the above equation can be rewritten as

\[ G_0 + \frac{G_1}{R - 1} = y^* + \frac{1}{R} y_1 + \frac{1}{R} y^* - b_0 \]  \hspace{1cm} (96)

In addition, the Euler equation in period 0 tells

\[ \beta \hat{R} = \frac{\Lambda_{g0}}{\Lambda_{g1}} = \frac{G_1}{G_0}. \]  \hspace{1cm} (97)
Similarly, for the private sector, the lifetime budget constraint is:

\[ C_0 + \frac{C_1}{R - 1} = y^* + \frac{1}{R} y_1 + \frac{1}{R} y^* - b_0 \]  \hspace{1cm} (98)

but the Euler equation in period 0 is a bit different:

\[ \frac{\beta \hat{R}}{1 + \mu_0} = \frac{\Lambda_0}{\Lambda_1} = \frac{C_1}{C_0}. \]  \hspace{1cm} (99)

By comparing equations (96), (97) with (98), (99), we immediately get \( C_1 < G_1 \) and \( C_0 > G_0 \) given that \( \mu_0 > 0 \), so that \( \Lambda_1 > \Lambda_{g1} \). It remains to show that a sufficiently small \( \epsilon > 0 \) transfer from the public sector to the private sector will increase aggregate welfare of households.

By assuming that \( \mu_0(\epsilon) < \mu_0(0) \), we obtain that revenue in period 1 satisfies \( y_1(\epsilon) > y_1(0) \). It means relaxing dividend constraint alleviates capital misallocation and pushes up aggregate output.

After we deduct \( \epsilon \) from the public sector to the private sector, if \( y_1(\epsilon) \) doesn’t change, then \( G_1 \) goes down by \( \frac{R(R-1)}{R^2} \epsilon \) and \( G_0 \) drops by \( \frac{R-1}{R} \epsilon \). However, since \( y_1(\epsilon) > y_1(0) \), we get \( |\Delta G_1| < \frac{R(R-1)}{R^2} \epsilon \) and \( |\Delta G_0| < \frac{R-1}{R} \epsilon \). The public welfare decreases by

\[ |\Delta G_0| \Lambda_{g0} + \frac{\beta}{1 - \beta} |\Delta G_1| \Lambda_{g1} < \frac{R - 1}{R} \epsilon \Lambda_{g0} + \frac{\beta}{1 - \beta} \frac{R(R-1)}{R^2} \epsilon \Lambda_{g1} = \frac{R}{R} \epsilon \Lambda_{g1} \]

Then we are left to prove that the private sector’s welfare increase would dominate the above.

First, if after the bailout transfer, in the Euler equation (99), \( \mu_0 \) doesn’t change, then \( \Delta C_1 > \frac{R(R-1)}{R[(1+\mu_0)(R-1)+1]} \epsilon \) and \( \Delta C_0 > \frac{(R-1)(1+\mu_0)}{(1+\mu_0)(R-1)+1} \epsilon \) as aggregate output \( y_1(\epsilon) > y_1(0) \). The private welfare increases by

\[ \Delta C_0 \Lambda_0 + \frac{\beta}{1 - \beta} \Delta C_1 \Lambda_1 > \frac{(R-1)(1+\mu_0)}{(1+\mu_0)(R-1)+1} \epsilon \Lambda_0 + \frac{\beta}{1 - \beta} \frac{R(R-1)}{R[(1+\mu_0)(R-1)+1]} \epsilon \Lambda_1 = \frac{R}{R} \frac{R}{(1+\mu_0)(R-1)+1} \epsilon \Lambda_1 \]

Going back to combine equations (96), (97) and also equations (98), (99), we obtain

\[ \frac{R}{R} \frac{R}{(1+\mu_0)(R-1)+1} \Lambda_1 = \frac{C_1}{R} \frac{R}{R} \frac{G_1}{R - 1}, \]

which means that \( \frac{\Lambda_{g1}}{R} = \frac{\Lambda_1}{(1+\mu_0)(R-1)+1} \). So we draw the conclusion that transferring \( \epsilon > 0 \) from the public sector to banks leads to aggregate welfare improving, conditional on in the Euler equation (99), \( \mu_0 \) doesn’t change. Then we apply the following Lemma to complete the proof.

**Lemma 9** In equations (98) and (99), welfare in the private sector increases when \( \mu_0 \) goes down.
Proof. Solving (98) and (99) to arrive at

\[
C_1 = \frac{(R - 1)\hat{R}}{R[(1 + \mu_0)(R - 1) + 1]} \left(y^* + \frac{1}{\hat{R}}y_1 + \frac{1}{\hat{R} R - 1} - b_0 \right)
\]

and

\[
C_0 = \frac{(R - 1)(1 + \mu_0)}{(1 + \mu_0)(R - 1) + 1} \left(y^* + \frac{1}{\hat{R}}y_1 + \frac{1}{\hat{R} R - 1} - b_0 \right).
\]

As a result, total private welfare is

\[
\log C_0 + \frac{\beta}{1 - \beta} \log C_1 = \text{constant} + \log \left(\frac{1 + \mu_0}{(1 + \mu_0)(R - 1) + 1} \right) + \frac{1}{R - 1} \log \left(\frac{1}{(1 + \mu_0)(R - 1) + 1} \right)
\]

Take derivative with respect to \( \mu_0 \) to obtain

\[
\frac{1}{1 + \mu_0} - \frac{R}{(1 + \mu_0)(R - 1) + 1} = - \frac{\mu_0}{(1 + \mu_0)[(1 + \mu_0)(R - 1) + 1]} < 0, \text{ whenever } \mu_0 > 0.
\]

It confirms that total private welfare increases when \( \mu_0 \geq 0 \) decreases. ■

Using the above Lemma, we are certain that with \( \mu(\epsilon) < \mu_0(0) \), private welfare will increase more than the case with fixed \( \mu_0 \) in equation (99).