Dynamic Matching in School Choice: Efficient Seat Reallocation After Late Cancellations

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Abstract

In many centralized school admission systems, a significant fraction of allocated seats are later vacated, often due to students obtaining better outside options. We consider the problem of reassigning these seats in a fair and efficient manner while also minimizing the movement of students between schools. Centralized admissions are typically conducted using the Deferred Acceptance (DA) algorithm, with a lottery used to break ties caused by indifferences in school priorities. We introduce the Permuted Lottery Deferred Acceptance (PLDA) mechanisms, which reassign vacated seats using a second round of Deferred Acceptance with a lottery given by a suitable permutation of the first round lottery numbers. We show that a mechanism based on a simple reversal of the first round lottery order performs the best among all PLDA mechanisms. We also characterize PLDA mechanisms as the class of truthful mechanisms satisfying some natural efficiency and fairness properties. Empirical investigations based on data from NYC high school admissions support our theoretical findings.

Keywords: dynamic matching, matching markets, school choice, deferred acceptance, tie-breaking, cancellations, reassignments.

1 Introduction

In many public school systems throughout the United States, students and families are required to submit preferences over the schools for which they are eligible. As this is done fairly early in the academic year, students typically do not know their options outside of the public school system at the time they submit preferences. As a consequence, a significant fraction of the students do not use their allotted seat in a public school. A well-designed reallocation process, run after students learn about their outside options, could lead to significant gains in overall welfare. Yet there is no known systematic way of reallocating unused seats. Our goal is to design an explicit reallocation mechanism run at a late stage of the matching process that reassigns these seats well.

During the past fifteen years, insights from matching theory have informed the design of school choice programs in several cities around the world. The formal study of this mechanism design approach to school choice originated in a paper of Abdulkadiroglu and Sönmez [2003]. They formulated a model in which students have strict preferences over a finite set of schools, each with a given capacity, and each school partitions the set of students into priority groups.

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There is now a vast and growing literature that explores many aspects of school choice systems and informs how they are designed in practice. However, most models considered in this literature are essentially static. Incorporating dynamic considerations, such as changes in student preferences, in designing assignment mechanisms is an important aspect that has not received much attention. Our work provides some initial theoretical results in this area and suggests that simple adaptations of one-shot mechanisms can work well in a more general setting.

We consider a two-stage model of school assignment with finitely many schools. Students initially submit their ordinal preferences over schools, and receive a first round assignment based on these preferences via Deferred Acceptance (DA). School preferences are given by weak priorities, and ties are broken via a single lottery ordering across all schools. Afterwards, some students may be presented with better outside options (such as admission to a private school), and may no longer be interested in the seat allotted to them. In the second round, students are invited to re-submit their (new) ordinal preferences over schools. The goal is to reallocate the seats so that the resulting assignment is fair, efficient, and so that the overall (two-stage) mechanism is strategy-proof and does not penalize students for participating in the second round.

The key idea we introduce is to apply a version of DA in which the initial assignment serves as a guarantee and couple the lottery numbers between the two rounds. This class of mechanisms—permuted lottery deferred acceptance mechanisms (PLDA)—enjoys strong efficiency and incentive properties. The mechanisms first break ties in school priorities by a single lottery ordering of the students, and a first-round assignment is computed by running DA; in the second round, each school first prioritizes students who were assigned to it in the first round over those who were not, and within each of the resulting two classes, students are prioritized according to their initial priorities at the school; finally, further ties across all schools are broken via a permutation of the (first-round) lottery numbers, and a second-round assignment is computed by running DA.

Our main result is that all PLDAs produce the same distribution over the final assignment when demand satisfies a technical condition we term the order condition. The order condition can be interpreted as all schools having the same relative overdemand in the two rounds, despite changes in student preferences. When the order condition is satisfied, the reverse lottery DA (RLDA) minimizes the number of reassigned students among all PLDAs. We illustrate this in Figure 1 using a simple example where students have common preferences and schools have a single priority group. (Our result, which applies to heterogeneous student preferences and arbitrary priorities at schools, is far more involved). We show that if the order condition holds, it can be verified by running RLDA. We also give an axiomatic justification for the class of PLDA mechanisms. In the case with no school priorities, they are equivalent to the class of mechanisms that are two-round strategy-proof while satisfying natural efficiency and fairness requirements.

Finally, we assess the performance of PLDAs in the presence of priorities using data from the New York City school system. We investigate a class of PLDAs that include RLDA, rerunning DA using the original lottery order (termed forward lottery deferred acceptance or FLDA), and rerunning DA using an independent random lottery. We find that all these mechanisms perform fairly similarly in terms of allocative efficiency, but RLDA reduces the number of reassigned students significantly. For instance, on the NYC public school system data set from 2004-2005, we find that FLDA results in about 7,200 reassignments out of a total of about 75,000 who remained in the public school system, whereas RLDA results in fewer than 3,100 reassignments (see Figure 4).

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1The student with the worst first round lottery number is given the highest priority in the second round.
Figure 1: Running DA with the same lottery creates a cascade of reassignments, whereas reversing the lottery minimizes reassignment.

In this example, there are 6 students with common preferences and 6 schools with a single priority group. All students prefer schools in the order $s_1 \succ s_2 \succ \cdots \succ s_6$. The student assigned to school $s_1$ in the first round leaves after the first round; otherwise all students find all schools acceptable in both rounds. Running DA with the same tie-breaking lottery reassigns all students to the school one better on their preference list, whereas reversing the tie-breaking lottery reassigns only the student initially assigned to $s_6$.

1.1 Related Work

In this subsection we survey related literature. The mechanism design approach to school choice was first formulated by Balinski and Sonmez [1999] and Abdulkadiroglu and Sonmez [2003]. Since then, many economists have worked closely with school authorities to redesign school choice systems [2]. These centralized mechanisms appear to outperform the uncoordinated and ad-hoc assignment systems that they replaced (Abdulkadiroglu et al. [2015]). A significant portion of the theoretical literature has focused on the relative merits of two canonical mechanisms—Deferred Acceptance and Top Trading Cycles (TTC)—and their variations. We refer the reader to recent surveys by Pathak [2011] and Abdulkadiroglu and Sonmez [2011] of the rich and growing (theoretical and empirical) literature on school choice problems.

One strand of literature explores how ties are broken within priority groups. Abdulkadiroglu et al. [2009] empirically compared single tie-breaking and multiple tie-breaking for DA, and recent papers of Arnosti [2015], Ashlagi and Nikzad [2016] and Ashlagi et al. [2015] study these tie-breaking rules analytically. We note that in our model, PLDA mechanisms use tie-breaking to reduce the number of reassignments while still satisfying natural efficiency and fairness properties [3].

A second strand of literature that is relevant to our model is the work of Abdulkadiroglu and Sonmez [1999] on house allocation models with existing tenants (or housing endowments). The second round in our model can be thought of as school seat allocation with existing endowments. Abdulkadiroglu and Sonmez [1999] prove that a generalization of TTC is strategy-proof, Pareto efficient, and individually rational [4] in this setting. However, directly applying the generalized TTC mechanism in the two-round setting suffers from some drawbacks. First, in our model the endowments are computed endogenously from the preferences. This means that a strategy-proof

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3Prior work has exploited indifferences to achieve other ends. For example, Erdil and Ergin [2008] show how to improve allocative efficiency, and Ashlagi and Shi [2014a] show how to increase community cohesion.

4An allocation is individually rational if every agent weakly prefers his assignment to his original endowment.
mechanism in the housing model is not necessarily strategy-proof in our two-round model, as a student has an incentive to manipulate their first round endowment. Secondly, while the TTC mechanism is natural in a model with endowments, it is less appropriate for assigning public school seats to students, where it is perfectly reasonable for a student to have priorities giving them a right to attend a school, but not to be able to trade these rights with each other.

Our work is among the few papers to consider a dynamic model for school admissions. Comte and Jehiel [2008] consider using DA to reassign agents who each hold a position at an organization. They show that a modified version of DA that prioritizes agents currently holding a position is strategy-proof, stable, and respects individual guarantees. Combe et al. [2016] study a more general model that also incorporates new agents, and show that the modified version of DA can be improved significantly in terms of overall welfare while also reducing the degree of instability. While the mechanism we use is a many-to-one version of modified DA, a critical distinction between this stream of work and ours is that in our model, the initial endowment is determined endogenously by preferences, and so students may want to manipulate their first round endowment. In addition, we are interested in exploiting indifferences in school priorities to reduce the number of reassignments, which is not addressed in this literature.

In recent work, Narita [2016] analyzes the preference data from NYC school choice system and observes that a significant proportion of preferences change after the initial match. Narita also considers modified DA in this setting and establishes its usefulness. However, he does not explore how mechanisms can minimize reassignment costs, which is a central component of our work.

Our work connects with some strands in the operations management literature. Ashlagi and Shil [2014b] consider the problem of optimal allocation without money, motivated also by the school choice problem. The class of mechanisms that emerge in our setting involve choosing a permutation of the initial lottery order, and we find that the reverse lottery minimizes the number of reassignments within this class. This is similar to the choice of a service policy in a queuing system, e.g., first-in-first-out (FIFO), last-in-first-out (LIFO), shortest-remaining-processing-time (SRPT), etc., wherein a particular policy is chosen in order to minimize an appropriate cost function such as the expected waiting time, e.g., see Lee and Srinivasan [1989]. Our continuum model parallels fluid limits and deterministic models employed in queueing (Whitt [2002]), revenue management (Talluri and Van Ryzin [2006]) and other contexts in the OM literature.

2 Model

We begin by informally describing our setting. We consider the problem of allocating seats at a finite set $S$ of schools to a set of students. Each school partitions the students into priority groups that are exogenously determined and publically known. Each student submits a strict preference ordering over the schools that she finds acceptable. A single lottery ordering of the students is used to resolve ties in the priority groups at all schools, resulting in an instance of the two-sided matching problem with strict preferences and priorities. Seats are initially allocated according to the student optimal deferred acceptance (DA) algorithm. We remark that the strict student preferences, weak school priorities, and use of DA with single tie-breaking are consistent with school choice systems (see e.g. [Abdulkadiroglu and Sonmez 2003]). Some students are subsequently presented with better outside options—such as admission to a private school that is not in $S$—and may no longer be interested in the seats allotted to them, effectively vacating those seats. After these outside options are revealed, each student submits her new ordinal

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5 A two-round mechanisms respects individual guarantees if each agent is assigned to a weakly better position in the second round.
preferences over schools, and a reassignment is computed. Since the reassignment occurs at a relatively late stage, moving students from one school to another is costly, potentially for both schools and students. Our goal is to design a procedure to reallocate students to schools that minimizes the amount of student movement with respect to the initial assignment, while satisfying appropriate notions of efficiency, fairness, and incentive compatibility.

The timeline for the mechanism design problem considered here is as follows: Students submit first round preference reports $\succ$, the mechanism designer obtains the first round assignment $\mu$ by running DA with uniform-at-random single tie-breaking, and the mechanism designer announces $\mu$. Then, students observe their outside option, and update their preferences accordingly. Finally, students submit their updated second round preference reports $\hat{\succ}$, the mechanism designer obtains the second round assignment $\hat{\mu}$ by running a reassignment mechanism $M$, and the mechanism designer announces $\hat{\mu}$. We illustrate this in Figure 2.

We describe two models for the problem of allocating seats to students. The discrete model (Section 2.1), which assumes a finite set of students, can be easily translated into implementable mechanisms and is used in the empirical analysis (Section 4). The continuum model (Section 2.2), which assumes a mass of students, is used in most of the theoretical analysis (Section 3.1). Intuitively, one could think of the continuum model as a reasonable approximation of the discrete model when the number of students is large.

Our continuum model can be viewed as two-round version of the model introduced by Azevedo and Leshno [2014].

### 2.1 Discrete Model

A finite set $\Lambda = \{1, 2, \ldots, m\}$ of students are to be assigned to a set $S = \{s_1, \ldots, s_n\}$ of schools. Each student can attend at most one school. For every school $s_i \in S$, let $q_i \in \mathbb{N}_+$ be the capacity of school $s_i$. Let $s_{n+1} \not\in S$ denote the outside option. We assume that the outside option has infinite capacity $q_{n+1} = \infty$. Each student $\lambda \in \Lambda$ has strict first round preferences $\succ^\lambda$ and second round preferences $\succ^\lambda$ over $S \cup \{s_{n+1}\}$. Each school $s_i$ has weak priorities $\succeq^i_S$ over $\Lambda$, which partition the students into priority groups. Equivalently, each student $\lambda$ has a priority group $p^\lambda_i \in \mathbb{N}$ at school $s_i$, and students in higher priority groups are preferred.

**Definition 1.** Preferences $(\succ, \succ^\lambda)$ are **consistent** if the second round preferences $\succ^\lambda$ are obtained from the first round preferences $\succ$ via truncation: (1) for every $s_i, s_j \in S$, $s_i \succ s_j$ iff $s_i \succ^\lambda s_j$, and (2) for every $s_i \in S$, $s_{n+1} \succ s_i$ implies $s_{n+1} \succ^\lambda s_i$.

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\[\text{Figure 2: Timeline of the two-round mechanism design problem}\]
In other words, preferences are consistent if the only change in preferences across the two rounds is each student’s relative ranking of \(s_{n+1}\), which weakly improves from the first round to the second, corresponding to an outside option (possibly) being realized between the two rounds. We say that a student is consistent if they have consistent preferences \((\succ^\lambda, \prec^\lambda)\).

Each student \(\lambda \in \Lambda\) is given a lottery number \(L(\lambda)\), drawn independently and uniformly from \([0,1]\). We remark that every ordinal ranking of students can be realized by some cardinal lottery function \(L\), and these random lottery numbers generate a permutation of the students uniformly at random.

An assignment specifies a school for each student. For a generic assignment \(\mu\), we let \(\mu(\lambda)\) denote the school to which student \(\lambda\) is assigned, and \(\mu(s_i)\) denote the set of students assigned to school \(s_i\). If \(\mu(\lambda) = s_i\) then student \(\lambda\) is said to be assigned to school \(s_i\); if \(\mu(\lambda) = s_{n+1}\), student \(\lambda\) is matched with her outside option and is said to be unassigned. An assignment is feasible if \(|\mu(s_i)| \leq q_i\) for all \(s_i\). In the rest of the paper, the term “assignment” is always used to mean a feasible assignment.

Next, we define what we mean by a reassigned student. All else being equal, we want to minimize the number of such students, as student movement is costly both for students and for school administration.

**Definition 2.** A student \(\lambda \in \Lambda\) is a reassigned student if she leaves a school in \(S\) for another school in \(S\). That is, \(\lambda\) is a reassigned student\(^8\) if \(\mu(\lambda) \neq \hat{\mu}(\lambda)\) and \(\mu(\lambda) \prec^\lambda s_{n+1}\).

### 2.2 Continuum Model

In the continuum model, a continuum mass of students \(\Lambda\) is to be assigned to a set \(S = \{s_1, \ldots, s_n\}\) of schools. Each student can attend at most one school. The set of students \(\Lambda\) has an associated measure \(\eta\), i.e., for any (measurable) \(A \subseteq \Lambda\), \(\eta(A)\) gives the mass of students in \(A\). The outside option is \(s_{n+1} \notin S\). As before, the capacities of the schools are \(q_1, \ldots, q_n \in \mathbb{R}_+\), and \(q_{n+1} = \infty\). A set of students of \(\eta\)-measure at most \(q_i\) can be assigned to school \(s_i\).

Each student \(\lambda \in \Lambda\) has a type \(\theta^\lambda\) and a first round lottery number \(L(\lambda) \in [0,1]\), which encode both student preferences and school priorities. A student is described by her type \(\theta = (\succ^\theta, \prec^\theta, p^\theta)\), where \(\succ^\theta\) and \(\prec^\theta\) are strict preferences over \(S \cup \{s_{n+1}\}\), which are, respectively, the student’s first and second round preferences, and \(p^\theta\) is an \(n\)-dimensional priority vector with \(p^\theta_{i}\) indicating (the number of) the priority group of student \(\theta\) at school \(s_i\). Let \(n_i\) be the number of priority groups at school \(s_i\). We assume that a student in the \(k\)th most preferred priority group at \(s_i\) is in priority group \(p_i = n_i - k\), so that \(p_i \in \{0,1,\ldots,n_i - 1\}\) and larger priority groups are preferred.

Let \(\Theta\) be the set of all student types. For each \(\theta \in \Theta\) let \(\zeta(\theta) = \eta(\{\lambda \in \Lambda : \theta^\lambda = \theta\}\) be the measure of all students with type \(\theta\). We say that \(\theta\) is consistent if the preferences \((\succ^\theta, \prec^\theta)\) are consistent (see Definition \(^\square\)), and otherwise we say that \(\theta\) is inconsistent. We assume that all students have consistent preferences.

**Assumption 1** (Consistent preferences). If \(\zeta(\theta) > 0\) then \(\theta\) is consistent.

As in the discrete model, we assume that the first round lottery numbers are i.i.d variables drawn uniformly from \([0,1]\) and do not depend on preferences. This means that for all \(\theta \in \Theta\) and intervals \((a, b)\) with \(0 \leq a \leq b \leq 1\), the proportion of students with type \(\theta\) who have lottery...
number in \((a, b)\) is equal to the length of the interval \((a, b)\): \[\eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, L(\lambda) \in (a, b)\}) = (b - a)\zeta(\theta)\].

An assignment specifies the set of students admitted at each school. For any assignment \(\mu\), we let \(\mu(\lambda)\) denote the school to which student \(\lambda\) is assigned, and \(\mu(s_i)\) denote the set of students assigned to school \(s_i\). We assume that \(\mu(s_i)\) is \(\eta\)-measurable and \(\eta(\mu(s_i)) \leq q_i\), for all \(s_i \in S \cup \{s_{n+1}\}\). We will again let \(\mu\) denote the first round assignment, and let \(\hat{\mu}\) denote the second round assignment.

2.3 Mechanisms

In this section, we formally define the class of mechanisms that we will be considering.

Let the students’ first and second round preference reports be denoted by \(\succ\) and \(\succsim\) respectively. A reassignment mechanism is a function that maps the realization of first round lotteries \(L(\lambda), \mu\) and the agents’ reports \(\succ\) and \(\succsim\) into a reassignment \(\hat{\mu}\). A two-round mechanism obtained from a reassignment mechanism \(M\) is a two-round mechanism where the first round mechanism is DA with uniform-at-random single tie-breaking, and the second round mechanism is \(M\). We emphasize that we intentionally keep the first round consistent with currently used mechanisms by fixing it to be DA with the same uniform-at-random lottery used for tie-breaking at all schools. It follows that the only freedom afforded the planner is the design of the second round mechanism.

We next describe some desirable properties of reassignment mechanisms. Any reassignment that requires taking away a student’s initial assignment against their will is impractical. Thus, we require our reassignment to respect first round guarantees:

**Definition 3.** A reassignment \(\hat{\mu}\) respects guarantees if every student prefers their second round allocation to their first round allocation, that is \(\hat{\mu}(\lambda) \succeq^L \mu(\lambda)\) for every \(\lambda \in \Lambda\).

Second, one of the main reasons for the success of DA in practice is the way it respects priorities: if a student is not assigned to a school she wants, it is because that school filled up with assigned students of higher priority. We require reassignments to respect priorities in this sense:

**Definition 4.** A reassignment \(\hat{\mu}\) respects priorities (subject to guarantees) if for every school \(s_i \in S\) and student \(\lambda \in \Lambda\) such that \(s_i \succ^\lambda \hat{\mu}(\lambda)\), we have \(\eta(\hat{\mu}(s_i)) = q_i\), and if in addition a student \(\lambda'\) satisfies \(\hat{\mu}(\lambda') = s_i \neq \mu(\lambda')\) then \(\lambda' \succeq^S \lambda\).

We next define a group of reassignment mechanisms that play a prominent role in this paper. We say \(P\) is a permutation if it is a Lebesgue measure preserving bijection from \([0, 1]\) to \([0, 1]\).

**Definition 5** (Permuted Lottery Deferred Acceptance (PLDA) Mechanisms). Let \(P\) be a permutation (that may depend on \(\zeta(\cdot)\)). Let \(L\) be the realization of first round lottery numbers, and let \(\mu\) be the first round assignment obtained by running DA with lottery \(L\). The permuted lottery deferred acceptance mechanism (PLDA) associated with \(P\) is a function mapping \((L, \mu, \succ, \succsim)\) into a reassignment \(\hat{\mu}^P\), which is obtained by running DA on \(\Lambda\) with student preferences \(\succsim\) and \(S\) and with school preferences \(\succeq^S\) that are determined as follows. For each \(s_i\):

- Student \(\lambda \in \Lambda\) for whom \(\mu(\lambda) = s_i\) are termed guaranteed at \(s_i\), and all other students are termed non-guaranteed at \(s_i\).

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\[\text{This can be justified via an axiomatization of the kind obtained in }\text{[Al-Najjar, 2004].}\]
- Schools prefer all guaranteed students to all non-guaranteed students, that is, for every student \( \lambda \in \Lambda \) guaranteed at \( s_i \) and student \( \lambda' \in \Lambda \) non-guaranteed at \( s_i \), we have \( \lambda \succ^S \lambda' \).
- Ties within each of the two groups—guaranteed and non-guaranteed—are broken first according to \( \succ^S \), and then according to the permuted lottery \( P \circ L \) (in favor of the student with the larger permuted lottery number).

Intuitively, these mechanisms involve running deferred acceptance twice. They use single tie-breaking in both rounds, explicitly relate the lotteries used in the two rounds via \( P \), and modify school priorities in order to guarantee that each student receives a (weakly) better assignment in the second round.

PLDA mechanisms are attractive in that they respect guarantees and priorities, assuming truthful reports of preferences. Moreover, although PLDA mechanisms are not generally strategy-proof, there is reason to expect that students will not strategize, since—as we point out in section 3.1—PLDAs satisfy a “strategy-proofness in the large” condition defined by Azevedo and Budish [2013].

Two special cases are worth highlighting. The RLDA (“reverse lottery”) mechanism uses the reverse permutation \( R(x) = 1 - x \); and the FLDA (“forward lottery”) mechanism, which preserves the original lottery order, uses the identity permutation \( F(x) = x \). In this paper, we provide strong evidence to support the use of the RLDA mechanism.

We end this section by defining permuted lottery deferred acceptance mechanisms in the continuum model in terms of cutoffs, in the style of Azevedo and Leshno [2014].

**Definition 6 (PLDA Mechanisms in the Continuum).** Let \( P \) be a permutation (that may depend on \( \zeta(\cdot) \)). The first round assignment \( \mu \) is defined by a vector of cutoffs \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_{+}^{n+1} \), where a student \( \lambda \) is given a first round score \( r^{\lambda} = p^{\lambda}_i + L(\lambda) \) at school \( s_i \) and assigned to her favorite school among those where her first round score exceeds the cutoff:

\[
\mu(\lambda) = \max\{s_i \in S \cup \{s_{n+1}\} : r^{\lambda} \geq C_i\}.
\]

In the second round, for each student \( \lambda \) with priority vector \( p^{\lambda} = p \), lottery number \( L(\lambda) = l \) and first round assignment \( \mu(\lambda) = s_i \), we associate a second round score \( \hat{r}^{\lambda} \) of \( n_i + P(l) \) at school \( s_i \), and a second round score of \( p_j + P(l) \) at all other schools \( s_j \), \( j \neq i \). The second round assignment \( \hat{\mu}_P \) is defined via a vector of cutoffs \( \hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n) \in \mathbb{R}_{+}^{n+1} \), where a student \( \lambda \) is assigned to her favorite school among those where her second round score weakly exceeds the cutoff:

\[
\hat{\mu}_P(\lambda) = \max\{s_i \in S \cup \{s_{n+1}\} : \hat{r}^{\lambda} \geq \hat{C}_i\}.
\]

The validity of such a definition in describing DA mechanisms follows from the arguments in Azevedo and Leshno [2014], and it is easy to verify that our model satisfies the technical conditions required in that paper. We say that a cutoff and assignment pair \((C, \mu)\) are market-clearing if \( \eta(\mu(s_i)) \leq q_i \) for all \( s_i \in S \cup \{s_{n+1}\} \), with equality if \( C_i > 0 \). It follows from the results in Azevedo and Leshno [2014] that the PLDA cutoffs and assignments \((C, \mu)\) and \((\hat{C}, \hat{\mu})\) are market-clearing.

Given cutoffs \( C \), we will find it useful to define cutoffs \( C_p \) for each priority type \( p \) by \( C_p,i = \lfloor C_i - p_i \rfloor \), where \( \lfloor x \rfloor \) is the largest integer smaller than or equal to \( x \). Intuitively, \( C_p,i \) is the lottery number a student \( \lambda \) with priority type \( p^{\lambda} = p \) needs in order to be able to go to school \( s_i \).
3 Main Results

We show that in the continuum model, under a certain technical condition, called the order condition, all PLDAs produce “type equivalent” assignments, and hence are equivalent in terms of welfare. Furthermore, we show that when this condition is satisfied, RLDA minimizes reassignments among all PLDAs. We also show that we can easily check for the order condition simply by running the RLDA mechanism. In Section 3.1, we provide axiomatic justification for PLDAs (in the case of no school priorities), showing that PLDAs are the mechanisms that satisfy certain natural efficiency and fairness properties. All proofs are provided in the appendix.

We begin by defining the order condition, which we will need to state our main results.

**Definition 7.** We say that a PLDA with permutation $P$ satisfies the local order condition on a set of primitives $(S, q, p, \Lambda, \eta)$ if for every priority class $p$ the first and second round cutoffs within that priority class are in the same order. That is, for all $s_i, s_j \in S \cup \{s_{n+1}\}$,

$$C_{p,i} > C_{p,j} \Rightarrow \hat{C}_{p,i}^P \geq \hat{C}_{p,j}^P.$$

We say that the order condition holds on a set of primitives $(S, q, p, \Lambda, \eta)$ if:

1. (Consistency across rounds) For every permutation $P$, the local order condition holds for PLDA with permutation $P$ on $(S, q, p, \Lambda, \eta)$; and

2. (Consistency across permutations) For every priority class $p$, for all pairs of permutations $P, P'$ and schools $s_i, s_j \in S \cup \{s_{n+1}\}$, it holds that $\hat{C}_{p,i}^P > \hat{C}_{p,j}^P \Rightarrow \hat{C}_{p,i}^{P'} \geq \hat{C}_{p,j}^{P'}$.

In other words, a permutation $P$ satisfies the local order condition on a set of primitives if the cutoff ordering is preserved from the first to the second round. The order condition holds on a set of primitives if the local order condition holds for all permutations and ties are broken consistently across permutations. We may interpret the order condition as an indication that the relative demand for the schools is consistent. Informally speaking, it means that the revelation of the outside options does not change the relative overdemand for the schools. We can view the requirement that it holds for all permutations as a form of robustness.

Next, we define type equivalence. For PLDAs, it simply means that the measure of each type with type $\theta \in \Theta$ assigned to each school is independent of the permutation.

**Definition 8.** Let $M$ and $M'$ be two-round mechanisms yielding probabilistic second round allocations $\hat{\mu}$ and $\hat{\mu}'$ respectively. Then the two allocations $\hat{\mu}$ and $\hat{\mu}'$ are said to be type equivalent if for all types $\theta \in \Theta$ and schools $s_i$,

$$\eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \hat{\mu}(\lambda) = s_i\}) = \eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \hat{\mu}'(\lambda) = s_i\}).$$

We say that the two mechanisms $M$ and $M'$ are type equivalent if they produce type equivalent second round allocations.

We remark that type equivalence depends only on the second round allocation, while reassignment measures the difference between the first and second round allocations.

We are now ready to state the main results of this section.

**Theorem 1** (Order condition implies type equivalence). If the order condition holds, all PLDA mechanisms produce type equivalent allocations.

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9 A reasonable utility model in the continuum would yield that type equivalence implies welfare equivalence.
We note that the assumption that students' ordinal preferences have full support follows from Assumption 2 which is that students' cardinal utilities have full support.

**Theorem 2** (Reverse lottery minimizes reassignment). If PLDAs produce type equivalent allocations under all permutations, RLDA minimizes the measure of reassigned students among PLDA mechanisms.

A natural question is whether the order condition is an algorithmically useful condition, since it requires knowing the outcome of all PLDAs. In fact, it is sufficient to show that the local order condition holds for the reverse lottery RLDA mechanism.

**Theorem 3** (Reverse lottery sufficient for order condition). Suppose that RLDA satisfies the local order condition on \((S, q, p, \Lambda, \eta)\). Then \((S, q, p, \Lambda, \eta)\) satisfies the order condition.

We emphasize that Theorem 3 states that if the local order condition holds for RLDA on some set of primitives, then the order condition holds, which is a statement about all permutations.

Our results present a strong case for using the RLDA mechanism when the main goals are to achieve allocative efficiency and minimize the number of reassigned students. Theorems 1 and 2 show that when the order condition holds RLDA is unequivocally optimal in the class of PLDA mechanisms, since all PLDA mechanisms give type-equivalent allocations and RLDA minimizes the number of reassigned students. Moreover, Theorem 3 shows that running RLDA also provides a quick and easy check for whether the order condition holds: if RLDA satisfies the local order condition, then the order condition holds, and the obtained second round allocation is optimal.

One important case covered by our results is the case of uniform dropouts and a single priority type. In this case, students either remain in the second round with the same preferences in both rounds, or drop out of the system entirely. If dropouts occur in an i.i.d. fashion, the order condition is satisfied. We provide direct proofs of Theorems 1 and 3 for this setting in Section B.1.

In Example 1, we give examples of when the order condition holds and does not hold, and illustrate the resulting implications for type equivalence. We illustrate these in Figure 2.

**Example 1.** Consider a setting with 2 schools with capacities \(q_1 = 2, q_2 = 5\), each with a single priority group. There is measure 4 of each of the four types of first round student preferences. Let \(\theta_i\) denote the student type that finds only school \(s_i\) acceptable, and \(\theta_{ij}\) denote the type that finds both acceptable and prefers \(s_i\) to \(s_j\). As we will be considering only students who either leave the system completely or keep the same preferences, we will also let these denote student types with the same preferences in the second round. If we run DA with single tie-breaking, the first round cutoffs are \((C_1, C_2) = \left(\frac{3}{4}, \frac{1}{2}\right)\).

Suppose that all type \(\theta_2\) students leave the system, and no other students receive an outside option. This frees up 2 units at \(s_2\). Under RLDA, the second round cutoffs are \((C_1^R, C_2^R) = (1, \frac{3}{4})\). In this case, the local order condition holds for RLDA, so the order condition holds and RLDA and RLDA are type equivalent. It is simple to verify that both FLDA and RLDA assign 

\[
\hat{\mu}^F(s_1, s_2) = \hat{\mu}^R(s_1, s_2) = ((1, 1, 0), (0, 2, 3)),
\]

Formally, this is the case where there exists \(\rho \in [0, 1]\) s.t. for every strict preference \(\succ\) over schools, it holds that 

\[
\zeta((\theta = (\succ^\theta, \succ^\theta), 1) \in \Theta : \succ^\theta = s_{n+1} \succ \ldots) = 1 - \rho\zeta((\theta = (\succ^\theta, \succ^\theta, 1) \in \Theta : \succ^\theta = s_{n+1} \succ \ldots)) \succ (1 - \rho)\zeta((\theta = (\succ^\theta, \succ^\theta), 1) \in \Theta : \succ^\theta = s_{n+1} \succ \ldots)).
\]

We note that this is essentially the case where dropouts are i.i.d. with probability \(\rho\). We remark that there are well-known technical measurability issue w.r.t. a continuum of random variables, but it should be noted that this issue can be handled—see, for example, [Al-Najjar 2004].
Figure 3: Example illustrates that FLDA and RLDA are type equivalent when the order condition holds, and not necessarily type equivalent when the order condition does not hold.

Students toward the left have larger lottery numbers. The patterned boxes above each column of students indicate the affordable sets for students in that column. When students who only want $s_2$ drop out, $s_1$ is still relatively more overdemanded, the order condition holds, and FLDA and RLDA are type equivalent.

When students who only want $s_1$ drop out, $s_2$ becomes relatively more overdemanded, and FLDA and RLDA give different ex ante assignments to students of every remaining type.

where the vector denotes the measure of students of type $(\theta_1, \theta_{12}, \theta_{21})$ assigned to the school.

Suppose that all type $\theta_1$ students leave the system, and no other students receive an outside option. This frees up 1 unit at $s_1$. Under RLDA, no new students are assigned to $s_2$, and the previously bottom ranked (but now top ranked) measure 1 of students who find $s_1$ acceptable are assigned to $s_1$. Hence the second round cutoffs are $(\hat{C}_1^R, \hat{C}_2^R) = (\frac{7}{8}, 1)$. In this case, the order condition does not hold. Type equivalence also does not hold, since the RLDA and FLDA assignments are

$$\hat{\mu}^R = ((1.5, 0.5, 0), (1, 2, 2)),$$
$$\hat{\mu}^F = ((2, 0, 0), (\frac{1}{3}, \frac{8}{3}, \frac{8}{3}))$$

where the vector denotes the measure of students of type $(\theta_{12}, \theta_{21}, \theta_2)$ assigned to the school. We note that in this case, FLDA is Pareto efficient, whereas RLDA is not.

3.1 Characterization

In this section, we consider a setting where each school has exactly one priority group. We show that the class of PLDA mechanisms is equivalent to the class of two-round mechanisms that satisfy a few natural fairness conditions, are two-round strategy-proof, respect guarantees and priorities, and are efficient among reassigned students. Note that, in the case with exactly one priority type, the first round corresponds to the random serial dictatorship (RSD) mechanism of Abdulkadiroğlu and Sönmez [1998], where the (random) order of students is given by the order of tie-breaking. We will only consider mechanisms that are non-atomic, that is, any single student changing their preferences has no effect on the assignment probabilities of other students.
We first specify what we mean by strategy-proofness and efficiency for a two-round mechanism with RSD in the first round. We remark that the two-round mechanism will give each student a probability distribution over first and second round allocation pairs. It will be helpful to think of each student’s induced probability distribution over the second round allocation for a fixed first round allocation. Formally, for a student $\lambda$ and a school $s_i$, we will be interested in $P(\hat{\mu}(\lambda) = s_i | \mu, \theta^\lambda = (\succ^\lambda, \succ^\lambda))$, which we call the student’s second round probability share in $s_i$ given a first round allocation $\mu$ and preferences $\succ, \succ'$. (In this section, since each school has exactly one priority group, we let a type be specified by just the preferences, $\theta = (\succ^\theta, \succ'^\theta)$.) We remark that we are not conditioning on the student’s first round lottery score.

In order to let students trade off between probabilities of assignment, we will need to assume some underlying cardinal utilities. Let $\Upsilon \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ be the set of pairs of first and second round cardinal utility vectors $(u, \hat{u})$ that correspond to consistent preferences, where $u_i, \hat{u}_i$ are the student’s utility from being assigned to $s_i$ in the first and second round respectively. Formally,$^{12}$

$$\Upsilon = \{ (u, \hat{u}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : u_i \geq u_j \iff \hat{u}_i \geq \hat{u}_j \forall i, j \leq n, \text{ and } u_i \leq u_{n+1} \Rightarrow \hat{u}_i \leq \hat{u}_{n+1} \forall i \leq n \}. $$

We assume that students’ consistent cardinal utilities have full support.

**Assumption 2 (Full support).** Student consistent cardinal utilities have full support. That is, for all sets $U \subseteq \Upsilon$ with positive Lebesgue measure, there exists a positive measure of students whose first and second round cardinal utilities pairs are in $U$.

We note that this is a strong assumption, and is stated in this manner for simplicity and brevity.$^{13}$

In the school choice setting, it is reasonable to assume that students will strategize for all the rounds for which they know their preferences for schools in $S$. Hence, it is desirable that at every point when a student (with consistent preferences) reports preferences, they receive the best expected utility, conditional on everything that has happened up to that point, by reporting truthfully. Specifically, let ex ante expected utility refer to the expected utility from the second round assignment at the beginning of the first round, conditional on first round reports, and let interim expected utility refer to the expected utility from the second round assignment at the beginning of the second round, conditional on first round guarantees and second round reports. We emphasize that all the uncertainty is created by the randomness in the mechanism, not by incomplete information about problem inputs and mechanism outputs. In particular, when calculating expected utilities, we assume that all students know the first and second round preference reports of all the other students, as well as the first round guarantees of all the other students.$^{14}$

**Definition 9.** A two-round mechanism is two-round strategy-proof if the following hold:

- Knowing the joint distribution of first round preferences, first round assignments and second round preferences, and assuming other students are truthful, no student (with consistent preferences) can obtain a better ex ante expected utility from their second round assignment by using any two-round strategy that involves lying in one or both rounds.

\[12\text{We note that } \Upsilon \text{ is isomorphic to } \mathbb{R}^{n+2} \text{ and inherits the Lebesgue measure via the isomorphism.}\]

\[13\text{We will only need support on certain specific subsets of cardinal utilities in order to prove our characterization result. Moreover, if we let the measure of students of any subset of types tend to } 0, \text{ our characterization result will still hold.}\]

\[14\text{We remark also that our model makes the most sense with a fixed outside option value in the first round that is then updated in the second round. We may interpret this as students knowing their valuations for each school both inside and outside the system, and assuming that the probability they will receive an outside school that they are not guaranteed is } 0, \text{ and reporting accordingly in the first round. We note that it is a dominant strategy for them to report in this manner.}\]
• Knowing the specific realization of first round assignments and the conditional distribution of second round preferences, and assuming other students are truthful, no student can obtain a better interim expected utility from their second round assignment by lying in the second round.

Note that a mechanism that uses a top-trading cycles like mechanism (which uses the first round allocation as an initial endowment) in the second round will not be two-round strategy proof, because students will benefit from manipulating their first round reports to obtain a more popular initial allocation which they can leverage to their advantage in the second round.

We remark that we require two-round strategy-proofness only for students whose true preference type is consistent. This is because reversals in student preferences can lead to conflicts between the desired first round assignment with respect to first round preferences, and the desired first round guarantee with respect to second round preferences. Moreover, it may be reasonable to assume that students who are sophisticated enough to strategize about misreporting in the first round to affect the guarantee structure in the second round will also know their second round preferences over schools in S at the beginning of the first round, and hence will have consistent preferences.\footnote{One obvious objection is that students may also obtain extra utility from staying at a school between rounds, or equivalently have a disutility for moving, creating inconsistent preferences where the school they are assigned in the first round becomes preferred to previously more desirable schools. We remark that Theorem \ref{thm:1} extends to the case when we consider students whose preferences incorporate an additional utility for staying put, provided that the utility is the same at every school for a given student, or satisfies a similar non-crossing property.}

Given that the second round occurs a significant period of time after the first round, in order to make the mechanism appear fair from the student perspective, we will want the second round assignment to be independent of first round preferences, aside from what is required to respect guarantees. A two-round mechanism is report-updating if a student’s second round assignment depends only on their first round lottery number, their first round assignment and their second round preference reports.

We also want our mechanism to treat students equally. A two-round mechanism is anonymous if it entirely ignores student identities. In particular, students with the same first round assignment and first and second round preference reports have the same distribution over second round assignments. A two-round mechanism where the first round is RSD satisfies the averaging axiom, if for a fixed first round allocation \( \mu \), every realization of the second round allocation \( \hat{\mu} \) leads to the same proportion of students with type \( \theta \) being assigned to a pair of schools \((s,s')\) in the two rounds. That is, for all \( \theta, s, s' \), there exists a constant \( c_{\theta,s,s'} \) such that \( \eta(\{ \lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) = s, \hat{\mu}(\lambda) = s' \}) = c_{\theta,s,s'} \).

We also want our two-round mechanism to be efficient for the students. In particular, we do not want any pair of students to be able to improve their utility by swapping probability shares in second round allocations. However, we also require that our two-round mechanism respects guarantees (see Definition \ref{def:guarantee}), and this suggests that we can require efficiency only among students who are reassigned in the second round.\footnote{An alternative would be to require Pareto efficiency including all students. This would lead to a smaller class of mechanisms that includes the forward lottery DA (FLDA) mechanism, but typically does not include the reverse lottery DA (RLDA) mechanism (see the end of Section \ref{sec:intro} for definitions of FLDA and RLDA).} This motivates the following definitions.

A two-round mechanism is non-wasteful if no student is assigned a school she prefers less to a school not at capacity, that is for each realization of \( \hat{\mu} \), student \( \lambda \in \Lambda \) and schools \( s_i, s_j \), if \( \hat{\mu}(\lambda) = s_i \) and \( s_j \not\sim^\lambda s_i \), then \( \eta(\hat{\mu}(s_j)) = q_j \). A Pareto-improving cycle of a two-round mechanism given a first round assignment \( \mu \) is an ordered set of types \( (\theta_1, \theta_2, ..., \theta_m) \in \Theta^m \) and schools \( (s_1, \tilde{s}_2, ..., \tilde{s}_m) \in S^m \) such that \( \eta(\{ \lambda : \theta^\lambda = \theta_i, \hat{\mu}(\lambda) = s_i \}) > 0 \) for all \( i \) and \( \tilde{s}_{i+1} \not\sim^\theta \tilde{s}_i \) for all \( i \) (where we define \( \tilde{s}_{m+1} = \tilde{s}_1 \)).
Definition 10. A two-round mechanism is Pareto efficient among reassigned students if each Pareto-improving cycle includes a student swapping a share in their first round guarantee. That is, for every Pareto-improving cycle of student types \((\theta_1, \theta_2, \ldots, \theta_m) \in \Theta^m\) and schools \((\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_m) \in S^m\), there exists \(i\) such that for all \(\lambda\), if \(\theta^\lambda = \theta_i\) and \(\hat{\mu}(\lambda) = s_i\) then \(\mu(\lambda) = \tilde{s}_i\).

Our characterization result is the following.

Theorem 4. Suppose student preferences are consistent and student cardinal utilities have full support (Assumption 2). A non-atomic two-round assignment mechanism where the first round respects guarantees and is:

1. non-wasteful;
2. two-round strategy-proof;
3. Pareto efficient among reassigned students;
4. anonymous;
5. averaging; and
6. report-updating;

if and only if the second round assignment is given by PLDA, or obtainable as a stable matching under the same second round school preferences as in PLDA. (Here the permutation \(P\) can depend on the measure of student preference types \(\zeta(\cdot)\).)

4 Empirical Analysis

In this section, we use data from New York City’s (NYC) school choice system to simulate and evaluate the performance of PLDA mechanisms under different permutations \(P\). The simulations indicate that our theoretical results are real-world relevant. Different choices of \(P\) are found to yield similar allocative efficiency: the number of students assigned to their \(k\)-th choice for each rank \(k\), as well as the number of students remaining unassigned, are very similar for different permutations \(P\). At the same time, the difference in the number of reassigned students is significant and is minimized under RLDA.

4.1 The Data and Simulations

We use data from the high school admissions process in NYC for the academic years 2004-05, 2005-06 and 2006-07, as follows:

1. First round preferences: In our simulation, we take the first round preferences \(\succ\) of every student to be the preferences they submitted in the main round of admissions. The algorithm used in practice is essentially strategy-proof (see Abdulkadirolu et al. [2005a]), so it is reasonable to assume that reported preferences are true preferences.

2. Second round preferences: In our simulation, students either drop out from the system entirely in the second round or maintain the same preferences. Students are considered to

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17 This is also random serial dictatorship.
18 We performed an initial cleanup of the data, such as removing preference entries which did not correspond to an existing school code.
19 The algorithm is not completely strategy-proof, since students may rank no more than 12 schools. However, only a very small percentage of students rank 12 schools.
20 However, there is some empirical evidence that students do not report their true preferences even in school choice systems with strategy-proof mechanisms, see e.g. Hassidim et al. 2015, Narita 2016.
Table 1: Simulation Results, 2004-2005 NYC High School admissions with priorities.

We show the mean number of students being reassigned, the mean percentage of students being reassigned, and the mean percentage getting their k-th choice or remaining unassigned, averaged across 50 realizations for each value of $\alpha$. All percentages are out of the total number of students, including those that drop out. The data contained 81,884 students and 653 schools. The percentage of students who dropped out was 9.22%. Variation across realizations in number of reassignments was approximately 100 students.

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<th>$k = 2$ (%)</th>
<th>$k = 3$ (%)</th>
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4.2 Results

The results of our computational experiments based on 2004-05 NYC high school admissions data appear in Figure 4 and Table 1. Allocative efficiency appears not to vary much across

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21 For a minority of the students (9.2% - 10.45%), their attendance in the following year could not be determined by our data, and hence we assume they drop out randomly at a rate equal to the rate of dropouts for the rest of the students (8.9% - 9.2%).

22 As per the final assignment produced by centralized allocation.

23 Unlike in the theoretical analysis, where we assumed no priorities, we take them into consideration here. We obtained similar results to the ones described below in simulations with no school priorities.

24 The results for 2005-06 and 2006-07 were similar.
values of $\alpha$; the number of students receiving their $k$-th choice for each $1 \leq k \leq 12$, as well as the number of unassigned students, vary by less than 1% of the total number of students. (Larger values of $\alpha$ give more students their first choice, but only slightly.) We further find that for most students, the likelihood of getting one of their top $k$ choices under FLDA and under RLDA are very close to each other. (Specifically, for 87% of students, these likelihoods differ empirically by less than 3% for all $k$. Here, the bound of 3% was chosen to suppress statistical variation in our simulations, that involved 19,800 trials each for FLDA and RLDA. FLDA and RLDA were compared since they represent the two extreme PLDAs. The same divergence metric was under 10% for 98.2% of students.) This is consistent with what we would expect based on our theoretical finding of type equivalence (Theorem 1) of the final allocation under different PLDA mechanisms.

Figure 4 shows that the mean number of reassignments is minimized at $\alpha = -\infty$ (RLDA) and increases with $\alpha$, which is consistent with our theoretical result in Theorem 2. The mean number of reassignments is as large as 7,200 under FLDA compared to just 3,100 under RLDA. Overall our findings suggest that RLDA would be the best choice of mechanism in the family of PLDA mechanisms considered.

5 Discussion

We have shown that the reverse lottery deferred acceptance mechanism (RLDA) is an attractive choice when the objective is to minimize the number of reassigned students. It attains similar allocative efficiency to all PLDA mechanisms when the order condition is satisfied (Theorem 1), minimizes reassigned students (Theorem 2), and the order condition itself can be verified by running the RLDA mechanism (3). In addition, RLDA also has the nice property of being equitable in an observable way. Specifically, under RLDA, students who receive a poor draw of the lottery in the first round are prioritized in the second round. This may make RLDA more palatable to students than other PLDA mechanisms. Indeed, the MIT dorm Random Hall uses a mechanism for assigning rooms that resembles the reverse lottery mechanism we have proposed (see Ran). Freshmen get their rooms through a serial dictatorship mechanism. At the end of the year (after seniors leave), they can claim the rooms vacated by the seniors, and each student is guaranteed to at least keep her current room. The initial lottery numbers (from their first
match) are reversed.

We also remark that our results suggest that PLDA mechanisms are an attractive class of mechanisms in more general settings, and the choice of mechanism within this class will vary with the policy goal. If, for instance, it were viewed as more equitable to allow more students to receive (possibly small) improvements to their first round assignment, then the FLDA mechanism that simply runs DA again would optimize over this objective. Moreover, our type-equivalence result (Theorem 1) shows that when the relative over-demand for schools stays the same this choice can be made without sacrificing allocative efficiency.

We axiomatically justified the class of PLDA mechanisms in settings where schools do not have priorities (Theorem 4). A natural question is whether a similar result holds for a model with school priorities. We find that in a model with priorities, natural extensions of our axioms continue to describe PLDA mechanisms, but also include undesirable generalizations of PLDA mechanisms. Specifically, suppose we modify the Pareto efficiency axiom so that any Pareto-improving cycle includes a student trading their second round assignment at some school $s$ to a student at a lower priority group at $s$ (where the highest priority group at $s$ are those students who have $s$ as a first round guarantee). Suppose we also add an axiom requiring that for each school $s$, the probability that a student who reports a top choice of $s$ then receives it in the first or second round is independent of their priority at other schools. This new set of axioms describes a class of mechanisms that includes the PLDA mechanisms, but there also exists an example market and a mechanism satisfying this new set of axioms such that the joint distribution over the two rounds of allocations does not match any PLDA. We believe that this new set of axioms describes a generalization of PLDA where the permutation of a student’s lottery number depends on their priority type. However, characterizing the class of mechanisms satisfying this natural set of axioms in the richer setting with school priorities remains an open question. It may also be possible to characterize PLDA mechanisms in a setting with priorities using a different set of axioms. We leave both questions for future research.

It is natural to ask what implications our results have for finite markets. By definition, PLDA mechanisms satisfy the efficiency, report-updating and anonymity requirements in finite markets as well. We believe that PLDA mechanisms are approximately strategyproof in large finite markets. In the second round it is clearly a dominant strategy to be truthful, and intuitively, for student $a$ to benefit from a first-round manipulation, his report should affect the second-round cutoffs in a manner that gives him a second-round allocation he would not have received otherwise. If the market is large enough, the cutoffs should converge to their limiting values, and the probability that student $a$ could benefit from such a manipulation would be negligible. A similar argument suggests that an approximate version of our characterization result (Theorem 4) should hold for finite markets with no priorities, as PLDA mechanisms satisfy an approximate version of the averaging axiom in large finite markets. Our results concerning type equivalence (Theorem 1), RLDA minimizing transfers (Theorem 2), and our sufficiency check for the order condition (Theorem 3) should also be approximately valid in the large market limit.

Another natural question is how to deal with inconsistent student preferences. Narita observed that in the current reapplication process in the NYC public school system, although only about 7% of students reapplied, about 70% of these reapplicants reported second round prefer-

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25 The MIT Random Hall matching is more complicated, because sophomores and juniors can also claim the vacated rooms, but the lottery only gets reversed at the end of freshman year. Afterward, if a sophomore switches room, her priority drops to the last place of the queue.

26 Specifically, consider a sequence of markets of increasing size. If the order condition holds in the continuum limit, this should lead to approximate type equivalence under all PLDAs and that RLDA approximately minimizes transfers among PLDAs in the finite markets as market size grows. Moreover, if the local order condition holds for the reverse lottery, then in large finite economies and for every permutation $P$, the set of students that violate the local order condition on PLDA($P$) will be small relative to the size of the market.
ences that were inconsistent with their first round reported preferences. We believe that some of our insights remain valid if a small fraction of students have an idiosyncratic change in preferences, or if a small number of new students enter in the second round. However, new effects may emerge if there is a systematic change in preferences, for instance if students derive utility from obtaining a slot in a school $s$ in the first round itself relative to getting a slot in $s$ in the second round. If student preferences are modified in this fashion, our axiomatization (Theorem 4) justifying PLDA mechanisms still holds, but an example shows that our results regarding type-equivalence and optimality of RLDA break down.

Finally, what insights do our results provide for situations in which allocation is done in three or more rounds? For instance, one could consider mechanisms under which the lottery is reversed (or permuted) after a certain number of rounds and thereafter remains fixed. At what stage should the lottery be reversed? Clearly, there are many other mechanisms that are reasonable for this problem, and we leave a more comprehensive study of this question for future work.

References


\[27\] For example, a student who was allocated her fourth choice in the first round may have a second round preference list that lists her top two choices followed by her fourth choice, i.e., her current allocation, because moving to her erstwhile third choice would not be worth the effort.


A List of Notation

A.1 Discrete Model

- $S = \{s_1, \ldots, s_n\}$: schools
- $s_{n+1}$: outside option
- $q_i$: capacity of school $i$
- $\Lambda$: finite set of students
- $\succ^\lambda$: first round preferences of student $\lambda$ over schools
- $\succ^\lambda$: second round preferences of student $\lambda$ over schools
- $\succeq^S_i$: weak priorities of school $s_i$ over students
- $p^\lambda$: vector giving the number of the priority group of student $\lambda$ at each school
- $L$: student lottery numbers

A.2 Continuum Model

- $\Lambda$: mass of students
- $\eta$: measure over $\Lambda$
- $\theta = (\succ^\theta, \succ^\theta, p^\theta)$: student types
- $\Theta$: space of student types $\theta$
- $\zeta(\theta)$: measure of students with type $\theta$ a given type
- $n_i$: the number of priority groups at school $s_i$

A.3 Mechanisms

- $P$: permutation
- $\mu$: first round assignment
- $\hat{\mu}$: second round assignment
- $\hat{\mu}P$: second round assignment from PLDA with permutation $P$
- $C$: first round cutoffs
- $\hat{C}^P$: second round cutoffs from PLDA with permutation $P$
- $C_p$: first round cutoffs restricted to priority type $p$
- $\hat{C}_p^P$: second round cutoffs from PLDA with permutation $P$ restricted to priority type $p$
A.4 Notation in the Appendix

- \( \hat{r}_i^\lambda = P(L(\lambda)) + n_i 1_{\{L(\lambda) \geq C_i\}} + p_i 1_{\{L(\lambda) < C_i\}} \): the amended scoring function of PLDA
- \( X_i = \{s_i, \ldots, s_{n+1}\} \): schools (weakly) after \( s_i \) in the cutoff ordering
- \( \gamma_i \): the proportion of students whose first round affordable set is \( X_i \)

A.5 Proofs for Uniform Dropouts (Section B.1)

- \( \rho \): probability that a student drops out
- \( \hat{C} \): constructed second round cutoffs
- \( f_P^i(x) \): proportion of students with \( s_i \) in their affordable set with permutation \( P \) and first and second round cutoffs \( (C_i, x) \)
- \( \gamma^P_i \): the fraction of students whose affordable set in the second round of PLDA with permutation \( P \) is \( X_i \)

A.6 Proof of Theorems 1 and 3

- \( \beta_{i,j} = \eta(\{\lambda \in \Lambda : \arg \max_{\tilde{\lambda}} X_{\tilde{\lambda}} = s_i\}) \): the measure of students who, when their set of affordable schools is \( X_{\tilde{\lambda}} \), will choose \( s_i \)
- \( E^\lambda(C) \): the set of schools affordable for type \( \lambda \) in the first round under PLDA with permutation \( P \)
- \( \hat{E}^\lambda(\hat{C}^P) \): the set of schools affordable for type \( \lambda \) in the second round under PLDA with permutation \( P \)
- \( \gamma^P_i = \eta(\{\lambda \in \Lambda : \hat{E}^\lambda_P(\hat{C}^P) = X_i\}) \): the fraction of students whose affordable set in the second round of PLDA with permutation \( P \) is \( X_i \)
- \( q_p \): restricted capacity vector for priority type \( p \)
- \( \Lambda_p \): set of students with priority type \( p \)
- \( \eta_p \): restriction of \( \eta \) to students with priority type \( p \)
- \( \mathcal{E}_p = (S, q_p, \Lambda_p, \eta_p) \): restricted primitives for priority type \( p \)
- \( s_{\sigma_p(i)} \): \( i \)th school under second round over-demand ordering for \( \mathcal{E}_p \)
- \( \hat{C}^P \): second round cutoffs defined for PLDA with the amended scoring function from the RLDA cutoffs \( \hat{C}^R \)
- \( \hat{C}^P_p \): second round cutoffs defined for PLDA on \( \mathcal{E}_p \) with the amended scoring function from the RLDA cutoffs \( \hat{C}^R \)
- \( \hat{n} \): smallest index of a school affordable to everyone

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A.7 Proof of Theorem 2

- $\ell_P = \eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) = s_i, \hat{\mu}_P(\lambda) \neq s_i\})$: the measure of students with type $\theta$ leaving school $s_i$ in the second round under PLDA with permutation $P$

- $e_P = \eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) \neq s_i, \hat{\mu}_P(\lambda) = s_i\})$: the measure of students with type $\theta$ entering school $s_i$ in the second round under PLDA with permutation $P$

A.8 Proof of Theorem 4

- $s_{\sigma(i)}$: $i$th school under second round over-demand ordering in non-atomic mechanism $M$ satisfying axioms (1)-(4)

- $\tilde{X}_i = \{s_{\sigma(i)}, s_{\sigma(i+1)}, \ldots, s_{\sigma(n+1)}\}$: schools (weakly) after $s_{\sigma(i)}$ in the second round over-demand ordering

- $\gamma_{i,j}$: proportion of students under the constructed PLSM whose first round affordable set was $X_i$ and second round affordable set was $\tilde{X}_j$

- $i(S') = \max\{j : s_j \in S'\}$: the maximum index of a school in $S'$

- $I_i = [C_i, C_j]$; the first round scores that give students first round affordable sets $\{X_{j+1}, X_{j+2}, \ldots, X_i\}$

- $\rho^\theta(I, S')$: the proportion of students with type $\theta$ who, under the mechanism $M$, have first round score in the interval $I$ and are assigned to a school in $S'$ in the second round

B Proofs

We begin with some general notation and definitions. Let $\mu$ be the initial allocation under RSD, and let $P$ be a permutation. We say that a school $s_i$ reaches capacity under a mechanism with output allocation $\mu$ if $\eta(\mu(s_i)) = q_i$.

We re-index the schools in $S \cup \{s_{n+1}\}$ so that $C_i \geq C_{i+1}$. Moreover, we assume that this indexing is done such that if the order condition is satisfied, then $\hat{C}_P^{i} \geq \hat{C}_P^{i+1}$ (where the cutoffs $\hat{C}_P$ are as defined by PLDA($P$)) holds simultaneously for all permutations $P$.

Throughout the appendix, for convenience, we slightly change the second round scoring function of a PLDA with permutation $P$ to be $\hat{r}_1^\lambda = P(L(\lambda)) + n_{\lambda}\mathbb{1}_{\{L(\lambda) \geq C_i\}} + p_{\lambda}\mathbb{1}_{\{L(\lambda) < C_i\}}$, meaning that we give each student a guarantee at any school for which they met the cutoff in the first round. By consistency of preferences, it is easily seen that this has no effect on the resulting assignment or cutoffs.

We say a student can afford a school in a round if her score in that round is at least as large as the school’s cutoff in that round. We say that the set of schools a student can afford in the second round (with the amended scoring function) is their affordable set.

Throughout the appendix, we let $X_i = \{s_i, \ldots, s_{n+1}\}$ be the set of schools at least as affordable as school $s_i$, and we let $\gamma_i$ be the proportion of students whose first round affordable set is $X_i$.

B.1 A Special Case: Uniform Dropouts

In this section, we consider the special case when there is exactly one priority group at each school, and students either leave the system uniformly at random, or retain the same priorities. We prove our theorems directly in this special case. In particular, we show that the
order condition holds, and give a direct proof that all PLDA mechanisms give type equivalent allocations.

Intuitively, we may think of this as the case when each student leaves the system with some probability \( \rho \). This can be interpreted as students leaving the city for some reason that is independent of the school choice system. In this setting, the second round problem is simply a rescaling of the first round. In particular, the measure of remaining students who were assigned to each school \( s_i \) in the first round is \((1-\rho)q_i\), the measure of students of each type \( \theta \) assigned to each school is scaled down by \(1-\rho\), the residual capacity of each school is \(pq_i\), and the measure of students of each type \( \theta \) remaining is scaled down by \(1-\rho\). Hence relative overdemand remains the same and schools fill in the same order regardless of the choice of permutation.

Throughout this section, since there are no priorities, we will let student types be defined either by \( \theta = (\succ^\theta, \preceq^\theta, 1) \) or simply \( \theta = (\succ^\theta, \preceq^\theta) \).

Formally, we define uniform dropouts with probability \( \rho \) as follows. For every strict preference \( \succ \) over schools, it holds that

\[
\zeta(\{\theta = (\succ^\theta, \preceq^\theta) \in \Theta : \succ^\theta = s \} \cup \{r \in \mathbb{R}^\Theta : \succ = r \}) = \rho \zeta(\{\theta = (\succ^\theta, \preceq^\theta) \in \Theta : \succ^\theta = s \}),
\]

and

\[
\zeta(\{\theta = (\succ^\theta, \preceq^\theta) \in \Theta : \succ^\theta = s \} \cup \{r \in \mathbb{R}^\Theta : \succ = r \}) = (1-\rho) \zeta(\{\theta = (\succ^\theta, \preceq^\theta) \in \Theta : \succ^\theta = s \}).
\]

We show first that Theorem \ref{thm:uniform_dropouts} holds in the setting with uniform dropouts. Specifically, we show that the local order condition holds for the reverse lottery, and that the (global) order condition holds. Since the argument for the reverse lottery is only slightly simpler than that for general permutations \( P \), we show directly that the order condition holds.

**Lemma 1.** In the case of uniform dropouts, the order condition holds.

**Proof of Lemma 7**. Let the first rounds cutoffs be \( C_1, C_2, \ldots, C_n \), where without loss of generality we index the schools such that \( C_1 \geq C_2 \geq \cdots \geq C_n \).

The main steps of the proof are as follows. We construct cutoffs \( \tilde{C}_i \) for school \( s_i \) by considering the proportion of students who receive \( s_i \) in their affordable set in the first round, and incorporating the guarantees, permutation and rescaling appropriately. We then show that these cutoffs are in the same order as the first round cutoffs. Finally, we use the fact that the cutoffs are in the same order to make inferences about aggregate demand and show that the cutoffs are market-clearing. It follows that the constructed cutoffs will be precisely the PLDA\((P)\) cutoffs.

Let

\[ f_i^P(x) = |\{l : l \geq C_i \text{ or } P(l) \geq x \}| \]

be the proportion of students who receive school \( s_i \) in their (second round) affordable set with the amended second round scoring functions under permutation \( P \) if the first and second round cutoffs are \( C_i \) and \( x \) respectively. Notice that \( f_i(x) \) is decreasing for all \( i \), \( f_i(0) = 1 \), \( f_i(1) = 1-C_i \), and if \( i < j \) then \( f_i(x) \leq f_j(x) \) for all \( x \in [0,1] \).

Let \( \tilde{C}_i^P \in [0,1] \) be defined by the equation

\[ f_i(\tilde{C}_i^P) = \frac{1}{1-\rho}(1-C_i), \]

and we let \( \tilde{C}_i^P = 0 \) if \( C_i < \rho \).

We show that \( \tilde{C}_1^P \geq \tilde{C}_2^P \geq \cdots \geq \tilde{C}_n^P \), and that these are the market-clearing DA cutoffs for the second round of PLDA with permutation \( P \).

We first show that the cutoffs \( \tilde{C} \) are in the right order, that is \( \tilde{C}_i^P \geq \tilde{C}_j^P \) for all \( i < j \).
Suppose $i < j$. If $\tilde{C}_i^P = 0$ then $C_j \leq C_i \leq \rho$ and so $\tilde{C}_j^P = 0 \leq \tilde{C}_i^P$ as required. Hence we may assume that $\tilde{C}_i^P, \tilde{C}_j^P > 0$. In this case, we have

$$\tilde{C}_j^P \leq \tilde{C}_i^P$$

$$\Leftrightarrow f_j \left( \tilde{C}_j^P \right) - (1 - C_j) \geq f_j \left( \tilde{C}_i^P \right) - (1 - C_j)$$

$$\Leftrightarrow \frac{\rho}{1 - \rho} (1 - C_j) \geq \left| \left\{ l : l < C_j, P(l) \geq \tilde{C}_i^P \right\} \right|.$$

Now

$$\left| \left\{ l : l < C_j, P(l) \geq \tilde{C}_i^P \right\} \right| \leq \left| \left\{ l : l < C_i, P(l) \geq \tilde{C}_i^P \right\} \right|$$

$$= \frac{\rho}{1 - \rho} (1 - C_i)$$

$$\leq \frac{\rho}{1 - \rho} (1 - C_j),$$

where the inequalities follow from the fact that $C_i \geq C_j$, and so $\tilde{C}_i^P \geq \tilde{C}_j^P$ as required.

We now show that $\tilde{C}^P$ is the set of market-clearing DA cutoffs for the second round of PLDA with permutation $P$.

We note that in DA, the cutoffs give each student an affordable set—the set of schools for which their score is above the cutoff—and that students choose their favorite school from their affordable set.

Recall that $X_i = \{ s_i, \ldots, s_{n+1} \}$ be the schools after $s_i$ in the first round cutoff ordering. Then $\gamma_i = C_i - C_{i+1}$ is the proportion of students whose first round affordable set is $X_i$. We note that since dropouts are uniform at random, this is the proportion of such students out of the total number of remaining students both before and after the dropouts occur.

Now $f_i \left( \tilde{C}_i^P \right)$ is the proportion of students whose second round affordable set contains $s_i$, and since $C_1 \geq C_2 \geq \cdots \geq C_n$ and $\tilde{C}_i^P \geq \tilde{C}_2^P \geq \cdots \geq \tilde{C}_n^P$, it follows that the affordable sets are nested. Hence the proportion of students (remaining after dropouts) whose second round affordable set is $X_i$ is given by

$$\gamma_i^P = f_{i+1} \left( \tilde{C}_{i+1}^P \right) - f_i \left( \tilde{C}_i^P \right) = \frac{C_i - C_{i+1}}{1 - \rho} = \frac{\gamma_i}{1 - \rho}.$$

For each student type $\theta = (\succ, \succ)$ and set of schools $S$, let $D^\theta(S)$ be the maximal school in $S$ under $\succ$, $D^\theta(S) \geq s \forall s \in S$, and let $\theta' = (\succ, \succ)$ be the student type consistent with $\theta$ that finds all schools unacceptable in the second round. Then for all $i$, a set of students of measure

$$\sum_{j \leq i} \sum_{\theta \in \Theta: D^\theta(X_j) = i} \gamma_j^P (1 - \rho) \zeta(\theta) = \sum_{j \leq i} \gamma_j \sum_{\theta \in \Theta: D^\theta(X_j) = i} \zeta(\theta)$$

choose to go to school $s_i$ in the second round under the second round cutoffs $\tilde{C}^P$, which is the same as the measure of the set of students who choose to go to school $s_i$ in the first round under first round cutoffs $C$. Since $C$ are market-clearing cutoffs, it follows that $\tilde{C}^P$ are too.

Hence $\tilde{C}^P = \tilde{C}^P$ satisfies $\tilde{C}_1^P \geq \cdots \geq \tilde{C}_n^P$, the local order condition is satisfied for all $P$, with schools indexed such that $C_1 \geq C_2 \geq \cdots \geq C_n$, and so the (global) order condition holds.

We next show that Theorem holds with uniform dropouts. Specifically, we show that all PLDA mechanisms give type equivalent allocations.
Lemma 2. In the case of uniform dropouts, all PLDA mechanisms produce type equivalent allocations.

Proof of Lemma 2. We showed in the proof of Lemma 1 that for all \( \lambda \), a set of students of measure
\[
\sum_{j \leq i} \sum_{\theta \in \Theta : D^p(X_j) = i} \gamma_j^P (1 - \rho) \zeta(\theta) = \sum_{j \leq i} \gamma_j \sum_{\theta \in \Theta : D^p(X_j) = i} \zeta(\theta)
\]
choose to go to school \( s_i \) in the second round under the second round cutoffs \( \hat{C}^P \), where
\[
\gamma_i^P = f_{i+1} \left( \hat{C}_{i+1}^P \right) - f_i \left( \hat{C}_i^P \right) = \frac{C_i - C_{i+1}}{1 - \rho} = \frac{\gamma_i}{1 - \rho}
\]
is the proportion of students with affordable set \( X_i \) in the second round with permutation \( P \) and cutoffs \( \hat{C}^P \).

It follows from this analysis that all PLDA mechanisms are ‘type equivalent’ to the first round assignment, in the following sense. For all sets of types \( T = \{(\succ, \succ), (\succ, \prec)\} \) of pairs of consistent types that either retain the same preferences or drop out, and for all schools \( s_i \), it holds that
\[
\eta \left\{ \lambda \in \Lambda : (\succ^\lambda, \prec^\lambda) \in T, \mu^P(\lambda) = s_i \right\} = \eta \left\{ \lambda \in \Lambda : (\succ^\lambda, \prec^\lambda) \in T, \mu(\lambda) = s_i \right\}.
\]
Since, under uniform dropouts, exactly one type from each set \( T \) remains in the system in round 2, and this covers all student types that remain in the system, it follows that \( \mu^P \) is type equivalent to \( \mu^{P'} \) for all permutations \( P, P' \).

B.2 Proof of Theorem 1

We first prove Theorem 1 in the case when all schools have one priority group. We then show that if the order condition holds, all PLDA mechanisms assign the same number of seats at a given school \( s_i \) to students of a given priority type \( p \). Hence we can reduce the general problem to the case when all schools have one priority group by restricting to the set of students with priority type \( p \). This shows that all PLDA mechanisms produce type equivalent allocations.

Lemma 3. Assume each school has a single priority group, \( p = 1 \). If the order condition holds, all PLDA mechanisms produce type equivalent allocations.

Proof. Let \( P \) be a permutation.

Assume the order condition holds. Then the schools in \( S \cup \{ s_{n+1} \} \) can be indexed so that \( C_i \geq C_{i+1} \) and \( \hat{C}_i^P \geq \hat{C}_{i+1}^P \) for all permutations \( P \) (simultaneously).

We first present the relevant notation that will be used in this proof. We are interested in sets of schools of the form \( X_i = \{ s_i, \ldots, s_{n+1} \} \). Let
\[
\beta_{i,j} = \eta(\{ \lambda \in \Lambda : s_i \text{ is the most desirable school in } X_j \text{ with respect to } \succ^\lambda \})
\]
be the measure of the students who, when their set of affordable schools is \( X_j \), will choose \( s_i \) (when following their second round preferences). Note that \( \beta_{i,j} = 0 \) for all \( j > i \).

Let \( E^\lambda(C) \) and \( \hat{E}_P^\lambda(\hat{C}^P) \) be the sets of schools affordable for type \( \lambda \) in the first and second round, respectively, when running PLDA with lottery \( P \). Note that for each student \( \lambda \in \Lambda \), there exists some \( i \) such that \( E^\lambda(C) = X_i \), and since the order condition is satisfied, there exists some \( j \leq i \) such that \( \hat{E}_P^\lambda(\hat{C}^P) = X_j \). The fact that \( \hat{E}_P^\lambda(\hat{C}^P) = X_j \) for some \( j \) is a result of the order condition; our modified scoring function guarantees that \( E^\lambda(C) \subseteq \hat{E}_P^\lambda(\hat{C}^P) \) (every school affordable in the first round is guaranteed in the second) and hence that \( j \leq i \).
Let $\gamma_i^P = \eta(\{\lambda \in \Lambda : \hat{E}_P^\lambda(\hat{C}^P) = X_i\})$ be the fraction of students whose affordable set in the second round of PLDA with permutation $P$ is $X_i$. We note that by definition of PLDA, $\eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \hat{E}_P^\lambda(\hat{C}^P) = X_i\}) = \zeta(\{\theta\})\gamma_i^P$, that is, the students whose affordable sets are $X_i$ “break proportionally” into types. For a school $i$, this means that the measure of students assigned to $s_i$ is therefore $\sum_{j} \beta_{i,j} \gamma_j^P$.

Let $P'$ be another permutation, and define $\gamma_i^{P'}$ similarly. We will prove by induction that there exist PLDA($P'$) cutoffs $\hat{C}^{P'}$ such that $\gamma_j^{P'} = \gamma_j^P$ for all $s_i \in S \cup \{s_{n+1}\}$. Note that by the proportional breaking into types of $\gamma_i^P$ and $\gamma_i^{P'}$, this will imply type equivalence.

Assume that the PLDA($P$) cutoffs $\hat{C}^P$ are chosen such that $\gamma_j^P = \gamma_j^P$ for all $j < i$, and $i$ is maximal such that this is true. Then we have that $\sum_{j \leq i-1} \beta_{i,j} \gamma_j^P = \sum_{j \leq i-1} \beta_{i,j} \gamma_j^{P'}$. Assume w.l.o.g. $\gamma_i^P > \gamma_i^{P'}$. It follows that $q_i \geq \sum_{i \leq j} \beta_{i,j} \gamma_j^P \geq \sum_{j \leq i} \beta_{i,j} \gamma_j^{P'}$, where the first inequality follows since $s_i$ cannot be filled beyond capacity. If the second inequality is strict, then the first inequality is also strict. Then $\hat{C}^{P'} = 0$. However, this means that all students can afford $s_i$ under $P'$, and therefore $\gamma_i^{P'} = 1 - \sum_{j<i} \gamma_j^{P'} = 1 - \sum_{j<i} \gamma_j^P \geq \gamma_i^P$, contradiction. If the second inequality is an equality, then $\beta_{i,i} = 0$ and no students demand school $i$ under the given affordable set structure. It follows that we can define the cutoff $\hat{C}_i^{P'}$ such that $\gamma_i^{P'} = \gamma_i^P$. This provides the required contradiction, completing the proof.

Now consider when schools have possibly more than one priority group. We show that if the order condition holds, then all PLDA mechanisms assign the same measure of students of a given priority type to a given school. It is not at all obvious that such a result should hold, since priority type and student preferences may be correlated, and the relative proportions of students of each priority type assigned to each school can vary widely. Nonetheless, the order condition imposes enough structure so that any given priority type is treated symmetrically across different PLDA mechanisms.

**Theorem 5.** If the order condition holds, then for all priority types $p$ and schools $s_i$ all PLDA mechanisms assign the same measure of students of priority type $p$ to school $s_i$.

**Proof.** Fix a permutation $P$. We show that if the order condition holds, then PLDA($P$) assigns the same measure of students of each priority type to each school $s_i$ as RLDA. The idea will be to define cutoffs on priority-type-specific economies, and show that these cutoffs are the same as the PLDA cutoffs. However, since cutoffs are not necessarily unique in the two-round setting, care needs to be taken to make sure that the individual choices for priority-type-specific cutoffs are consistent across priority types.

The proof runs as follows. We first define an economy $E_p$ for each priority type $p$ that gives only as many seats as assigned to students of priority type $p$ under RLDA. We then invoke the order condition and Theorems 3 and 1 to show that all PLDA mechanisms are type equivalent on each $E_p$. We also use the order condition to argue that it is sufficient to consider affordable sets, and also to select ‘minimal’ cutoffs. Then we construct cutoffs $C_{p,i}$ using the economies $E_p$ and show that they are (almost) independent of priority type. Finally, we show that this means $C_{p,i}$ also define PLDA cutoffs on the large economy $E$ and conclude that PLDA($P$) assigns the same measure of students of each priority type to each school $s_i$ as RLDA.

(1) **Defining little economies $E_p$ on each priority type.**

Fix a priority type $p$. Let $q_p$ be a restricted capacity vector, where $q_{p,i}$ is the measure of students of priority type $p$ assigned to school $s_i$ under RLDA. Let $\Lambda_p$ be the set of students $\lambda$ such that $p^\lambda = p$, and let $\eta_p$ be the restriction of the distribution $\eta$ to $\Lambda_p$. Let $E_p$ denote the primitives

\[\text{Note that } \eta \Lambda = 1, \text{ as } \eta \text{ is a probability distribution over } \Lambda.\]
(S, q_p, Λ_p, η_p). Recall that \( \hat{C}^R \) are the second round cutoffs of RLDA on \( \mathcal{E} \). It follows from the definition of \( \mathcal{E}_p \) that \( \hat{C}^R_p \) are also the second round cutoffs for RLDA on \( \mathcal{E}_p \).

Let \( \tilde{C}^P_p \) be the second round cutoffs of PLDA(\( P \)) on \( \mathcal{E}_p \). We show that the cutoffs \( \tilde{C}^P_p \) defined on the little economy are the same as the consistent second round cutoffs \( \hat{C}^P_p \) for PLDA with permutation \( P \) on the large economy \( \mathcal{E} \), that is \( \tilde{C}^P_p = \hat{C}^P_p \).

(2) Implications of the order condition

We have assumed that the order condition holds.

This has a number of implications for PLDA mechanisms on the little economies \( \mathcal{E}_p \). For all \( p \), the local order condition holds for RLDA on \( \mathcal{E}_p \). Hence by Theorem \( 3 \) the little economies \( \mathcal{E}_p \) each satisfy the order condition. Moreover, by Theorem \( 1 \) all PLDA mechanisms on \( \mathcal{E}_p \) are type equivalent. Finally, if we can show that for every permutation \( P \), PLDA(\( P \)) assigns the same measure of students of each priority type to each school \( s_i \) as RLDA, then \( \mathcal{E} \) satisfies the order condition if and only if for all \( p \) the little economy \( \mathcal{E}_p \) satisfies the order condition.

The order condition also allows us to determine aggregate student demand from the proportions of students who have each school in their affordable set. In general, if affordable sets break proportionally across types, and if for each subset of schools \( S' \subseteq S \) we know the proportion of students whose affordable set is \( S' \), then we can determine aggregate student demand. The order condition implies that for any pair of permutations \( P, P' \), the affordable sets from both rounds are nested in the same order under both permutations. In other words, for each priority type \( p \) there exists a permutation \( \sigma_p \) such that the affordable set of any student in any round of any PLDA mechanism is of the form \( s_{\sigma_p(i)}, s_{\sigma_p(i+1)}, \ldots, s_{\sigma_p(n)}, s_{n+1} \). Hence when the order condition holds, to determine the proportion of students whose affordable set is \( S' \), it is sufficient to know the proportion of students who have each school in their affordable set.

Another more subtle implication of the order condition is the following. In the second round of PLDA, for each permutation \( P \) and school \( s_i \) there will generically be an interval that \( \hat{C}^P_i \) can lie in and still be market-clearing. The intuition is that there will be large empty intervals corresponding to students who had school \( s_i \) in their first round affordable set, and whose second round lottery changed accordingly. When the order condition holds, we can without loss of generality assume that as many of the cutoffs for a given priority type are 0 or 1 as possible, and the order condition will still hold.

Formally, for cutoffs \( C \) we can equivalently define priority-type-specific cutoffs \( C_{p,i} = [C_i - p_i] \). Note the cutoffs \( C_p \) are consistent across priority types, namely

- Cutoffs match for two priority types with the same priority group at a school,
  
  \[ p_i = p_i' \Rightarrow C_{p,i} = C_{p',i} \text{ and } \hat{C}_{p,i} = \hat{C}_{p',i}; \]

- There is at most one marginal priority group at each school,
  
  \[ C_{p,i}, C_{p',i} \in (0, 1) \Rightarrow p_i = p_i'. \]

Moreover, if cutoffs \( C_p \) are consistent across priority types, then there exist cutoffs \( C \) from which they arise.

Suppose that we set as many of the priority-type specific cutoffs \( \hat{C}^P \) to 0 as possible, i.e. we let \( \hat{C}^P_i \) be 0 if all students have \( s_i \) in their affordable set, 1 if none do, and keep it the same otherwise. Then under this new definition, \( C, \hat{C}^P \) satisfies the local order condition consistently with all other PLDAs.

Specifically, let

\[ f^P_{p,i}(x) = |\{l : l \geq C_{p,i} \text{ or } P(l) \geq x\}| \]
be the proportion of students of priority type \( p \) who have school \( s_i \) in their affordable set if the first and second round cutoffs are \( C_{p,i} \) and \( x \) respectively. Notice that \( f \) is decreasing in \( x \). If \( f^{P}_{p,i} \left( \hat{C}^{P}_{p,i} \right) = 1 \) we set \( \hat{C}^{P}_{p,i} = 0 \), and otherwise we keep \( \hat{C}^{P}_{p,i} \) unchanged.

Since \( E \) satisfies the order condition, for all \( p \) there exists an ordering \( \sigma_{p} \) such that \( C_{\sigma_{p}(1)} \geq C_{\sigma_{p}(2)} \geq \cdots \geq C_{\sigma_{p}(n)} \) and \( \hat{C}^{P'}_{\sigma_{p}(1)} \geq \hat{C}^{P'}_{\sigma_{p}(2)} \geq \cdots \geq \hat{C}^{P'}_{\sigma_{p}(n)} \) for all permutations \( P' \). We show that the order condition implies that the newly defined cutoffs \( \hat{C}^{P'} \) satisfy \( \hat{C}^{P'}_{\sigma_{p}(1)} \geq \hat{C}^{P'}_{\sigma_{p}(2)} \geq \cdots \geq \hat{C}^{P'}_{\sigma_{p}(n)} \). This is because the order condition implies that \( f \) is increasing in \( i \), i.e. for each \( p, i < j \) and \( x \) it holds that \( f^{P}_{p,\sigma_{p}(i)}(x) \leq f^{P}_{p,\sigma_{p}(j)}(x) \). Hence if we set \( \hat{C}^{P}_{p,\sigma_{p}(i)} \) to be 0 then for all \( j > i \)

\[
 f^{P}_{p,\sigma_{p}(j)} \left( \hat{C}^{P}_{p,\sigma_{p}(j)} \right) \geq f^{P}_{p,\sigma_{p}(j)} \left( \hat{C}^{P}_{p,\sigma_{p}(i)} \right) \quad \text{(since } f \text{ is decreasing)}
\]

\[
 \geq f^{P}_{p,\sigma_{p}(i)} \left( \hat{C}^{P}_{p,\sigma_{p}(i)} \right) \quad \text{(since } f \text{ is increasing in } i)
\]

\[
 = 1,
\]

so we set \( \hat{C}^{P}_{p,\sigma_{p}(j)} \) to be 0. In other words, we have essentially taken a set of the smallest cutoffs and set them all to 0.

(3) **Cutoffs \( \hat{C}^{P}_{p,i} \) are (almost) independent of priority type.**

We now show that \( \hat{C}^{P}_{p,i} \) depends on \( p \) only via \( p_i \), and does not depend on \( p_j \) for all \( j \neq i \). We will use the fact that \( E_{p} \) satisfies the order condition, and characterize \( \hat{C}^{P}_{p,i} \) using an affordable set argument.

Now since \( E_{p} \) satisfies the order condition, all PLDA mechanisms on \( E_{p} \) are type equivalent, and the proportion of students who have each school in their affordable set is the same across all PLDA mechanisms. In other words, for all permutations \( P, P' \), priority types \( p \) and schools \( s_i \) it holds that \( f^{P}_{p,i} \left( \hat{C}^{P}_{p,i} \right) = f^{P'}_{p,i} \left( \hat{C}^{P'}_{p,i} \right) \).

We write this in terms of RLDA as follows. Note that

\[
 f^{R}_{p,i}(x) = (1 - x) + \min\{x, (1 - C_{p,i})\}.
\]

- **Case 1:** \( f^{R}_{p,i} \left( \hat{C}^{R}_{p,i} \right) = 0 \).
  Then no students with priority type \( p \) have \( s_i \) in their second round affordable set, so \( \hat{C}^{P}_{p,i} = 1 \).

- **Case 2:** \( f^{R}_{p,i} \left( \hat{C}^{R}_{p,i} \right) \in (0, 1) \).
  Then \( \hat{C}^{P}_{p,i} \) satisfies the following equation in terms of \( \hat{C}^{R}_{p,i}, C_{p,i} \) and \( P \):

\[
 f^{P}_{p,i} \left( \hat{C}^{P}_{p,i} \right) = f^{R}_{p,i} \left( \hat{C}^{R}_{p,i} \right) = 2 - C_{p,i} - \hat{C}^{R}_{p,i}.
\]

We note that an application of the intermediate value theorem shows that this equation always has a solution in \([0, 1]\), since \( f^{P}_{p,i}(0) = 1 - C_{p,i}, f^{P}_{p,i}(1) = 1 \), \( f_{p,i} \) is continuous and decreasing on \([0, 1]\), and we are in the case when \( 1 - C_{p,i} < \hat{C}^{R}_{p,i} < 1 \).

Hence \( \hat{C}^{P}_{p,i} \) is defined by \( f^{P}_{p,i} \) and \( f^{R}_{p,i} \).

- **Case 3:** \( 0 \leq \hat{C}^{R}_{p,i} \leq 1 - C_{p,i} \).
  In this case, all students with priority type \( p \) have \( s_i \) in their second round affordable set and we set \( \hat{C}^{P}_{p,i} = 0 \).
In all three cases, the value of \( \tilde{C}^p_{p,i} \) depends on \( f^P_{p,i}(\cdot), f^R_{p,i}(\cdot) \) and \( \tilde{C}^P_{p,i} \), which depend only on \( C_{p,i} \) and the permutations \( P \) or \( R \). Since \( C_{p,i} \) depends on \( p \) only through \( p_i \), it follows \( \tilde{C}^P_{p,i} \) depends on \( p \) only through \( p_i \).

Hence if \( p, p' \) are two priority vectors such that \( p_i = p'_i \), then \( \tilde{C}^P_{p,i} = \tilde{C}^P_{p',i} \), so the \( \tilde{C}^P_i \) are consistent across priority types. Hence the \( \tilde{C}^P_{p,i} \) define cutoffs \( \tilde{C}^P_i \) that are independent of priority type.

(4) \( \tilde{C}^P_i \) are the PLDA cutoffs.

Finally, we remark that \( \tilde{C}^P_i \) are market-clearing cutoffs. This is because we have shown that for each priority type \( p \), the number of students assigned to each school \( s_i \) is the same under RLDA and under the demand induced by the cutoffs \( \tilde{C}^P_i \), and we know that the RLDA cutoffs are market-clearing for \( E \).

Hence \( \tilde{C}^P_i \) give the assignments for PLDA on \( E \), and since \( \tilde{C}^P_i \) was defined individually for each priority type \( p \) on \( E_p \) it follows that PLDA(\( P \)) assigns the same measure of students of each priority type to each school \( s_i \) as RLDA.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Fix a priority type \( p \). By Theorem 3 for every school \( s_i \), all PLDA mechanisms assign the same measure \( \eta_p,\lambda \) of students of priority type \( p \) to school \( s_i \).

Consider the subproblem with primitives \( E_p = (S, q_p, \Lambda_p, \eta_p) \). By Lemma 3 for all \( \theta \in \Theta \) and \( s_i \),

\[
\eta_p(\{ \lambda \in \Lambda_p : \theta^\lambda = \theta, \mu_p(\lambda) = s_i \}) = \eta_p(\{ \lambda \in \Lambda_p : \theta^\lambda = \theta, \tilde{\mu}_p(\lambda) = s_i \}).
\]

Since \( \eta_p \) is the restriction of \( \eta \) to \( \lambda_p \), it follows that all PLDA mechanisms are globally type equivalent.

B.3 Proof of Theorem 2

Proof of Theorem 2. Fix \( \theta = (\succ^\theta, \succeq^\theta, p^\theta) \in \Theta \) and school \( s_i \in S \). We will show that \( R \) minimizes the measure of reassigned students with type \( \theta \) who were assigned to school \( s_i \) in the first round.

The idea is that reversing the lottery shortcuts improvement chains within a particular type, moving one student many schools up their preference list instead of moving many students each a few schools up their preference list. We make this formal.

Let \( P \) be a permutation. Let the measure of students with type \( \theta \) leaving school \( s_i \) in the second round under PLDA(\( P \)) be denoted by \( \ell_P = \eta(\{ \lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) = s_i, \mu(\lambda) \neq s_i \}) \). Let the measure of students with type \( \theta \) entering school \( s_i \) underpermuation \( P \) be denoted \( e_P = \eta(\{ \lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) \neq s_i, \mu(\lambda) = s_i \}) \). Due to type equivalence, there is a constant \( c \), independent of \( P \), such that \( \ell_P = e_P - c \). If \( s_{i+1} \succ^\theta s_i \), then there are no reassigned students with type \( \theta \) entering school \( s_i \) under either PLDA(\( P \)) or RLDA, since all students of type \( \theta \) prefer to be unassigned. So assume \( s_i \succ^\theta s_{i+1} \).

We will show that \( \ell_R \leq \ell_P \) for all permutations \( P \). If \( e_R = 0 \) and no students of type \( \theta \) enter \( s_i \) under RLDA, then \( \ell_R = e_R - c \leq e_P - c = \ell_P \) for all permutations \( P \), and fewer students of type \( \theta \) leave \( s_i \) under RLDA. So assume \( e_R > 0 \). We claim that in this case \( \ell_R = 0 \), which will complete our proof. We will show that having both \( e_R > 0 \) and \( \ell_R > 0 \) means that students who entered \( s_i \) in the second round of RLDA had worse first and second round lottery numbers than students who left \( s_i \) in the second round of RLDA, which contradicts the reversal of the lottery.

Since \( e_R > 0 \), there exists some student \( \lambda \in \Lambda \) with type \( \theta^\lambda = \theta \) for which \( s_i = \tilde{\mu}_p(\lambda) \succ^\theta \mu(\lambda) \). By consistency, we have \( s_i \succ^\theta \mu(\lambda) \), and therefore \( \lambda \) could not afford (meet the cutoff for) \( s_i \) in
the first round. If \( \ell_R = 0 \) we are done. If \( \ell_R > 0 \), there exists some student \( \lambda' \in \Lambda \) with type \((\succ^{\lambda'}, \succeq^{\lambda'}, p^{\lambda'}) = \theta^{\lambda'}\) for which \( s_j = \hat{\mu}_R(\lambda') \succeq^\theta \mu(\lambda') = s_i \).

By definition, \( \lambda' \) could afford \( s_i \) in the first round and \( \lambda \) could not, and hence \( L(\lambda') > L(\lambda) \). Note that since \( s_i \succeq^\theta s_{n+1} \), then \( s_j \succeq^\theta s_{n+1} \), and thus consistency gives that \( s_j >^\theta s_i \).

Now since \( \lambda' \) received a better second round assignment under RLDA than \( \lambda \), \( \hat{\mu}_R(\lambda') >^\theta \hat{\mu}_R(\lambda) \), and both \( \lambda \) and \( \lambda' \) were reassigned under RLDA, it follows that \( R(L(\lambda')) > R(L(\lambda)) \).

This contradicts that \( L(\lambda') > L(\lambda) \), and therefore \( \ell_R = 0 \), completing our proof. \( \square \)

### B.4 Proof of Theorem 3

**Proof of Theorem 3.** In what follows, we will fix a permutation \( P \) and show that the PLDA mechanism with permutation \( P \) satisfies the order condition and is type equivalent to the reverse lottery RLDA mechanism.

1. **Every school has a single priority group**

   We first consider the case when \( n_i = 1 \) for all \( i \), that is, every school has a single priority group. Recall that the schools are indexed according to the first round overdemand ordering, so that \( C_1 \geq C_2 \geq \cdots \geq C_n \geq C_{n+1} \). Since the order condition holds for the reverse lottery \( R \), let us assume that they are also indexed according to the second round overdemand ordering under RLDA, so that \( \hat{C}_1 \geq \hat{C}_2 \geq \cdots \geq \hat{C}_n \geq \hat{C}_{n+1} \).

   The idea will be to construct a set of cutoffs \( \hat{C}^P \) directly from the permutation \( P \) and the cutoffs \( \hat{C}^R \), show that the cutoffs are in the correct order \( \hat{C}^P_1 \geq \hat{C}^P_2 \geq \cdots \geq \hat{C}^P_n \geq \hat{C}^P_{n+1} \) and show that the cutoffs \( \hat{C}^P \) and resulting assignment are market-clearing when school preferences are given by the amended scoring function with permutation \( P \).

2a. **Definitions**

   As in the proof of Theorem 1, let \( \beta_{i,j} = \eta(\{\lambda \in \Lambda : \arg\max_{\succeq^\lambda} X_j = s_i\}) \) be the measure of students who, when their set of affordable schools is \( X_j \), will choose \( s_i \). Let \( E^\lambda(C) \) be the set of schools affordable for type \( \lambda \) in the first round under PLDA with any permutation, let \( \hat{E}^\lambda(\hat{C}^R) \) be the set of schools affordable for type \( \lambda \) in the second round under RLDA, and let \( \hat{E}^\lambda(\hat{C}^P) \) be the set of schools affordable for type \( \lambda \) in the second round under PLDA with permutation \( P \).

   Let \( \gamma_i^R = \eta(\{\lambda \in \Lambda : \hat{E}^\lambda(C^R) = X_i\}) \) be the fraction of students whose affordable set in the second round of RLDA is \( X_i \), and let \( \gamma_i^P = \eta(\{\lambda \in \Lambda : \hat{E}^\lambda(C^P) = X_i\}) \) be the fraction of students whose affordable set in the second round of PLDA with permutation \( P \) is \( X_i \).

   Let \( \hat{n} \) be the smallest index such that \( s_{\hat{n}} \) does not reach capacity when not offered to all the students. In other words, \( \hat{n} \) is the smallest index such that every student has school \( s_{\hat{n}} \) in their (second round) affordable set under RLDA, i.e. \( s_{\hat{n}} \in \hat{E}^\lambda(C^R) \). Since the order condition holds for RLDA, we may equivalently express \( \hat{n} \) in terms of cutoffs as the smallest index such that \( (1 - C_{\hat{n}}) + (1 - \hat{C}_{\hat{n}}^R) \geq 1 \). Such \( \hat{n} \) always exists, since every student has the outside option \( s_{n+1} \) in their total affordable set.

2b. **Defining cutoffs for PLDA**

   Let us define cutoffs \( \hat{C}^P \) as follows. For \( i \geq \hat{n} \) let \( \hat{C}_i^P = 0 \). For each permutation \( P \), define a function

   \[
   f_i^P(x) = |\{l : l \geq C_i \text{ or } P(l) \geq x\}|
   \]

   representing the proportion of students who have \( s_i \) in their (second round) affordable set with first and second round cutoffs \( C_i, x \) under the amended scoring function with permutation \( P \). Since \( P \) is measure-preserving, \( f_i^P(x) \) is continuous and monotonically decreasing in \( x \).
For $i < \hat{n}$, we inductively define $\tilde{C}^P_i$ to be the largest real smaller than $\tilde{C}^P_{i-1}$ satisfying
\[ f_i^P(\tilde{C}^P_i) = f_i^R(\tilde{C}^R_i) \]
(2)
where we define $\tilde{C}^P_0 = 1$. Now $f_i^P(0) = 1 \geq f_i^R(\tilde{C}^R_i)$, and
\[
\begin{align*}
  f_i^P(\tilde{C}^P_{i-1}) &= f_i^P(\tilde{C}^P_{i-1}) + |\{l \mid l \in [C_i, C_{i-1}) \text{ and } P(l) \geq \tilde{C}^P_{i-1}\}| \\
  &\leq f_i^R(\tilde{C}^R_{i-1}) + (C_{i-1} - C_i) \\
  &= (1 - C_i) + (1 - \tilde{C}^R_{i-1}) \\
  &\leq f_i^R(\tilde{C}^R_i)
\end{align*}
\]
where in the first equality we are using that $C_{i-1} \geq C_i$, the first inequality follows from the definition of $\tilde{C}^P_{i-1}$, and the last inequality since $\tilde{C}^R_{i-1} \geq \tilde{C}^R_i$.

It follows from the intermediate value theorem that the cutoffs $\tilde{C}^P_i$ are well-defined and satisfy $\tilde{C}^P_1 \geq \tilde{C}^P_2 \geq \cdots \geq \tilde{C}^P_{\hat{n}} \geq \tilde{C}^P_{\hat{n}+1}$.

(1c) The constructed cutoffs clear the market

We show that the cutoffs $\tilde{C}^P_i$ and resulting assignment (from letting students choose their favorite school out of those for which they meet the cutoff) are market-clearing when the second round scores are given by $\tilde{r}_i^\lambda = P(L(\lambda)) + n_i 1_{\{\lambda \geq C_i\}} + p_i 1_{\{\lambda \geq C_i\}}$. We call the mechanism with this second round assignment $M^P$.

The idea is that since the cutoffs $\tilde{C}^P_i$ are decreasing in the same order as $C_i$ and $\tilde{C}^R_i$, the (second round) affordable sets are nested in the same order under both sets of second round cutoffs. It follows that aggregate student demand is uniquely specified by the proportion of students with each school in their affordable set, and that we have defined these to be equal, $f_i^P(\tilde{C}^P_i) = f_i^R(\tilde{C}^R_i)$. It follows that $\tilde{C}^P_i$ are market-clearing and give the PLDA($P$) cutoffs, so PLDA($P$) satisfies the order condition (with the indices indexed in the same order as with RLDA). We make the affordable set argument explicit below.

Consider the proportion of lottery numbers giving a second round affordable set $X_i$. Since $\tilde{C}^R_1 \geq \tilde{C}^R_2 \geq \cdots$, under RLDA this is given by
\[
\gamma_i^R = f_{i+1}^R(\tilde{C}^R_{i+1}) - f_i^R(\tilde{C}^R_i),
\]
if $i < \hat{n}$ and 0 if $i \geq \hat{n}$, where we define $f_0^0(x) = 1$ for all $P$ and $x$. Similarly, since $\tilde{C}^P_1 \geq \tilde{C}^P_2 \geq \cdots$, under $M^P$ this is given by
\[
\gamma_i^P = f_{i+1}^P(\tilde{C}^P_{i+1}) - f_i^P(\tilde{C}^P_i)
\]
if $i < \hat{n}$ which is precisely $\gamma_i^R$, and 0 if $i \geq \hat{n}$.

Hence, for all $i < \hat{n}$, the measure of students assigned to school $s_i$ under both RLDA and $M^P$ is $\sum_{j \leq i} \beta_{i,j} \gamma_j^R = q_i$, and for all $i \geq \hat{n}$, the measure of students assigned to school $s_i$ is $\sum_{j \leq \hat{n}} \beta_{i,j} \gamma_j^R < q_i$. It follows that the cutoffs $\tilde{C}^P_i$ are market-clearing cutoffs when the second round scores are given by $\tilde{r}_i^\lambda = P(L(\lambda)) + n_i 1_{\{\lambda \geq C_i\}} + p_i 1_{\{\lambda \geq C_i\}}$, so PLDA($P$) = $M^P$ satisfies the order condition.

(2) Some school has more than one priority group.

Now consider when schools have possibly more than one priority group. We show that if RLDA satisfies the order condition, then PLDA with permutation $P$ assigns the same number of students of each priority type to each school $s_i$ as RLDA, and within each priority type assigns
the same number of students of each preference type to each school as RLDA. We do this by first assuming that PLDA with permutation \( P \) assigns the same number of students of each priority type to each school \( s_i \) as RLDA, and showing that this gives consistent cutoffs.

We note that this proof is uses very similar arguments to the proof of Theorem 5.

(2a) **Defining little economies \( \mathcal{E}_p \) on each priority type.**

Fix a priority type \( p \). Let \( q_p \) be a restricted capacity vector, where \( q_{p,i} \) is the measure of students of priority type \( p \) assigned to school \( s_i \) under RLDA. Let \( \Lambda_p \) be the set of students \( \lambda \) such that \( p^\lambda = p \), and let \( \eta_p \) be the restriction of the distribution \( \eta \) to \( \Lambda_p \). Let \( \mathcal{E}_p \) denote the primitives \((S, q_p, \Lambda_p, \eta_p)\).

Let \( \tilde{C}_p^P \) be the second round cutoffs of PLDA(\( P \)) on \( \mathcal{E}_p \). By definition \( \hat{C}_p^P \) are the second round cutoffs of RLDA on \( \mathcal{E}_p \). We show that the cutoffs \( \tilde{C}_p^P \) defined on the little economy are the same as the consistent second round cutoffs \( \hat{C}_p^P \) for PLDA(\( P \)) on the large economy \( \mathcal{E} \), that is \( \tilde{C}_p^P = \hat{C}_p^P \).

(2b) **Implications of RLDA satisfying the local order condition**

Since RLDA satisfies the local order condition on \( \mathcal{E} \), RLDA also satisfies the local order condition on \( \mathcal{E}_p \) for all \( p \). It follows from (1) that the order condition holds on each of the little economies \( \mathcal{E}_p \). Hence by Theorem 1 all PLDA mechanisms on \( \mathcal{E}_p \) are type equivalent. Moreover, as in the proof of Theorem 5, the order condition on \( \mathcal{E}_p \) also allows us to determine aggregate student demand in \( \mathcal{E}_p \) from the proportions of students who have each school in their affordable set.

Finally, as in the proof of Theorem 5 we may assume that for each \( p \) and school \( s_i \) the cutoff \( \tilde{C}_{p,i}^P \) is the minimal real satisfying

\[
f_{p,i}^P (\tilde{C}_{p,i}^P) = f_{p,i}^R (\tilde{C}_{p,i}^R)
\]

where for each permutation \( P \)

\[
f_{p,i}^P(x) = |\{ l : l \geq C_{p,i} \text{ or } P(l) \geq x \}|
\]

is the proportion of students of priority type \( p \) who have school \( s_i \) in their affordable set if the first and second round cutoffs are \( C_{p,i} \) and \( x \) respectively.

It follows that \( \tilde{C}_{p,i}^P \) depends on \( p \) only via \( p_i \), and does not depend on \( p_j \) for all \( j \neq i \). This is because the value of \( \tilde{C}_{p,i}^P \) depends on \( f_{p,i}^P(\cdot) \), \( f_{p,i}^R(\cdot) \) and \( \tilde{C}_{p,i}^R \), which depend only on \( C_{p,i} \) and the permutations \( P \) or \( R \). Moreover, \( C_{p,i} \) depends on \( p \) only through \( p_i \). Hence if \( p, p' \) are two priority vectors such that \( p_i = p'_i \), then \( \tilde{C}_{p,i}^P = \tilde{C}_{p',i}^P \), so the \( \tilde{C}_{p,i}^P \) are consistent across priority types. Hence the \( \tilde{C}_{p,i}^P \) define cutoffs \( \tilde{C}_i^P \) that are independent of priority type.

(3) \( \tilde{C}_i^P \) are the PLDA cutoffs.

Finally, we show that \( \tilde{C}_i^P \) are market-clearing cutoffs. By (1), for each priority type \( p \), the number of students assigned to each school \( s_i \) is the same under RLDA and under the demand induced by the cutoffs \( \tilde{C}_i^P \), and we know that the RLDA cutoffs are market-clearing for \( \mathcal{E} \).

Hence \( \tilde{C}_i^P \) give the assignments for PLDA on \( \mathcal{E} \), and since \( \tilde{C}_i^P \) was defined individually for each priority type \( p \) on \( \mathcal{E}_p \) it follows that PLDA(\( P \)) assigns the same measure of students of each priority type to each school \( s_i \) as RLDA.
B.5 Proof of Theorem 4

Proof of Theorem 4. Let a permuted lottery stable matching mechanism (PLSM) be a reassignment mechanism that is non-atonic and outputs a matching that is stable under the same second round school preferences as in PLDA. In a slight abuse of notation, we will also let PLSM refer to the two-round mechanism defined by this reassignment mechanism. We need to show that PLSM satisfies the axioms and that any mechanism satisfying the axioms is a PLSM.

We first recall the cutoff characterization of the set of stable matchings for given student preferences and responsive school preferences (encoded by student scores \( r \)), as provided by Azevedo and Leshno [2014]. Namely, if \( C \in \mathbb{R}^{n+1} \) is a vector of cutoffs, let the assignment \( \mu \) defined by \( C \) be given by assigning each student of type \( \lambda \) to her favorite school among those where her score weakly exceeds the cutoff, \( \mu(\lambda) = \max_{\lambda' < \lambda} \{ s_i \in S \cup \{s_{n+1}\} : r_{i}^{\lambda} \geq C_i \} \). We say that \( C \) is market-clearing if under the assignment \( \mu \) defined by \( C \), every school with a positive cutoff is exactly at capacity, \( \eta(\mu(s_i)) \leq q_i \) for all \( s_i \in S \cup \{s_{n+1}\} \), with equality if \( C_i > 0 \). Then the set of all stable matchings is precisely given by the set of assignments defined by market-clearing vectors Azevedo and Leshno [2014].

Recall also that under PLDA with permutation \( P \), a student of type \( \lambda \) has a second round score \( \tilde{r}_i^\lambda = P(L(\lambda)) + 1_{\{L(\lambda) \geq C_i\}} \) at school \( s_i \) for each school \( s_i \in S \cup \{s_{n+1}\} \). In a slight abuse of notation, we will sometimes let \( \tilde{C}^P \) refer to the second round cutoffs from some fixed PLSM with permutation \( P \) (not necessarily corresponding to the student-optimal stable matching given by PLDA).

(1) Any PLSM satisfies the axioms:

Fix a permutation \( P \) and some PLSM with permutation \( P \). We first show that this particular PLSM satisfies all the axioms.

Let \( \eta \) be a distribution of students, and let \( \tilde{C}^P \) be the second round cutoffs corresponding to the assignment given by the PLSM for this distribution of student types.

PLSM respects guarantees because fewer students are guaranteed at each school than the capacity of the school. PLSM is non-wasteful because the second round finds a stable matching where all schools find all students acceptable, which is non-wasteful.

We now show that the PLSM is two-round strategyproof. Since students are non-atonic, no student can change the cutoffs \( \tilde{C}^P \) by changing their first or second round reports. Hence it is a dominant strategy for all students to report truthfully in the second round. Moreover, for any student of type \( \lambda \), the only difference between having a first round guarantee at a school \( s_i \) and having no first round guarantee is that in the former, \( \hat{r}_i^\lambda \) increases by 1. This means that having a guarantee at a school \( s_i \) changes the student’s second round assignment in the following way. She receives school \( s_i \) when she would have otherwise received a seat in some school \( s_j \) that she reported preferring less to \( s_i \), and her second round assignment is unchanged otherwise. Therefore students want their first round guarantee to be the best under their second round preferences, so it is a dominant strategy for students with consistent preferences to report truthfully in the first round.

PLSM is Pareto efficient among reassigned students, since we use single tie-breaking and the output is stable with respect to the second round lotteries \( \hat{\tilde{r}} \). This is easily seen via the cutoff characterization. Without guarantees, if a student gets a assigned a seat at a school \( s_i \) when they prefer \( s_{i+1} \), then \( \tilde{C}_{i}^P < \tilde{C}_{i+1}^P \), so there cannot be a cycle in the preferences of such students.

We make the above intuition formal. Fix the first round allocation \( \mu \). Suppose there exists a Pareto-improving cycle, that is a set of students \( \lambda_1, \lambda_2, \ldots, \lambda_m \in A \) and schools \( s_1, s_2, \ldots, s_m \in S \), such that \( \mathbb{P}(\hat{\mu}(\lambda_i) = s_i|\mu, \hat{\tilde{r}}>\hat{\tilde{s}}) > 0 \) for all \( i \) and \( s_{i+1} \hat{\sim}_{\lambda_i} s_i \) for all \( i \) (where we define \( s_{m+1} = s_1 \)). Suppose for the sake of contradiction that there is no \( i \) such that \( \mu(\lambda_i) = s_i \) for some \( i \).
(or \( \lambda_i \succ_s \lambda_{i-1} \) for some \( i \)). Then, if \( l_i \) is the first round lottery number of student \( \lambda_i \) at school \( s_i \) (which is a random variable), and \( \hat{r}_i \) is the second round lottery number of student \( \lambda_i \) at school \( s_i \), then \( \hat{r}_i = P(l_i) \) for all \( i \). Moreover, since \( s_{i+1} \sim \lambda_i, s_i \) but \( P(\hat{\mu}(\lambda_i) = s_i | \mu, \succ, \hat{\succ}) > 0 \), it follows that the second round lottery number is in the interval \([\hat{C}_{i}^{P}, \hat{C}_{i+1}^{P}]\) (where we define \( \hat{C}_{i}^{P} = \hat{C}_{i}^{P} \)) with positive probability. Since \( P \) is fixed, we may also assume that the cutoffs \( \hat{C}_{i}^{P} \) are fixed and so this interval is non-empty for all \( i \). But this is clearly a contradiction.

Averaging follows from the continuum model, which preserves the relative proportion of students with different reported types under random lotteries and permutations of random lotteries. Anonymity and report-updating are easily checked.

(2) Any mechanism satisfying the axioms is a PLSM:

We now show that any mechanism \( M \) satisfying the axioms is a PLSM in a particular sense. We will show specifically that if we assume that each instantiation of \( M \) provides type equivalent output, then \( M \) is type equivalent to a PLSM. Moreover, if we assume that conditional on their reports, students’ assignments under \( M \) are uncorrelated, we are able to explicitly construct a PLSM that provides the same output as \( M \). We provide a sketch of the proof before fleshing out the details.

Fix a distribution of student types \( \zeta \). Since the first round of our mechanism \( M \) is deferred acceptance with uniform-at-random single tie-breaking and \( M \) is anonymous, this gives a distribution \( \eta \) of students that is the same (up to relabeling of students) at the end of the first round. For a fixed labeling of students, it also gives a distribution over first round assignments \( \mu \) and a distribution over second round assignments \( \hat{\mu} \).

We first invoke averaging to assume that all ensuing constructions of aggregate cutoffs and measures of students assigned to pairs of schools in the two rounds are deterministic. Specifically, since the first round assignment \( \mu \) is given by uniform-at-random single tie-breaking, and the mechanism satisfies the averaging axiom, we may assume that each pathwise realization of the mechanism gives type equivalent (two-round) allocations. Hence, for the majority of the proof we perform our constructions of aggregate cutoffs and measures of students pathwise, and assume that any realization produces the same cutoffs and measures of students. (In particular, the quantities \( \hat{C}_{i}, \rho_{i,j}, \gamma_{i,j} \) that we will later define will be the same across all realizations.)

Sketch of Proof:

We use Pareto efficiency among reassigned students to construct a first round overdemand ordering \( s_1, s_2, \ldots, s_n, s_{n+1} \) and a permutation \( \sigma \) giving the second round overdemand ordering \( s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n+1)} \) as in [Ashlagi and Shi 2014a], where school \( s \) comes before \( s' \) in an ordering for the first (second) round if there exists a non-zero measure of students who prefer school \( s \) to \( s' \) in the first (second) round but who are assigned to \( s' \) in the first (second) round. (In the case of the second round ordering, we require that these students’ second round assignments \( s' \) not be the same as their first round guarantee.) The existence of these orderings follow from the fact that the first round mechanism, DA with a single priority class and uniform-at-random single tie-breaking, is Pareto efficient, combined with the fact that the two-round mechanism is Pareto efficient among reassigned students. We let \( X_i = \{s_i, s_{i+1}, \ldots, s_{n+1}\} \) denote the set of schools after \( s_i \) in the first round overdemand ordering, and \( \tilde{X}_i = \{s_{\sigma(i)}, s_{\sigma(i+1)}, \ldots, s_{\sigma(n+1)}\} \) denote the set of schools after \( s_{\sigma(i)} \) in the second round overdemand ordering.

We next note that instead of assignments \( \mu \) and \( \hat{\mu} \), we can think of giving students first and second round affordable sets \( E(\lambda), \tilde{E}(\lambda) \) so that \( \mu \) and \( \hat{\mu} \) are given by letting each student choose their favorite school in their affordable set for that round. We use two-round strategy-proofness, report-updating and anonymity to show that two students of different types face the same joint distribution over first and second round affordable sets. This allows us to construct the permutation \( P \) by constructing proportions \( \gamma_{i,j} \) of students whose first round affordable set
was \( X_i \) and (second round) affordable set was \( \tilde{X}_j \). This is the most technical step in the proof, and so we separate it into several steps. We first define a ‘prefix property’ and show that it holds (Lemma 4.4). We then use this prefix property to show that it is possible to construct the proportions \( \gamma_{i,j} \) so that they are both the same for every student type \( \theta \), and consistent with the proportion of students in each type assigned to each pair of schools jointly over the first and second round assignments.

We then construct the lottery \( L \) and verify that if second round school preferences \( \succeq^S \) are given by first prioritizing all guaranteed students over non-guaranteed students and subsequently breaking ties according to the permuted lottery \( P \circ L \), then defining the first and second round affordable sets via cutoffs given by \( \gamma_{i,j} \) is consistent with the first round assignments and gives a second round assignment that is stable with respect to second round school and student preferences.

**Formal Proof:**

We now present the formal proof. Since we are assuming that the considered mechanism \( M \) is strategy-proof, we assume that students report truthfully and so consider preferences instead of reported preferences. We will explicitly specify when we are considering the possible outcomes of a single student misreporting.

Let the schools be numbered \( s_1, s_2, \ldots, s_n \) such that \( C_i \geq C_{i+1} \) for all \( i \). The intuition is that this is the order in which they reach capacity in the first round. We observe that all reassignments are index-decreasing. That is, for all \( s, s' \), if there exists a non-zero measure of students who are assigned to \( s \) in the first round and \( s' \) in the second round, and \( s' \neq s_{n+1} \), then \( s = s_i \) and \( s' = s_j \) for some \( i \geq j \). This follows since the mechanism respects guarantees, student preferences are consistent, and the schools are indexed in order of increasing first-round affordability. Throughout this section we will denote the outside option \( s_{n+1} \) either by \( s_0 \) or \( \emptyset \), to make it more evident that indices are decreasing.

Next, we define a permutation \( \sigma \) on the schools. We think of this as giving a second round overemand (or inverse affordability) ordering, where the schools fill in the order \( s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)} \) in the second round, and which we will eventually show gives the same outcome as a PLSM with cutoffs \( \hat{C}_{\sigma(1)} \geq \hat{C}_{\sigma(2)} \geq \ldots \). For all \( s, s' \), if there exists a non-zero measure of students with consistent preferences who have second round preference reports \( \succ \) such that \( s \succ s' \), and are not assigned to \( s \) in the first round, but are assigned to \( s' \) in the second round, then \( s = s_{\sigma(i)}, s' = s_{\sigma(j)} \) for some \( i < j \). We assume that \( \sigma \) is the unique permutation satisfying this that is maximally order preserving. That is, for all pairs of schools \( s_i, s_j \) for which no non-zero measure of students of the above type exist, \( \sigma(i) < \sigma(j) \) iff \( i < j \). We also define \( \sigma(n+1) = n+1 \).

An ordering \( \sigma \) with the required properties exists since the mechanism is Pareto efficient among reassigned students, so every trading cycle involves a student swapping a share in their first round guarantee.

Let \( S' \) be a set of schools, and let \( \succ \) be a preference ordering over all schools. We say that \( S' \) is a prefix of \( \succ \) if \( s' \succ s \) for all \( s' \in S', s \not\in S' \). For a set of schools \( S' \), let \( i(S') = \max \{ j : s_j \in S' \} \) be the maximum index of a school in \( S' \). We may think of \( i(S') \) as the index of the most affordable school in \( S' \) in the first round.

For a student type \( \theta = (\succ, \tilde{\succ}) \), an interval \( I \subseteq [0, 1] \) and a set of schools \( S' \), let \( \rho^\theta(I, S') \) be the proportion of students with type \( \theta \) who, under the mechanism \( M \), have first round lottery in the interval \( I \) and are assigned to a school in \( S' \) in the second round. When \( S' = \{ s' \} \) we will sometimes write \( \rho^\theta(I, s') \) instead of \( \rho^\theta(I, \{ s' \}) \). In this section, for brevity, when defining preferences \( \succ \) we will sometimes write \( \succ : [s_{i_1}, s_{i_2}, \ldots, s_{i_k}] \) instead of \( s_{i_1} \succ s_{i_2} \succ \cdots \succ s_{i_k} \).

\footnote{Here we are assuming that this proportion is the same for every realization of the first round of \( M \). This requires non-atomicity and anonymity.}
We now construct the permutation \( P \) as follows. For all pairs of indices \( i, j \), we define a scalar \( \gamma_{i,j} \), which we will show can be thought of as the proportion of students (of any type) whose first round affordable set is \( X_i \) and whose second round affordable set is \( \tilde{X}_j \).

Now, for all pairs of indices \( i, j \) such that \( \sigma(j) < i \), we define student preferences \( \theta_{i,j} = (\succ_{i,j}, \succsim_{i,j}) \) such that

\[
\succ_{i,j} : [s_{\sigma(j)}, s_{i-1}, s_i, s_{n+1}] \quad \text{and} \quad \succsim_{i,j} : [s_{\sigma(j)}, s_{n+1}],
\]

with all other schools unacceptable. (We remark that in the case where \( \sigma(j) = i - 1 \), the first two schools in this preference ordering coincide.) We note that the full support assumption implies that there is a positive measure of such students. Let \( \rho_{i,j} \) be the proportion of students of type \( \theta_{i,j} \) whose first round assignment is \( s_i \) and whose second round assignment is school \( s_{\sigma(j)} \). Intuitively, \( \rho_{i,j} \) is the proportion of students who can deduce that their lottery number is in the interval \([C_i, C_{i-1}]\), and whose second round affordable set contains \( \tilde{X}_j \).

For a fixed index \( i \), we define \( \gamma_{i,j} \) for \( j = 1, 2, \ldots, n \) to be the unique solutions to the following \( n \) equations:

\[
\begin{align*}
\gamma_{i,j} &= 0 & \text{for all } j \text{ such that } \sigma(j) \geq i, \\
\gamma_{i,1} + \cdots + \gamma_{i,j} &= \rho_{i,j} & \text{for all } j \text{ such that } \sigma(j) < i.
\end{align*}
\]

We may intuitively think of \( \gamma_{i,j} \) as the proportion of students of type \( \theta_{i,j} \) whose first round lottery is in \([C_i, C_{i-1}]\) and whose second round affordable set contains \( s_{\sigma(j)} \) but not \( s_{\sigma(j-1)} \). If \( \sigma(j) \geq i \) then this would include all students whose first round lottery is in \([C_i, C_{i-1}]\), and so we define \( \gamma_{i,j} = 0 \).

We also define \( \gamma_{i,n+1} \) to be

\[
\gamma_{i,n+1} = C_{i-1} - C_i - \sum_{j=1}^{n} \gamma_{i,j}.
\]

Since transfers are index-decreasing, we may intuitively think of \( \gamma_{i,n+1} \) as the proportion of students of type \( \theta_{i,j} \) assigned to school \( s_i \) in the first round whose only available school in the second round comes from their first round guarantee.

We define the lottery \( P \) from \( \gamma_{i,j} \) as follows. We break the interval \([0, 1]\) into \((n+1)^2\) intervals, \( \tilde{I}_{i,j} \), where the interval \( \tilde{I}_{i,j} \) has length \( \gamma_{i,j} \), and the intervals are ordered in decreasing order of the first index \( i \),

\[
\tilde{I}_{n+1,n+1}, \tilde{I}_{n+1,n}, \ldots, \tilde{I}_{1,1}. \tag{30}
\]

The interval \( \tilde{I}_{i,j} \) can be thought of as the lottery numbers of students whose first round lottery is in \([C_i, C_{i-1}]\) and whose second round affordable set contains \( s_{\sigma(j)} \) but not \( s_{\sigma(j-1)} \) (whose only available school in the second round comes from their first round guarantee, if \( j = n + 1 \)).

The permutation \( P \) maps the intervals back into \([0, 1]\) in decreasing order of the second index \( j \),

\[
P(\tilde{I}_{n+1,n+1}), P(\tilde{I}_{n,n+1}), \ldots, P(\tilde{I}_{2,1}), P(\tilde{I}_{1,1}). \tag{31}
\]

In Figure 5 we show an example for two schools.

We note that \( \sum_{j=1}^{n+1} \gamma_{i,j} = C_{i-1} - C_i \), which is the proportion of students whose first round affordable set is \( X_i \). We may interpret \( \gamma_{i,j} \) to be the proportion of students who can deduce that their lottery number is in the interval \([C_i, C_{i-1}]\), and whose second round affordable set is \( \tilde{X}_j \), and so \( \sum_{j=1}^{n+1} \gamma_{i,j} \) is the proportion of students whose second round affordable set is \( \tilde{X}_j \).

---

30Specifically, let \( \tilde{I}_{i,j} = [C_{i-1} - \sum_{j' \leq j} \gamma_{i,j'}, C_{i-1} - \sum_{j' < j} \gamma_{i,j'}] \).

31Specifically, let \( \hat{C}_{\sigma(j)} = 1 - \sum_{j' \leq j} \gamma_{i',j} \), and let \( P(\tilde{I}_{i,j}) = [\hat{C}_{\sigma(j-1)} - \sum_{i' \leq i} \gamma_{i',j}, \hat{C}_{\sigma(j-1)} - \sum_{i' < i} \gamma_{i',j}] \).
with first round scores in \( \tilde{\theta}_i \) and second round affordable set \( \tilde{X}_j \), and that this PLSM mechanism gives the same joint distribution over first and second round assignments as \( M \). To do this, we first show that this distribution of first and second round affordable sets gives rise to the correct joint first and second round assignments over all students. We then use anonymity to construct \( L \) in such a way so as to have the correct first and second round assignment joint distributions for each student. Finally, we verify that these second round affordable sets give a stable matching under the second round school preferences given by \( P \).

**(2b) Equivalence of the joint distribution of assignments given by affordable sets and \( M \):**

Fix student preferences \( \theta = (\succ, \tilde{\succ}) \). We show that if we let \( \gamma_{i,j} \) be the proportion of students with preferences \( \theta \) who have first round affordable set \( X_i \) and second round affordable set \( \tilde{X}_j \), then we obtain the same joint distribution over assignments in the first and second rounds for students with preferences \( \theta \) as under mechanism \( M \). In doing so, we will use the following lemma about prefixes.

The ‘prefix lemma’ states that for every set of schools \( S' \), there exist certain intervals of the form \( I^l_i = [C_i, C_j] \) such that for any two student types whose top set of acceptable schools under second round preference reports is \( S' \), the proportion of students with lotteries in \( I^l_i \) who are upgraded to a school in \( S' \) in the second round is the same for each type.

We define a prefix of preferences \( \succ \) to be a set of schools \( S' \) that is a top set of acceptable schools under \( \succ \), that is, for all \( s' \in S' \) and \( s \notin S' \), it holds that \( s' \succ s \).

**Lemma 4.** [Prefix Property] Let \( s = s_j \) be a school, and let \( S' \not\ni s \) be a set of schools such that \( i(S') \not\ni j \). Let \( \theta = (\succ, \tilde{\succ}) \) and \( \theta' = (\succ', \tilde{\succ}') \) be consistent preferences such that \( S' \) is a prefix of \( \succ, \tilde{\succ} \) and some students with preferences \( \theta \) are assigned to school \( s \) in the first round, and similarly \( S' \) is a prefix of \( \succ', \tilde{\succ}' \) and some students with preferences \( \theta' \) are assigned to school \( s \) in the first round.

Then the proportion of such students of type \( \theta \) whose first round lotteries are in the interval \([C_j, C_i(S')]) \) and who are assigned to a school in \( S' \) in the second round is the same as the proportion of such students of type \( \theta' \) whose first round lotteries are in the interval \([C_j, C_i(S')]) \) and who are assigned to a school in \( S' \) in the second round. Equivalently, \( \rho^\theta([C_j, C_i(S')], S') = \rho^\theta([C_j, C_i(S')], S') \).

**Sketch of proof of Lemma 4.** The idea of the proof is to use the full support assumption to identify students who are essentially indifferent between schools in \( S' \), and then use report-updating and two-round strategy-proofness to show that they are indifferent between reporting...
either $\theta$ or $\theta'$. This shows that the conditional probabilities of being assigned to $S'$ are the same for students of type $\theta$ or $\theta'$ (conditional on certain first round assignments). We then invoke anonymity to argue that proportions of types of students assigned to a certain school are given by the conditional probabilities for individual students of being assigned to that school. We present the full proof at the end of Section 3.1

We now show that the mechanism $M$ and the affordable set distribution $\gamma_{i,j}$ produce the same joint distribution of assignments.

(2b.i.) Students with two acceptable schools:
To give a bit of the flavor of the proof, we first consider student preferences $\theta$ of the form $\succ: [s_{i_1}, s_{i_2}, s_{n+1}]$ and $\succ': [s_{i_1}, s_{n+1}]$, where all other schools are unacceptable. There are five ordered pairs of schools that students of this type can be assigned to in the two rounds. Namely, if we let $(s, s')$ denote assignment to $s$ in the first round and $s'$ in the second round, then the ordered pairs are $(s_{i_1}, s_{i_1})$, $(s_{i_2}, s_{i_1})$, $(s_{i_2}, s_{n+1})$, $(s_{n+1}, s_{i_1})$, and $(s_{n+1}, s_{n+1})$. Since the proportion of students with each round assignment is fixed, it suffices to show that the mechanism $M$ and the mechanism that assigns first and second round affordable set distributions according to $\gamma_{i,j}$ produce the same proportion of students assigned to $(s_{i_2}, s_{i_1})$, and the same proportion of students assigned to $(s_{n+1}, s_{i_1})$.

The proportion of students with preferences $\theta$ who are assigned to $(s_{i_2}, s_{i_1})$ and $(s_{n+1}, s_{i_1})$ under $M$ are given by $\rho^\theta([C_{i_2}, C_{i_1}], s_{i_1})$ and $\rho^\theta([0, \max\{s_{i_2}, s_{n+1}\}], s_{i_1})$ respectively. We want to show that this is the same as the proportion of students with preferences $\theta$ who are assigned to $(s_{i_2}, s_{i_1})$ and $(s_{n+1}, s_{i_1})$ respectively when first and second round affordable sets are given by the affordable set distribution $\gamma_{i,j}$. We remark that when $i_1 > i_2$ this holds vacuously, since all the terms are 0. Hence, since for any school $s$ the proportion of students with preferences $\theta$ who are assigned to $s$ in the first round does not depend on $\theta$, it suffices to consider the case $i_1 < i_2$.

Let $\theta' = (\succ', \succ')$ be the preferences given by $\succ': [s_{i_1}, s_{i_2+1}, \ldots, s_{i_2}, s_{i_1}, s_{n+1}]$ and $\succ': [s_{i_1}, s_{n+1}]$, where only the schools with indices between $i_1$ and $i_2$ are acceptable in the first round, only $s_{i_1}$ is acceptable in the second round, and all other schools are unacceptable.

Recall that for all $i > i_1$, $\theta_{i,\sigma^{-1}(i_1)} = (\succ_{i,\sigma^{-1}(i_1)}, \succ_{i,\sigma^{-1}(i_1)})$ are the student preferences such that $\succ_{i,\sigma^{-1}(i_1)}: [s_{i_1}, s_{i-1}, s_{i}, s_{n+1}]$ and $\succ_{i,\sigma^{-1}(i_1)}: [s_{i_1}, s_{n+1}]$, with all other schools unacceptable, and that $\rho_{i,\sigma^{-1}(i_1)}$ is the proportion of students of type $\theta_{i,\sigma^{-1}(i_1)}$ whose first round assignment is $s_{i_1}$ and whose second round assignment is school $s_{i_1}$. (We note that in the case when $i = i_1 + 1$, we let the first two schools under the preference ordering $\succ_{i,\sigma^{-1}(i_1)}$ coincide.)

Then the proportion of students with preferences $\theta$ who are assigned to $(s_{i_2}, s_{i_1})$ under $M$ is given by $\rho^\theta([C_{i_2}, C_{i_1}], s_{i_1})$, where

$$\rho^\theta([C_{i_2}, C_{i_1}], s_{i_1}) = \rho^\theta([C_{i_2}, C_{i_1}], s_{i_1}) \quad \text{by the prefix property (Lemma 3)}$$

$$= \sum_{i_1 < i_1 \leq i_2} \rho^\theta([C_{i_1}, C_{i_1}], s_{i_1})$$

$$= \sum_{i_1 < i_1 \leq i_2} \rho^\theta_{i,\sigma^{-1}(i_1)}([C_{i_1}, C_{i_1}], s_{i_1}) \quad \text{by report-updating}$$

$$= \sum_{i_1 < i_1 \leq i_2} \rho_{i,\sigma^{-1}(i_1)} \quad \text{by the definition of } \rho_{i,\sigma^{-1}(i_1)}$$

$$= \sum_{i_1 < i_1 \leq i_2} \sum_{j \leq \sigma^{-1}(i_1)} \gamma_{i,j} \quad \text{by the definition of } \gamma_{i,j},$$

which is precisely the proportion of students with preferences $\theta$ who are assigned to $(s_{i_2}, s_{i_1})$ if the first and second round affordable sets are given by $\gamma_{i,j}$.
Similarly, let \( \theta'' \) be the preferences given by \( \succ'' : [s_i, s_{i+1}, \ldots, s_n, s_{n+1}] \) and \( \succ'' : [s_{i'}, s_{n+1}] \), where only \( s_i \) and the schools with indices greater than \( i \) are acceptable in the first round, only \( s_{i'} \) is acceptable in the second round, and all other schools are unacceptable.

Then the proportion of students with preferences \( \theta \) who are assigned to \((s_{n+1}, s_i)\) under \( M \) is given by \( \rho^\theta([0, C_{\max\{i_1, i_2\}}]), s_{i_1}) \), where

\[
\rho^\theta([0, C_{\max\{i_1, i_2\}}], s_{i_1}) = \rho''([0, C_{\max\{i_1, i_2\}}], s_{i_1}) \quad \text{(by the prefix property (Lemma 4))}
\]
\[
= \sum_{\max\{i_1, i_2\} < i \leq n} \rho''([C_i, C_{i-1}], s_{i_1})
\]
\[
= \sum_{\max\{i_1, i_2\} < i \leq n} \rho''([C_i, C_{i-1}], s_{i_1}) \quad \text{(by report-updating)}
\]
\[
= \sum_{\max\{i_1, i_2\} < i \leq n} \rho''([C_i, C_{i-1}], s_{i_1}) \quad \text{(by the definition of \( \rho''(i, \sigma^{-1}(i)) \))}
\]
\[
= \sum_{i, j : \max\{i_1, i_2\} \leq i \leq n, j \leq \sigma^{-1}(i)} \gamma_{i, j} \quad \text{(by the definition of \( \gamma_{i, j} \))},
\]

which is precisely the proportion of students with preferences \( \theta \) who are assigned to \((s_{n+1}, s_i)\) if the first and second round affordable sets are given by \( \gamma_{i, j} \).

\( 2b.ii. \) **Students with general preferences:**

We now consider general (consistent) student preferences \( \theta \) of the form \( (\succ, \succ') \), where

\[ \succ : [s_{i_1}, s_{i_2}, \ldots, s_{i_k}, s_{n+1}] \quad \text{and} \quad \succ' : [s_{i_1}, s_{i_2}, \ldots, s_{i_l}, s_{n+1}], \]

for some \( k > l \) and where all other schools are unacceptable. We wish to show that for every pair of schools \( s, s' \in \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}, s_{n+1}\} \), the mechanism \( M \) and the mechanism that assigns first and second round affordable set distributions according to \( \gamma_{i, j} \) produce the same proportion of students assigned to \((s, s')\). It suffices to show that for every prefix \( S' \) of the preferences \( \succ' \) and every school \( s \in \{s_{i_2}, \ldots, s_{i_k}, s_{n+1}\} \), the mechanism \( M \) and the mechanism that assigns first and second round affordable set distributions according to \( \gamma_{i, j} \) produce the same proportion of students assigned to \( s \) in the first round and some school in \( S' \) in the second round. We say that the students are assigned to \((s, S')\).

Fix a prefix \( S' \) of \( \succ' \) and a school \( s = s_{i_j}, 1 < j \leq k \). Let \( m \leq k \) be such that \( S' = \{s_{i_1}, s_{i_2}, \ldots, s_{i_m}\} \). If \( j \leq m \) then \( s \in S' \), so in any mechanism that respects guarantees, the proportion of students assigned to \((s, S')\) is the same as the proportion of students assigned to \( s \) in the first round.

If \( j > m \) and \( i_j \leq i(S') \), then in the first round, whenever the school \( i_j \) is available in the first round, so is the preferred school \( i(S') \); thus for any school \( s' \), the proportion of students assigned to \( s \) in the first round is 0. It follows that in any mechanism that respects guarantees, the proportion of students assigned to \((s, S')\) is 0.

From here on, we may assume that \( j > m \) (ie. \( s \not\in S' \)) and \( i_j > i(S') \). Since \( i_j > i(S') \), the proportion of students with preferences \( \theta \) who are assigned to \((s, S')\) under \( M \) is given by \( \rho^\theta([C_{i_j}, C_{i(S')}], S') \). Let \( i(\sigma(S')) \) be the index \( i \) such that \( s_i \in S' \) and \( \sigma^{-1}(i) \) is maximal, that is, the index of the school in \( S' \) that is most affordable in the second round.

Let \( \theta' = (\succ', \succ') \) be the preferences given by

\[ \succ' : [s_{i(\sigma(S'))}, s', s_{i(S')} + 1, s_{i(S')} + 2, \ldots, s_{i_j - 1}, s_{i_j}, s_{n+1}] \quad \text{and} \quad \succ' : [s_{i(\sigma(S'))}, s', s_{n+1}] \]

for all \( s' \in S' \setminus \{s_{i(\sigma(S'))}\} \). It may be helpful to think of this as all preferences of the form

\[ \succ' : [S', s_{i(S')} + 1, s_{i(S')} + 2, \ldots, s_{i_j - 1}, s_{i_j}, s_{n+1}] \quad \text{and} \quad \succ' : [S', s_{n+1}], \]
where the school \( s_{i(\sigma(S'))} \) comes first and otherwise the schools in \( S' \) are ordered arbitrarily.

We remark that only the schools in \( S' \) and the schools with indices between \( i(S') \) and \( i_j \) are acceptable in the first round, only the schools in \( S' \) are acceptable in the second round, and all other schools are unacceptable. Since \( j > m, i_j > i(S') \), and the preferences \( \theta \) are consistent, the preferences \( \theta' \) are well-defined. Let \( \theta'' = (\succ'', \succ') \) be the preferences given by \( \succ'' = \succ' \) and \( \succ' = [s_{i(\sigma(S'))}, s_{n+1}] \).

Recall that for all \( i > i(\sigma(S')) \), \( \theta_{i,\sigma^{-1}(i(\sigma(S')))} = (\succ_{i,\sigma^{-1}(i(\sigma(S')))}, \succ_{i,\sigma^{-1}(i(\sigma(S')))}) \) are the student preferences such that

\[
\succ_{i,\sigma^{-1}(i(\sigma(S')))}: [s_{i(\sigma(S'))}, s_{i-1}, s_i, s_{n+1}] \text{ and } \succ_{i,\sigma^{-1}(i(\sigma(S')))}: [s_{i(\sigma(S'))}, s_{n+1}],
\]

with all other schools unacceptable. Additionally, recall that \( \rho_{i,\sigma^{-1}(i(\sigma(S')))} \) is the proportion of students of type \( \theta_{i,\sigma^{-1}(i(\sigma(S')))} \) whose first round assignment is \( s_i \) and whose second round assignment is school \( s_{i(\sigma(S'))} \).

Let \( \hat{S} = \{s_{i_1}, s_{i_2}, \ldots, s_{i_{j-1}}\} \), and let \( \hat{i}(\hat{S}) \) be the index \( i \) such that \( i \in \hat{S} \) and \( \sigma^{-1}(i) \) is maximal, that is, the index of the school preferable to \( s \) under \( \succ \) that is most affordable in the second round.

Then the proportion of students with preferences \( \theta \) who are assigned to \( (s, S') \) under \( M \) is given by \( \rho^\theta([C_{i_j}, C_{i(\hat{S})}], S') \), where \( \rho^\theta([C_{i_j}, C_{i(\hat{S})}], S') \) (by the prefix property (Lemma 4) with prefix \( S' \))

\[
\rho^\theta([C_{i_j}, C_{i(\hat{S})}], S') = \sum_{i(\hat{S}) < i \leq i_j} \rho^\theta([C_i, C_{i-1}], S')
\]

(by the definition of the second round over-demand ordering)

\[
= \sum_{i(\hat{S}) < i \leq i_j} \rho^\theta([C_i, C_{i-1}], s_{i(\sigma(S'))}) \quad \text{(by the prefix property with prefix \{s_{i(\sigma(S'))}\})}
\]

\[
= \sum_{i(\hat{S}) < i \leq i_j} \rho^\theta([C_i, C_{i-1}], s_{i(\sigma(S'))}) \quad \text{(by report-updating)}
\]

\[
= \sum_{i(\hat{S}) < i \leq i_j} \rho_{i,\sigma^{-1}(i(\sigma(S')))}([C_i, C_{i-1}], s_{i(\sigma(S'))}) \quad \text{(by the definition of } \rho_{i,\sigma^{-1}(i(\sigma(S')))} \text{)}
\]

\[
= \sum_{i(\hat{S}) < i \leq i_j} \sum_{j' \leq \sigma^{-1}(i(\sigma(S')))} \gamma_{i,j'} \quad \text{(by the definition of } \gamma_{i,j'} \text{)}
\]

which is precisely the proportion of students with preferences \( \theta \) who are assigned to \( (s, S') \) if the first and second round affordable sets are given by \( \gamma_{i,j'} \).

(3) Constructing the lottery \( L \)

Fix a student \( \lambda \) who reports first and second round preferences \( \theta = (\succ, \succ') \). Suppose that \( \lambda \) is assigned to schools \( (s_i, s_j) \) in the first and second rounds respectively. We first characterize all first and second budget sets consistent with the over-demand orderings that could have led to this assignment. Let \( \hat{i} \) be the smallest index \( i' \) such that \( \max_\succ X_{i'} = s_i \), let \( \hat{j} \) be the smallest index \( j' \) such that \( \max_\succ X_{j'} = s_j \), and let \( \hat{j} \) be the largest index \( j' \) such that \( \max_\succ X_{j'} = s_j \). Then the set of first and second budget sets that student \( \lambda \) could have been assigned by the mechanism is given by \( \{X_{i'}, X_{j'} \cup \{s_i\} : \hat{i} \leq i' \leq i, \hat{j} \leq j' \leq \hat{j}\} \).

(We remark that the asymmetry in these definitions is due to the existence of the first round guarantee in the second round budget sets.)
Conditional on $\lambda$ being assigned to schools $(s_i, s_j)$ in the first and second rounds respectively, we assign a lottery number $L(\lambda)$ to $\lambda$ distributed uniformly over the union of intervals $\bigcup_{i',j':i'\leq i,j'\leq j'} I_{i',j'}$,

$$(L(\lambda) \mid (\mu(\lambda), \tilde{\mu}(\lambda)) = (s_i, s_j)) \sim \text{Unif} \left( \bigcup_{i',j':i'\leq i,j'\leq j'} I_{i',j'} \right),$$

independent of all other students’ assignments.

We show that this is consistent with the first round of the mechanism being RSD. We have shown in (1) that if for each pair of reported preferences $\theta = (\succ, \succ^*) \in \Theta$, a uniform proportion $\gamma_{i',j'}$ of students with reported preferences $\theta$ are given first and second round budget sets $X_{i'}', \{s'\} \cup \bar{X}_{j'}$ (where $s' = \max_{\succ} X_i$ is the first round assignment of such students), we obtain the same distribution of assignments as $M$. Since $M$ is anonymous and satisfies the averaging axiom, and since $|\tilde{I}_{i',j'}| = \gamma_{i',j'}$, it follows that each student’s first round lottery number is distributed as $\text{Unif}[0,1]$.

(4) **Constructing the PLSM and verifying stability**

Given the constructed lottery $L$, we construct the second round cutoffs $\hat{C}_i$ for the PLSM and verify that the allocation $\hat{\mu}$ is feasible and stable with respect to the schools’ second round preferences, as defined by $P \circ L$ and the guarantee structure. Specifically, in PLSM, each student with first round score $l$ and first round allocation $s$ has a second round score $\hat{r}_i = P(l) + \mathbb{1}(s = s_i)$ at each school $s_i \in S \cup \{s_n + 1\}$, and students are assigned to their favorite school $s_i$ at which their second round score exceeds the school’s second round cutoff, $\hat{r}_i \geq \hat{C}_i$.

Recall that the schools are indexed so that $C_1 \geq C_2 \geq \cdots \geq C_{n+1}$, and that the permutation $\sigma$ is chosen so that the second round overdemand ordering is given by $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n+1)} = s_{n+1}$, and so it should follow that the second round cutoffs $\hat{C}_i$ satisfy $\hat{C}_{\sigma(1)} \geq \hat{C}_{\sigma(2)} \geq \cdots \geq \hat{C}_{\sigma(n+1)}$.

By the characterization of stable allocations given in Azevedo and Leshno [2014], it suffices to show that if each student with first round allocation $s$ and second round lottery number in $[\hat{C}_{\sigma^{-1}(i)}, \hat{C}_{\sigma^{-1}(i-1)}]$ is assigned to their favorite school in $\{s\} \cup \bar{X}_i$, then the resulting assignment $\hat{\mu}$ is equal to the second round assignment $\hat{\mu}$ of our mechanism $M$, and satisfies that $\eta(\hat{\mu}^{-1}(s_i)) \leq q_i$ for any school $s_i$, and $\eta(\hat{\mu}^{-1}(s_i)) = q_i$ if $\hat{C}_i > 0$.

For fixed $i, j$, let $\hat{C}_{\sigma(j)} = 1 - \sum_{i',j':i'\leq i,j'\leq j'} \gamma_{i',j'}$ and let $\hat{C}_{i,\sigma(j)} = \hat{C}_{\sigma(j-1)} - \sum_{i' \leq i} \gamma_{i',j'}$. (We remark that since $\gamma_{i,j}$ refers to the $i$th school to fill in the first round, $s_i$, and the $j$th school to fill in the second round, $s_{\sigma(j)}$, the $\hat{C}$ are indexed slightly differently to $\gamma_{i,j}$.)

We use the averaging assumption and the equivalence of assignment probabilities that we have shown in (1) to conclude that if $\hat{\mu}$ is the assignment given by running DA with round scores $\hat{r}$ and cutoffs $\hat{C}$, then $\hat{\mu} = \tilde{\mu}$.

This is fairly evident, but we also show it explicitly below. Specifically, consider a student $\lambda \in \Lambda$ with first round lottery number $L(\lambda)$ and reported preferences $\theta = (\succ, \succ^*)$. Let $i, j$ be such that $L(\lambda) \in \bigcup_{i',j':i'\leq i,j'\leq j'} I_{i',j'}$, where $i$ is the smallest index $i'$ such that $\max_{\succ} X_{i'} = s_i$, let $j$ be the smallest index $j'$ such that $\max_{\succ} \bar{X}_{j'} \cup \{s_i\} = s_j$, and $j$ is the largest index $j'$ such that $\max_{\succ} \bar{X}_{j'} \cup \{s_i\} = s_j$. Then, because of the way in which we have constructed the lottery $L$, $(\mu(\lambda), \tilde{\mu}(\lambda)) = (s_i, s_j)$.

Moreover, since

$$P(L(\lambda)) \in P(\bigcup_{i',j':i'\leq i,j'\leq j'} I_{i',j'}) = \bigcup_{i',j':i'\leq i,j'\leq j'} P(I_{i',j'}),$$

where $P(I_{i',j'}) \in [\hat{C}_{\sigma(j')}, \hat{C}_{\sigma(j'-1)}]$, it holds that under $\hat{\mu}$, student $\lambda$ receives their favorite school
in \( \{s_i\} \cup \hat{X}_j \) for some \( j \leq j' \leq \bar{j} \), which is the school \( s_j \). Hence \( \hat{\mu}(\lambda) = \hat{\mu}(\lambda) = s_j \).

It follows immediately that the allocation \( \hat{\mu} \) is feasible, since it is equal to the feasible assignment \( \hat{\mu} \).

Finally, let us check that the allocation is stable. Suppose that \( \hat{C}_j > 0 \). We want to show that \( \eta(\hat{\mu}^{-1}(s_j)) = q_j \). First note that it follows from the definition of \( \hat{C}_j \) that

\[
1 > \sum_{\nu', j': j' \leq \sigma^{-1}(j)} \gamma_{\nu', j'} = \sum_{\nu'} \rho_{\nu', \sigma^{-1}(j)}.
\]

Consider student preferences \( \theta = (\cdot, \cdot) \) given by \( \succ : [s_j, s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{n+1}] \). Then \( \sum_{\nu'} \rho_{\nu', \sigma^{-1}(j)} \) is the proportion of students of type \( \theta \) who are assigned to school \( s_j \) in the second round, which, by assumption, is also the probability that a student with preferences \( \theta \) is assigned to \( s_j \) in the second round. But since \( M \) is Pareto efficient among reassigned students, and hence non-wasteful, this means that \( \eta(\hat{\mu}^{-1}(s_j)) = q_j \).

\[ \square \]

**Proof of Lemma** Here, we prove the prefix property.

We first observe that since the mechanism is report-updating, any schools reported to be acceptable but ranked below \( s \) in the first round are inconsequential. Moreover, since \( M \) respects guarantees and by two-round strategy-proofness, any schools reported to be acceptable but ranked below \( s \) in the second round are inconsequential. We show this below.

Suppose there are two student preferences \( \theta = (\succ, \cdot) \) and \( \theta' = (\succ', \cdot) \) that differ only in the second round preferences, and \( \succ' \) is given by truncating the preferences \( \succ \) at the school \( s \) and making all schools worse than \( s \) under \( \succ \) unacceptable. Let \( \succ' : [s_{i_1}, s_{i_2}, \ldots, s_{i_k} = s, s_{n+1}] \) and let \( S' = \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\} \). Since the mechanism respects guarantees, students reporting \( \theta \) and \( \theta' \) are assigned only to schools in \( S' \). Suppose for the sake of contradiction that the mechanism \( M \) treats students with reported preferences \( \theta \) and \( \theta' \) who are assigned to \( s \) in the first round differently, that is, the distribution of \( (\hat{\mu}(\lambda) \mid \mu(\lambda) = s, \lambda \) reports preferences \( \theta) \) is different to the distribution of \( (\hat{\mu}(\lambda) \mid \mu(\lambda) = s, \lambda \) reports preferences \( \theta') \). Let \( j \) be the smallest index for which the probability of assignment differs,

\[
\mathbb{P}(\hat{\mu}(\lambda) = s_{i_j} \mid \mu(\lambda) = s, \lambda \ reports \ \theta) \neq \mathbb{P}(\hat{\mu}(\lambda) = s_{i_j} \mid \mu(\lambda) = s, \lambda \ reports \ \theta').
\]

Suppose the left hand side is larger than the right hand side by some amount \( \Delta \). Consider a student with true preferences \( \theta' \) whose cardinal utilities for schools in \( \{s_{i_1}, s_{i_2}, \ldots, s_{i_j}\} \) are \( v, v - \varepsilon, \ldots, v - (j - 1)\varepsilon \), for all other schools in \( S' \) are \( \frac{\varepsilon}{j+1}, \ldots, \frac{\varepsilon}{i}, \) and for all other schools is 0. Then by misreporting \( \succ' \) in the second round, they increase their interim expected utility by at least \( \Delta(v - (j - 1)\varepsilon) \geq \varepsilon \), which for sufficiently small \( \varepsilon \) is positive, contradicting two-round strategy-proofness. Hence the left hand side is smaller than the right hand side, and a similar argument for a student with true preferences \( \theta \) shows that the probabilities must in fact be equal.

Hence it suffices to prove the lemma for first round preference orderings \( \succ \) and \( \succ' \) for which \( s \) is the last acceptable school. We note that the above argument proves the prefix property for any two sets of preferences \( \theta, \theta' \) and first round assignment \( s \) where the first round preferences are identical. When the first round preferences are different, because students cannot misreport their first round preferences after the fact, we will need to use two-round strategy-proofness for a number of additional student preferences in order to show that the prefix property holds. When the second round preferences are different, we will need to consider more fine-grained information that students have about the lottery number, and use two-round strategy-proofness
for a number of different student preferences based on this fine-grained information. We provide
the formal argument below.

Let \( i_1, \ldots, i_k \) be the indices of the schools in \( S' \), in increasing order. We observe that
\( i_k = i(S') \). Recall that \( s = s_j \), where \( i_k < j \).

Since we wish to prove that the lemma holds for all pairs \( \theta, \theta' \) satisfying the assumptions, it
suffices to show that the lemma holds for a fixed preference \( \theta \) and vary only \( \theta' \). Therefore, we may,
without loss of generality, fix the preferences \( \theta \) to satisfy that

\[
\succ: [s_{i(S')}, s_{i_1}, \ldots, s_{i_{k-1}}, s = s_j, s_{n+1}] \quad \text{and} \quad \succ\!: [s_{i(S')}, s_{i_1}, \ldots, s_{i_{k-1}}, s_{n+1}]
\]

and all other schools unacceptable. That is, the worst school in \( S' \) is top ranked, then all other
schools in \( S' \) in order. In the first round \( s = s_j \) is also acceptable, and in the second round only
schools in \( S' \) are acceptable.

We remark that given the first round ordering, the worst school in \( S' \) and the school \( s \) (namely
\( s_{i(S') \}) \) and \( s_j \) are the only acceptable schools to which students of type \( \theta \) will be assigned in
the first round. Moreover, it follows from the structure of the preferences \( \theta \) and \( \theta' \) that the
proportion of students with preferences \( \theta \) who can deduce that their score is in \( [C_j, C_{i(S')}] \) is
precisely \( C_{i(S')} - C_j \), and similarly for \( \theta' \). Similarly, the proportion of students with preferences
\( \theta \) (or \( \theta' \)) who can deduce that their lottery number is in \( [C_{i(S')}, 1] \) is precisely \( 1 - C_{i(S')} \). (Note
that students with preferences \( \theta' \) may be able to deduce that their lottery number falls in a
subinterval of the interval we have specified. However, this does not affect our statements.) We
remark that the set of students with preferences \( \theta \) and lottery number in \([C_{i(S')}, 1]\) is precisely
the set of students with preferences \( \theta \) who are assigned to a school in \( S' \) in the first round, and
similarly for \( \theta' \).

To compare the proportion of students of types \( \theta \) and \( \theta' \) whose scores are in \( [C_j, C_{i(S')}] \) and
who are assigned to \( S' \) in the second round, we define a third student type \( \theta'' \) as follows. Let
\( \theta'' = (\succ', \succ) \) be a set of preferences where the first round preferences are the same as the first
round preferences of \( \theta' \), and the second round preferences are the same as the second round
preferences of type \( \theta \).

Let \( \lambda \) be a student with preferences \( \theta \), and similarly let \( \lambda' \) be a student with preferences
\( \theta' \). We use the two-round strategy-proofness of the mechanism to show that \( \lambda \) has the same
probability of being assigned to some school in \( S' \) in the second round as if she had reported
type \( \theta'' \), and similarly for \( \lambda' \). Since the proportion of students of either type being assigned to
a school in \( S' \) in the first round is the same and the mechanism respects guarantees, this is
sufficient to prove the prefix property.

Formally, let \( \rho \) be the probability that \( \lambda \) is assigned to some school in \( S' \) in the second round
if she reports truthfully, conditional on being able to deduce that her first round score is in
\([C_j, C_{i(S')}]\), and let \( \rho' \) be the probability that \( \lambda' \) is assigned to some school in \( S' \) in the second
round if she reports truthfully, conditional on being able to deduce that her first round score is in
\([C_j, C_{i(S')}]\). (We note that given her first round assignment \( \mu'(\rho') \), the student \( \rho' \) may actually
be able to deduce more about her first round score, and so the interim probability that \( \rho' \) is
assigned to some school in \( S' \) in the second round if she reports truthfully is not necessarily\( \rho' \).) Let \( \rho'' \) be the probability a student with preferences \( \theta'' \) and first round score in \( [C_j, C_{i(S')}] \)
chosen uniformly at random is assigned to some school in \( S' \) in the second round. It follows
from the design of the first round and from anonymity that \( \rho \) is the probability that a student
with preferences \( \theta \) and lottery number in \([C_j, C_{i(S')}]\) chosen uniformly at random is assigned to
some school in \( S' \) in the second round, and similarly for \( \rho' \).

Proving the lemma is equivalent to proving \( \rho = \rho' \). We show that \( \rho = \rho'' = \rho' \). Note that
the first equality is between preferences that are identical in the second round, and the second
equality is between preferences that are identical in the first round.
We first show that $\rho = \rho''$, that is, changing just the first round preferences does not affect the probability of assignment to $S'$. This is almost immediate from report-updating and Bayesian incentive compatibility, since the second round preferences under $\theta$ and $\theta''$ are identical. (This also illustrates the power of the report-updating assumption, since it implies that manipulating first round reports to obtain a more fine-grained knowledge of the lottery number does not help, since assignment probabilities are conditionally independent of the lottery number.) We run through the full argument below.

Consider a student $\lambda_0$ with preferences $\theta$ whose second round cardinal utilities over schools in $S'$ are $v, v - \varepsilon, v - 2\varepsilon, \ldots, v - (|S'| - 1)\varepsilon$, for some small $\varepsilon$ and some large $v$, and are 0 for all other schools and the outside option.

Let $\pi$ be the probability that a student with preferences $\theta$ who is unassigned in the first round is assigned to a school in $S'$ in the second round. We note that since the last acceptable school under preferences $\theta$ and $\theta'$ is $s = s_j$, the set of students with preferences $\theta$ who are unassigned in the first round is equal to the set of students with preferences $\theta$ with lottery number in $[0, C_j]$, and similarly the set of students with preferences $\theta''$ who are unassigned in the first round is equal to the set of students with preferences $\theta''$ with lottery number in $[0, C_j]$. Hence, the report-updating property and the fact that $\theta$ and $\theta''$ have the same second preferences gives us that $\pi$ is also the probability that a student with preferences $\theta''$ who is unassigned in the first round is assigned to a school in $S'$ in the second round.

By truthfully reporting preferences $\theta$, student $\lambda_0$ has expected ex ante utility of at most

$$(1 - C_i(S'))v + (C_i(S') - C_j)\rho v + C_j\pi v,$$

where the first term is an upper bound on their expected utility from having a lottery number in $[C_i(S'), 1]$ in the first round, the second term is an upper bound on their expected utility from having a lottery number in $[C_j, C_i(S')]$ in the first round, and the third term is an upper bound on their expected utility from having a lottery number in $[0, C_j]$ in the first round.

By misreporting preferences $\theta''$, they have expected ex ante utility of at least

$$(1 - C_i(S'))(v - |S'|\varepsilon) + (C_i(S') - C_j)\rho''(v - |S'|\varepsilon) + C_j\pi(v - |S'|\varepsilon).$$

From two-round strategy-proofness for $\lambda_0$ and taking $\varepsilon$ to zero, it follows that $\rho \geq \rho''$. A symmetric argument for a student with preferences $\theta''$ with the same second round cardinal utilities gives the reverse inequality, and hence $\rho = \rho''$.

We now show that $\rho = \rho''$. This is a little more involved, but essentially relies on breaking the set of students with first round score in $[C_j, C_i(S')]$ into smaller subsets, depending on their first round assignment, and using Bayesian incentive compatibility for students who have high value for schools in $S'$ and low value outside of $S'$ to show that in each subset, the probability of an arbitrary student being assigned to a school in $S'$ in the second round is the same for students with either set of preferences $\theta'$ or $\theta''$.

We first introduce some notation for describing the first round preferences of $\theta'$ and $\theta''$. Let $\{j_1 \leq \cdots \leq j_m\}$ be the indices between $i(S')$ and $j$ corresponding to schools that a student with preferences $\theta'$ and lottery number in $[C_j, C_i(S')]$ could have been assigned to in the first round. Formally, we define them to be the indices $k$ for which $s_k \notin S'$, $i(S') < k \leq j$, $s_k \succ' s_j$ and $s_k$ is relevant in the first round overdemand ordering, that is, $k' < k$ for all $k'$ such that $s_k \succ' s_{k'}$.

We observe that $j_m = j$.

For $l = 1, \ldots, m$, let $\rho'_l$ be the probability that a student with preferences $\theta'$ who was assigned to school $s_{j_l}$ is assigned to a school in $S'$ in the second round.

The set of students with preferences $\theta'$ assigned to school $s_{j_l}$ in the first round is precisely the set of students with preferences $\theta'$ whose first round lottery number is in $[C_{j_l}, C_{j_{l-1}}]$ and similarly
the set students of with preferences $\theta''$ assigned to school $s_{ji}$ in the first round is precisely the set of students with preferences $\theta''$ whose first round lottery number is in $[C_{ji}, C_{ji-1}]$. It follows that $(C_i(S') - C_j) = \sum_{l=1}^{m}(C_{ji-1} - C_{ji})$, and that

$$(C_i(S') - C_j)\rho' = \sum_{l=1}^{m}(C_{ji-1} - C_{ji})\rho'_l.$$ 

Let $\rho''_l$ be the probability that a student with preferences $\theta''$ who was assigned to school $s_{ji}$ is assigned to a school in $S'$ in the second round. Then it also holds that

$$(C_i(S') - C_j)\rho'' = \sum_{l=1}^{m}(C_{ji-1} - C_{ji})\rho''_l.$$ 

We show now that $\rho''_l = \rho'_l$ for all $l$, which implies that $\rho' = \rho''$.

Consider a student $\lambda_l$ with preferences $\theta'$ assigned to school $s_{ji}$ in the first round whose second round cardinal utilities are, for some small $\varepsilon$ and some large $v, v-\varepsilon, v-2\varepsilon, \ldots, v-|S'| \varepsilon$ in the appropriate order for schools in $S, \varepsilon, \varepsilon, \ldots$ for the other acceptable schools, and 0 for all other schools and the outside option.

By truthfully reporting preferences $\theta'$, student $\lambda_l$ has expected interim utility of at most

$$\rho'_l v + (1 - \rho'_l)\varepsilon,$$

and by misreporting preferences $\theta''$, they have expected interim utility of at least

$$\rho''_l(v - |S'|\varepsilon),$$

so from second round strategy-proofness for $\lambda_l$ and taking $\varepsilon$ to 0, it follows that $\rho'_l \geq \rho''_l$.

Similarly, consider a student $\lambda''_l$ with preferences $\theta''$ with the same first round guarantee $s_{li}$, second round cardinal utilities over schools in $S'$ also $v, v-\varepsilon, \ldots, v-|S'|\varepsilon$ in the order corresponding to preferences $\succ$, and second round cardinal utilities for other acceptable schools $\varepsilon, \varepsilon, \ldots$. Second round strategy-proofness for these students gives the reverse inequality $\rho'_l \leq \rho''_l$, and hence $\rho'_l = \rho''_l$. This completes the proof of the lemma. \qed

C Example of non-strategy-proofness of PLDA

In this section, we provide an example illustrating that when non-atomicity does not hold, PLDA mechanisms are not necessarily strategyproof.

**Example 2.** Consider a setting with $n = 2$ schools and $m = 4$ students. Each school has capacity 1 and a single priority class. For readability, we let $\emptyset$ denote the outside option, $\emptyset = s_{n+1} = s_3$. The students have the following preferences:

1. $s_1 \succ_1 \emptyset \succ_1 s_2$ and $\emptyset \succ_1 s_1 \succ_1 s_2$,
2. $s_1 \succ_2 s_2 \succ_2 \emptyset$, second round preferences identical,
3. $s_2 \succ_3 s_1 \succ_3 \emptyset$, second round preferences identical,
4. $s_2 \succ_4 \emptyset \succ_4 s_1$, second round preferences identical.
We show that the two-round mechanism where the second round is the reverse lottery deferred acceptance mechanism is not strategy-proof.

Assume student 2’s utility is $M$ for $s_1$, $\epsilon$ for $s_2$, and 0 for $s_3$ in both rounds, where $M >> \epsilon > 0$. Consider the lottery which yields $1 \succ^B 2 \succ^B 3 \succ^B 4$. If the students report truthfully, the first round assignment is 

$$\mu(A) = (\mu(1), \mu(2), \mu(3), \mu(4)) = (s_1, s_2, \emptyset, \emptyset),$$

and the reassignment is 

$$\hat{\mu}(A) = (\hat{\mu}(1), \hat{\mu}(2), \hat{\mu}(3), \hat{\mu}(4)) = (\emptyset, s_2, s_1, \emptyset).$$

However, consider what happens if student 2 reports $s_1 \succ^r \emptyset \succ^r s_2$ in both rounds. Then, the first round assignment becomes $\mu(A) = (s_1, \emptyset, s_2, \emptyset)$, and the reassignment becomes 

$$\hat{\mu}(A) = (\emptyset, s_1, s_2, \emptyset),$$

which is a strictly beneficial change for student 2 (and, in fact, weakly beneficial for all students).

We remark that this reassignment was not stable in the second round when students reported truthfully, since, in that case school $s_2$ had second round preferences $2 \succ^S s_4 \succ^S s_3 \succ^S s_1$, and so school $s_2$ and student 4 form a blocking pair.

Consider now the expected utility of student 2 from reporting truthfully and her expected utility from misreporting, when all other students report truthfully and the expectation is over the first round lottery order. With probability $\frac{1}{4!}$, the lottery order is $1 \succ^B 2 \succ^B 3 \succ^B 4$, in which case student 2 can change her assignment from $s_2$ to $s_1$ by reporting $s_2$ as unacceptable. Moreover, one can verify that for any lottery order, if student 2 received $s_1$ in the first or second round under truthful reporting, then she also received $s_1$ in the same round by misreporting. Hence, by misreporting in this particular fashion, student 2 increases her probability of receiving $s_1$ by at least $\frac{1}{4!}$. Thus, for $M$ sufficiently large with respect to $\epsilon$, this violates strategy-proofness.

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32 This is because any stable matching in which student 2 is assigned $s_1$ remains stable after student 2 truncates. Indeed, student 2 is not part of any unstable pair, as she got her first choice, and any unstable pair not involving student 2 remains unstable under the true preferences, as only student 2 changes her preferences.