Matching while Learning

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We consider the problem faced by a service platform that needs to match supply with demand, but also to learn attributes of new arrivals in order to match them better in the future. We introduce a benchmark model with heterogeneous workers and jobs that arrive over time. Job types are known to the platform, but worker types are unknown and must be learned by observing match outcomes. Workers depart after performing a certain number of jobs. The payoff from a match depends on the pair of types and the goal is to maximize the steady-state rate of accumulation of payoff.

Our main contribution is a complete characterization of the structure of the optimal policy in the limit that each worker performs many jobs. The platform faces a trade-off for each worker between myopically maximizing payoffs (*exploitation*) and learning the type of the worker (*exploration*). This creates a multitude of multi-armed bandit problems, one for each worker, coupled together by the constraint on availability of jobs of different types (*capacity constraints*). We find that the platform should estimate a shadow price for each job type, and use the payoffs adjusted by these prices, first, to determine its learning goals and then, for each worker, (i) to balance learning with payoffs during the "exploration phase", and (ii) to myopically match after it has achieved its learning goals during the "exploitation phase."

Keywords: matching, learning, two-sided platform, multi-armed bandit, capacity constraints.

1 INTRODUCTION

This paper considers a central operational challenge faced by platforms that serve as matchmakers between supply and demand. Such platforms face a fundamental *exploration-exploitation* trade-off: on the one hand, efficient operation involves making matches that generate the most value ("exploitation"); on the other hand, the platform must continuously learn about newly arriving participants, so that they can be efficiently matched ("exploration"). In this paper, we develop a structurally simple and nearly optimal approach to resolving this trade-off.

In the model we consider, there are two groups of participants: *workers* and *jobs*. The terminology is inspired by online labor markets (e.g., Upwork for remote work, Handy for housecleaning, Thumbtack and Taskrabbit for local tasks, etc.); however, our model can be viewed as a stylized abstraction of many other matching platforms as well. Time is discrete, and new workers and jobs arrive at the beginning of every time period. Workers depart after performing a specified number of jobs. Each time a worker and job are matched, a (random) payoff is generated and observed by the platform, where the payoff distribution depends on the worker type and the job type.

As our emphasis is on the interaction between matching and learning, our model has several features that focus our analysis in this paper. First, the platform centrally controls matching: at the beginning of each time period, the platform matches each worker in the system to an available job. Second, strategic considerations are not modeled; this remains an interesting direction for future work. Finally, we focus on the goal of maximizing the steady-state rate of payoff generation.[1]

[1]This is a reasonable proxy for the goal of a platform that, say, takes a fraction of the total surplus generated through matches. More generally, we believe that this is a benchmark problem whose solution informs algorithmic design for settings with other related objectives, such as revenue maximization.

We now describe the learning challenge faced by the platform. In most platforms, more is known about one side of the platform than the other; accordingly, we assume job types are known, while the type of a new worker is unknown. The platform learns about workers' types through the payoffs obtained when they are matched to jobs. However, because the supply of jobs is limited, using jobs to learn can reduce immediate payoffs, as well as deplete the supply of jobs available to the rest of the marketplace. Thus the presence of capacity constraints forces us to carefully design both exploration and exploitation in any algorithm if we aim to optimize the rate of payoff generation.

Our main contribution in this paper is the development of a matching policy that is nearly payoff optimal. Our algorithm is divided into two phases in each worker's lifetime: *exploration* (identification of the worker type) and *exploitation* (optimal matching given the worker's identified type). We refer to our policy as *DEEM: Decentralized Explore-then-Exploit for Matching*.

To develop intuition for our solution, consider a simple example with two types of jobs ("easy" and "hard") and two types of workers ("expert" and "novice"). Experts can do both types of tasks well; but novices can only do easy tasks well. Suppose that there is a limited supply of easy jobs: more than the mass of novices available, but less than the total mass of novices and experts. In particular, to maximize payoff the platform must learn enough to match some experts to hard jobs.

DEEM has several key features, each of which can be understood in the context of this example. *First*, DEEM has a natural decentralization property: it determines the choice of job for a worker based only on that worker's history. This decentralization is arguably essential in large-scale online platforms, where matching is typically carried out on an individual basis, rather than centrally. In order to accomplish this decentralization, it is essential for the algorithm to account for the externality to the rest of the market when a worker is matched to a given job. For example, if easy jobs are relatively scarce, then matching a worker to such a job makes it unavailable to the rest of the market. Our approach is to "price" this externality: we find *shadow prices* for the capacity constraints, and adjust per-match payoffs downward using these prices.

Second, our algorithm design specifies *learning goals* that ensure an efficient balance between exploration and exploitation. In particular, in our example, we note that there are two kinds of errors possible while exploring: misclassifying a novice as an expert, and vice versa. Occasionally mislabeling experts as novices is not catastrophic: some experts need to do easy jobs anyway, and so the algorithm can account for it in the exploitation phase. Thus, relatively less effort can be invested in minimizing this error. However, mistakenly labeling novices as experts *can* be catastrophic: in this case, novices will be matched to hard jobs in the exploitation phase, causing substantial loss of payoff; thus the probability of such errors must be kept very small. A major contribution of our work is to precisely identify the correct learning goals in the exploration phase, and then design DEEM to meet these learning goals while maximizing payoff generation.

Third, DEEM involves a carefully constructed exploitation phase to ensure that capacity constraints are met while maximizing payoffs. A naive approach during the exploitation phase would match a worker to any job type that yields the maximum externality-adjusted payoff corresponding to his type label. It turns out that such an approach leads to significant violations of capacity constraints, and hence poor performance. The reason is that in a generic capacitated problem instance, one or more worker types are indifferent between multiple job types, and suitable tie-breaking is necessary to achieve good performance. In our theoretical development, we achieve this by modifying the solution to the static optimization problem with known worker types, whereas our practical implementation of DEEM achieves appropriate tie-breaking via simple but dynamically updated shadow prices. Both solutions are decentralized.

Our main result (Theorem 5.1) shows that DEEM achieves essentially optimal regret as the number of jobs N performed by each worker during their lifetime grows, where regret is the loss in payoff relative to the maximum achievable with known worker types. In our setting, a lower bound on the regret is $C \log N/N(1 + o(1))$ for some $C \in [0, \infty)$ that is a function of system parameters. DEEM achieves this level of regret to leading order when $C > 0$, while it achieves a regret of $O(\log\log N/N)$ if $C = 0$. Our theory is complemented by implementation and simulation that demonstrate a natural and implementable heuristic that translates our work into practice.[2] In particular, our simulations reveal substantial benefit from jointly managing capacity constraints and learning, as we do in DEEM.

The remainder of the paper is organized as follows. After discussing related work in Section 2, we present our model and outline the optimization problem of interest to the platform in Section 3. In Section 4, we discuss the three key ideas above in the design of DEEM, and present its formal definition. In Section 5, we present our main theorem, and discuss the optimal regret scaling. In Section 6 we present a sketch of the proof of the main result. In Section 7, we discuss practical implementation of DEEM, and use simulations to compare the performance of the resulting heuristic with well-known multi-armed bandit algorithms. We conclude in Section 8. All the proofs of the results appearing in the paper are presented in the Appendices.

2 RELATED LITERATURE

A foundational model for investigating the exploration-exploitation tradeoff is the stochastic multi-armed bandit (MAB) problem [Audibert and Munos, 2011, Bubeck and Cesa-Bianchi, 2012, Gittins et al., 2011]. The goal is to find an adaptive expected-regret-minimizing policy for choosing among arms with unknown payoff distributions (where regret is measured against the expected payoff of the best arm) [Agrawal and Goyal, 2011, Auer et al., 2002, Lai and Robbins, 1985]. The closest work in this literature to our paper is by Agrawal et al. [1989]; in their model, they assume the joint vector of arm distributions can only take on one of finitely many values. This introduces correlation across different arms; depending on certain identifiability conditions, the optimal regret is either $\Theta(1/N)$ or $\Theta(\log N/N)$. In our model, the analog is that job types are arms, and for each worker we solve a MAB problem to identify the true type of a worker, within a finite set of possible worker types.

Our work is also related to recent literature on MAB problems with capacity constraints; we refer to these broadly as *bandits with knapsacks*. The formulation is the same as the classical MAB problem, with the modification that every pull of an arm depletes a vector of resources which are limited in supply [Badanidiyuru et al., 2013]. The formulation subsumes several related problems in revenue management under demand uncertainty [Babaioff et al., 2015, Besbes and Zeevi, 2009, 2012, Sauré and Zeevi, 2013, Wang et al., 2014], and budgeted dynamic procurement [Badanidiyuru et al., 2012, Singla and Krause, 2013]; there have been a variety of extensions [Agrawal and Devanur, 2014, Badanidiyuru et al., 2014], with recently a significant generalization of the problem to a contextual bandit setting, with concave rewards and convex constraints [Agrawal and Devanur, 2015, Agrawal et al., 2015]. There is considerable difference between our model and bandits with knapsacks. Bandits with knapsacks consider a single MAB problem over a fixed time horizon. Our setting on the other hand can be seen as a system with an ongoing arriving *stream* of MAB problems, one per worker; these MAB problems are coupled together by the capacity constraints on arriving jobs. Indeed, as noted in the introduction, a significant structural point for us is to solve these problems in a decentralized manner, to ease their implementation in large-scale online platforms.

[2] Notably, our implementation does not require knowledge of the arrival rates of jobs and workers.

We conclude by discussing some other directions of work that are related to this paper. There are a number of recent pieces of work that consider efficient matching in dynamic two-sided matching markets [Akbarpour et al., 2014, Anderson et al., 2015, Baccara et al., 2015, Damiano and Lam, 2005, Das and Kamenica, 2005, Hu and Zhou, 2015, Kadam and Kotowski, 2015, Kurino, 2005]; a related class of dynamic resource allocation problems, online bipartite matching, is also well studied in the computer science community (see [Mehta, 2012] for a survey). Similar to the current paper, Fershtman and Pavan [2015] also study matching with learning, mediated by a central platform. Relative to our model, their work does not have constraints on the number of matches per agent, while it does consider agent incentives. Finally, a recent work [Massoulie and Xu, 2016] studies a pure learning problem in a setting similar to ours with capacity constraints on each type of server/expert; while there are some similarities in the style of analysis, that paper focuses exclusively on learning the exact type, rather than balancing exploration and exploitation as we do in this paper.

3 THE MODEL AND THE OPTIMIZATION PROBLEM

In this section we first describe our model. In particular, we describe the primitives of our platform ("workers" and "jobs"), and givea formal specification of the matching process we study. We conclude by precisely defining the optimization problem of interest that we solve in this paper.

3.1 Preliminaries

Workers and jobs. For convenience we adopt the terminology of *workers* and *jobs* to describe the two sides of the market. We assume a fixed set of job types $\mathcal{J}$, and a fixed set of worker types $\mathcal{I}$.

A key point is that the model we consider is a continuum model, and so the evolution of the system will be described by *masses* of workers and jobs.[3] In particular, at each time step, a mass $\hat{\rho}(i) > 0$ of workers of type i and a mass $\mu(j) > 0$ of jobs of type j arrive. In what follows, we model the scenario where type uncertainty exists only for workers; i.e., the platform will know the types of arriving jobs exactly, but will need to learn the types of arriving workers. We also assume for now that the arrival rates of jobs and workers are known to the platform; later in Section 7, we discuss how the platform might account for the possibility that these parameters are unknown.

Matching and the payoff matrix. If a mass of workers of type i is matched to a mass of jobs of type j, we assume that a fraction $A(i, j)$ of this mass of matches generates a reward of 1 (per unit mass), while a fraction $1 - A(i, j)$ generates a reward of zero (per unit mass). This formal specification is meant to capture a large-system model in a setting where matches between type i workers and type j jobs generate a Bernoulli($A(i, j)$) payoff. We do not concern ourselves with the division of payoffs between workers and employers in this paper; instead we assume that the platform's goal is to maximize the total rate of payoff generation.[4] We call the matrix A the *payoff matrix*; throughout, we assume that no two rows of A are identical.[5]

A key assumption in our work is that the platform *knows* the matrix A. In particular, we are considering a platform that has enough aggregate information to understand compatibility between different worker and job types; however, for any *given* worker newly arriving to the platform, the platform does not know the worker's type. Thus, from the perspective of the platform, there will be uncertainty in payoffs in each period because although the platform knows that a given mass of workers of type i exist in the platform, the identity of the workers of type i is not known.

[3] Formally, this can be seen as a continuum scaling of a discrete system; see, e.g., [Dai, 1995, Maglaras and Zeevi, 2003, 2005]
[4] This would be the case, e.g., in a platform where the operator takes a fixed percentage of the total payoff generated from a match.
[5] This mild requirement simply ensures that it is possible, in principle, to distinguish between each pair of worker types.

We define an "empty" job type κ, such that all worker types matched to κ generate zero reward, i.e., $A(i, \kappa) = 0$ for all i. We view κ as representing the possibility that a worker goes unmatched, and thus assume that an unbounded capacity of job type κ is available, i.e., $\mu(\kappa) = \infty$.

Worker lifetimes. We imagine that each arriving worker lives in the system for N time steps,[6] and has the opportunity to be matched to a job in each time step (so each job takes one unit of time to complete). We assume the platform knows N.

Note that we have $\rho(i) = \hat{\rho}(i)N$ as the total mass of workers of type i in the system at each time step. For our theoretical analysis, we later consider a scaling regime where $N \to \infty$, and $\hat{\rho}(i) \to 0$, while $\rho(i)$ remains fixed. In this regime worker lifetimes grow to infinity, and arrival rates scale down, but the total mass of workers of each type available in each time period remains fixed.

Generalized imbalance. Throughout our technical development, we make a mild structural assumption on the problem instance, defined by the tuple (ρ, μ, A). This is captured by the following definition. We say that arrival rates $\rho = (\rho(i))_{i \in \mathcal{I}}$ and $\mu = (\mu(j))_{j \in \mathcal{J}}$ satisfy the *generalized imbalance condition* if there is no pair of nonempty subsets of worker types and job types $(\mathcal{I}', \mathcal{J}')$, such that the total worker arrival rate of $\mathcal{I}'$ exactly matches the total job arrival rate of $\mathcal{J}'$. Formally,

$$\sum_{i \in \mathcal{I}'} \rho(i) \neq \sum_{j \in \mathcal{J}'} \mu(j) \quad \forall \mathcal{I}' \subseteq \mathcal{I}, \mathcal{J}' \subseteq \mathcal{J}, \mathcal{I}' \neq \phi \,.$$

The generalized imbalance condition holds generically.[7] Note that this condition does not depend on the matrix A.

Worker history. To define the state of the system and the resulting matching dynamics, we need the notion of a worker history. A *worker history* is a tuple $H_k = ((j_1, x_1), \ldots, (j_k, x_k))$, where j_m is the job type this worker was matched to at her m-th time step in the system, for $1 \leq m \leq k$; and $x_m \in \{0, 1\}$ is the corresponding reward obtained. Note that since workers live for N jobs, the histories will have $k = 0, \ldots, N-1$. We let ϕ denote the empty history (for $k = 0$).

3.2 System dynamics

Our goal is to model the following process. The operator observes, at any point in time, the distribution of histories of workers in the platform, and also knows the job arrival rate μ. The matching policy of the platform amounts to determining what mass of workers of each type of history will be matched to which type of jobs. Ultimately, for this process to generate high payoffs over time, the platform must choose jobs to learn worker types in order to optimize payoffs.

With this intuition in mind, we now give a formal specification of our system dynamics.

System profile. A *system profile* ν is a joint measure over worker histories and worker types; i.e., $\nu(H_k, i)$ is the mass of workers in the system with history H_k and type i. The evolution of the system is a discrete-time dynamical system $\nu_0, \nu_1, \nu_2, \ldots$, where each ν_t is a system profile.[8]

Matching policy. To describe the dynamics we assume that the platform uses a *matching policy* to match the entire mass of workers to jobs in each time step (we think of unmatched workers as being matched to the empty job type κ). We assume that any mass of jobs left unmatched in a given period disappears at the end of that period (our results do not depend on this assumption).

[6] Our analysis and results generalize to random worker lifetimes that are i.i.d. across workers of different types, with mean N and any distribution such that the lifetime exceeds $N/\text{polylog}(n)$ with high probability.

[7] The set $(\hat{\rho}, \mu)$ for which the condition holds is open and dense in $\mathbb{R}_{++}^{|\mathcal{I}|+|\mathcal{J}|}$, where $\mathbb{R}_{++}$ are the strictly positive real numbers.

[8] The platform cannot directly observe the system profile, but can infer it. The platform observes the mass of workers with each possible history $\left(\sum_{i \in \mathcal{I}} \nu(H_k, i)\right)_{H_k}$. It can then infer $\nu(H_k, i)$'s individually by using knowledge of arrival rates $\hat{\rho}(i)$'s, and the A matrix (which allows it to calculate the likelihood of seeing the sequence of outcomes in H_k under the worker type i), together with Bayes' rule.

Suppose that the system starts at time $t = 0$ with no workers in the system before this time.[9] A *matching policy* $\pi_0, \pi_1, \dots$ for the system specifies, at each time t, given a system profile ν_t, the mass of workers with each history that is matched to jobs of each type. In particular, let $\pi_t(H_k, j|\nu_t)$ denote the fraction of workers with history H_k matched to jobs of type j at time t, given a system profile ν_t. (Thus $\sum_j \pi_t(H_k, j|\nu_t) = 1$ for all t, H_k, and ν_t.) Note that the matching policy acts on each worker's history, not on the true type of each worker: this is because the platform is assumed to not know worker types, except as learned through the history itself.

Dynamics. These features completely determine the evolution of the system profile $\{\nu_t\}$. Observe that $\nu_t(H_k, i)\pi_t(H_k, j|\nu_t)$ is the total mass of workers of type i with history H_k who are matched to jobs of type j at time t, given policy π_t and system profile ν_t. For all i, j, and t, we have

$$\nu_{t+1}(\phi, i) = \hat{\rho}(i); \tag{1}$$

$$\nu_{t+1}((H_k, (j, 1)), i) = \nu_t(H_k, i)\pi_t(H_k, j|\nu_t)A(i, j), \qquad k = 0, \dots, N-2; \tag{2}$$

$$\nu_{t+1}((H_k, (j, 0)), i) = \nu_t(H_k, i)\pi_t(H_k, j|\nu_t)(1 - A(i, j)), \; k = 0, \dots, N-2. \tag{3}$$

Decentralization through worker-history-only (WHO) policies. Note that, in general, policies may be time-varying, and may have complex dependence on the system profile ν_t. We consider a much simpler class of policies that we call *worker-history-only (WHO) policies.* These are policies where there exists a π such that

$$\pi_t(H_k, j|\nu_t) = \pi(H_k, j).$$

In other words, in a WHO policy, the fraction of workers with history H_k who are matched to jobs of type j does not depend on either time or on the full system profile. Thus WHO policies are *decentralized.*

An obvious concern at this point is that a policy cannot allocate more jobs of type j than there are. We formalize this capacity constraint in (8) below: in particular, a WHO policy does not exceed the capacity of any job type in any period if and only if it satisfies (8).

Let Π^N denote the class of WHO policies, for a given N. In Section D.1 in Appendix D, we establish that it suffices to restrict attention to policies in Π^N that satisfy (8).

Remark 1. *For any feasible policy, there exists a WHO policy satisfying capacity constraints that achieves a payoff accumulation rate arbitrarily close to that of the former policy. In particular, WHO policies satisfying capacity constraints suffice to achieve the highest possible payoff accumulation rate.*

Steady state of a WHO policy π. First, suppose that there are no capacity constraints, and consider the system dynamics (1)–(3), assuming the system initially starts empty. The dynamics (1)–(3) yields a unique steady state that can be inductively computed for $k = 0, 1, \dots$:

$$\nu_\pi(\phi, i) = \hat{\rho}(i); \tag{4}$$

$$\nu_\pi((H_k, (j, 1)), i) = \nu_\pi(H_k, i)\pi(H_k, j)A(i, j), \; k = 0, \dots, N-2; \tag{5}$$

$$\nu_\pi((H_k, (j, 0)), i) = \nu_\pi(H_k, i)\pi(H_k, j)(1 - A(i, j)), \; k = 0, \dots, N-2. \tag{6}$$

We refer to the measure ν_π as the *steady state* induced by the policy π.

Routing matrix of a WHO policy π. If the system is in steady state, then at any time period, π induces a steady-state fraction $x_\pi(i, j)$ of the mass of workers of type i that are assigned to type j jobs. We have $x_\pi(i, j) = \frac{\sum_H \nu_\pi(H,i)\pi(H,j)}{\sum_H \nu_\pi(H,i)} = \frac{\sum_H \nu_\pi(H,i)\pi(H,j)}{\rho(i)}$. We call $\{x_\pi(i, j)\}_{\mathcal{I}\times\mathcal{J}}$ the *routing matrix* achieved by the policy π. This is a (row) stochastic matrix; i.e., each row sums to 1. Observe

[9] In what follows we ultimately consider a steady-state analysis of the dynamical system, and initial conditions will be irrelevant as long as the initial mass of workers is bounded.

that the mass of demand for jobs of type j from workers of type i in any time period is $\rho(i)x_\pi(i,j)$, and the total mass of demand for jobs of type j in any time period is $\sum_{i\in\mathcal{I}}\rho(i)x_\pi(i,j)$.

Let $\mathcal{X}^N = \left\{ x_\pi : \pi \in \Pi^N \right\} \subseteq [0,1]^{|\mathcal{I}|\times|\mathcal{J}|}$ be the set of routing matrices achievable (when each worker does N jobs) by WHO policies. (Again, we note that capacity constraints are ignored in the definition of $\mathcal{X}^N$.) In Appendix D, we show that $\mathcal{X}^N$ is a convex polytope (see Proposition D.4).

3.3 The optimization problem

Our paper focuses on maximization of the *steady-state rate of payoff accumulation*, subject to the capacity constraints. This leads to the following optimization problem:

$$\text{maximize} \quad W^N(\pi) \triangleq \sum_{i\in\mathcal{I}}\rho(i)\sum_{j\in\mathcal{J}} x_\pi(i,j)A(i,j) \tag{7}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{I}}\rho(i)x_\pi(i,j) \le \mu(j) \qquad \forall j\in\mathcal{J}\,; \tag{8}$$

$$x_\pi \in \mathcal{X}^N. \tag{9}$$

The objective is the steady-state rate of payoff accumulation per time period, expressed in terms of the routing matrix induced by a (WHO) policy π. The constraint is the capacity constraint: the system will be stable if and only if the total "demand" for jobs of type j is not greater than the arrival rate of jobs of type j.

Since $\mathcal{X}^N$ is a convex polytope, this is a linear program, albeit a complex one. The complexity of this problem is hidden in the complexity of the set $\mathcal{X}^N$, which includes all possible routing matrices that can be obtained using WHO policies. The remainder of our paper is devoted to solving this problem and characterizing its value, by considering an asymptotic regime where $N\to\infty$.

3.4 The benchmark: Known worker types

We evaluate our performance relative to a natural benchmark: the maximal rate of payoff accumulation possible if worker types are perfectly *known* upon arrival. In this case, *any* stochastic matrix is feasible as a routing matrix. Let $\mathcal{D}$ denote the set of all stochastic matrices:

$$\mathcal{D} = \left\{ x\in\mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|} : x(i,j)\ge 0;\ \sum_{j\in\mathcal{J}} x(i,j) = 1 \right\}. \tag{10}$$

Note that any routing matrix in $\mathcal{D}$ is implementable by a simple policy under known worker types: given a desired routing matrix x, route a fraction $x(i,j)$ of workers of type i to jobs of type j.

Thus, with known worker types, the maximal rate of payoff accumulation is given by the solution to the following optimization problem:

$$\text{maximize} \quad \sum_{i\in\mathcal{I}}\rho(i)\sum_{j\in\mathcal{J}} x(i,j)A(i,j) \tag{11}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{I}}\rho(i)x(i,j) \le \mu(j) \qquad \forall j\in\mathcal{J}\,; \tag{12}$$

$$x\in\mathcal{D}. \tag{13}$$

We let V^* denote the maximal value of the preceding optimization problem, and let x^* denote the solution. This linear program is a special case of the "static planning problem" that arises frequently in the operations literature (see, e.g. [Ata and Kumar, 2005]). The problem can also be viewed as a version of the assignment problem due to Shapley and Shubik [Shapley and Shubik, 1971], in which the resources are divisible.

3.5 Regret

We evaluate the performance of a given policy in terms of its *regret* relative to V^*. In particular, given N and a WHO policy π satisfying (8), we define the regret of π as $V^* - W^N(\pi)$.

We focus on the asymptotic regime where $N \to \infty$, and try to find policies that have "small" regret in this regime. This asymptotic regime allows us to identify structural aspects of policies that perform well. In Appendix D (see Proposition D.3), we show that it is relatively "easy" to design policies that achieve a vanishing regret (and even regret that is within a constant factor of the smallest possible). The idea is straightforward: informally, when N is large, policies that "explore" for a vanishing fraction of worker lifetimes will be able to learn the worker's true type sufficiently well to yield a rate of payoff accumulation such that regret converges to zero in the limit.

For this reason, our analysis focuses on a more refined notion of asymptotic optimality. In particular, we focus on developing policies that achieve a nearly optimal *rate* at which the regret $V^* - W^N(\pi_N)$ approaches zero. This is formalized in Theorem 5.1 below.

3.6 A note on terminology

Note that, intuitively, WHO policies have the feature that decisions are taken on the basis of the history of a given worker, not on the basis of the system profile as a whole. In the sequel, *we will typically refer to $\pi(H_k, j)$ as "the probability that a worker of history H_k is matched to a job of type j."* We use this terminology to make the presentation more intuitive, since the intention is that our algorithms be implemented at the level of each individual worker's history. However, to formalize all our arguments, we emphasize that our proofs translate $\pi(H_k, j)$ as the fraction of workers of history H_k matched to a job type j; this correspondence applies throughout the technical development.

4 DEEM: A PAYOFF-MAXIMIZING POLICY

In this section we present the design of a sequence of policies π_N^* that achieves a nearly optimal rate of convergence of regret $V^* - W^N(\pi_N^*)$. We refer to our policy design as *DEEM: Decentralized Explore-then-Exploit for Matching*. Our main result, stated in the next section, is Theorem 5.1: there we exactly quantify the regret performance of DEEM (an upper bound on its regret), and characterize it as nearly optimal (a lower bound on the regret of any feasible WHO policy).

To begin to understand the challenges involved, consider the example in Figure 1. In this example, there are two types of workers: "novice" and "expert," with a mass of $\rho = 0.5$ of each present in steady state. There are two types of jobs: "easy" and "hard," each arriving at rate 0.6.

We make several observations regarding this example that inform our subsequent work.

(1) *The benchmark.* In this example, the optimal solution to the benchmark problem (11)–(13) with *known* types routes all novices to easy jobs, a mass 0.1 of experts to easy jobs, and a mass 0.4 of experts to hard jobs. Of course, our problem is that we do not know worker types on arrival.

(2) *Capacity constraints affect an optimal WHO policy's need to learn.* If easy and hard jobs are in infinite supply, then the WHO policy π that matches all workers to easy jobs is optimal. However, with the finite supply of available easy jobs, some workers must do hard jobs. But which workers?

Clearly, for payoff optimality, an optimal policy should aim to match *experts* to hard jobs. But this is only possible if it first learns that a worker is an expert. Because of the structure of A, the type of a worker can only be learnt by matching it to hard jobs; those who perform well on these jobs are experts, and those who fail are novices.

(3) *Minimizing regret requires learning up front.* Assigning workers of unknown type to hard jobs necessarily incurs regret relative to the benchmark. Indeed, novices unknowingly matched to hard jobs lead to a regret of 0.8 per unit mass of such workers in each period. Minimizing this

regret therefore requires that the algorithm not only learn worker types, but also do so relatively early in their lifetime, so that workers identified as experts can be assigned many hard jobs.

In our work, this leads to a structure where we separate our policy into *exploration* and *exploitation* phases: the policy first tries to learn a worker's type, and then "exploits" by assigning this worker to jobs while assuming that the learned type is correct. The exploration phase will be of length $O(\log N)$, which is short relative to the worker's lifetime.

(4) *Some mistakes in the exploration phase are worse than others.* There are two kinds of mistakes that the policy can make while learning: it can mistakenly identify novices as experts, and it can mistakenly identify experts and novices. These mistakes differ in their impact on regret.

Suppose that at the end of the exploration phase, the algorithm misclassifies a novice as an expert. This has a dire impact on regret: the novice is then assigned to hard jobs in the exploitation phase, and as noted above, this incurs a regret of 0.8 per unit mass (of workers misclassified this way) per unit time. Thus we must work hard in the exploration phase to avoid such errors.

On the other hand, suppose that at the end of the exploration phase, the algorithm misclassifies an expert as a novice. This mistake is far less consequential: workers misclassified in this way will be assigned to easy jobs. But a mass 0.1 of experts must be assigned to easy jobs even in the benchmark solution with known types. Therefore, as long as this misclassified mass is not too large, we can adjust for it in the exploitation phase.

This discussion highlights the need to precisely identify the *learning goals* of the algorithm: to minimize regret, how strongly does each worker type need to be distinguished from others? A major contribution of our work is to demonstrate an optimal construction of learning goals for regret minimization. As noted above, the capacity constraints fundamentally influence the learning goals of the algorithm.

In the remainder of the section, we describe key ideas behind the construction of our policy, highlighted by the issues raised in the preceding example. We formally describe DEEM in Section 4.4. We state our main theorem in Section 5.

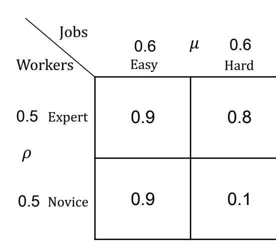

Fig. 1. An example

4.1 Key idea 1: Use shadow prices as an "externality adjustment" to payoffs

We begin by first noticing an immediate difficulty that arises in using WHO policies in the presence of capacity constraints. WHO policies are decentralized, i.e., they act only on the history of the worker; as such, they cannot use *aggregate* state information about the system, that conveys whether capacity constraints are being met or not. In order to solve (7)–(9), therefore, we need to find a way to "adjust" for capacity constraints despite the fact that our policy acts only at the level of worker histories.

Our key insight is to use *shadow prices* for the capacity constraints to adjust payoffs; we then measure regret with respect to these adjusted payoffs. Recall that (7)–(9) is a linear program. Let p^N be the optimal shadow prices (dual variables) for the capacity constraints (8). Then by standard duality results, it follows that the policy that is optimal for (7)–(9) is also optimal for the following unconstrained optimization problem:

$$\text{maximize}_{x_\pi \in \mathcal{X}^N} \sum_{i \in \mathcal{I}} \rho(i) \sum_{j \in \mathcal{J}} x_\pi(i,j)(A(i,j) - p^N(j)). \tag{14}$$

Thus one may attempt to account for capacity constraints using shadow prices[10] $p^N(j)$.

The challenge here is that the set $\mathcal{X}^N$ is quite complex, and thus characterizing the optimal shadow prices of (7)–(9) is not a reasonable path forward. Instead, we use the optimal shadow prices in the benchmark linear program with *known types* (11)–(13) to adjust payoffs; we then measure regret with respect to these adjusted payoffs (the practical heuristic we implement uses a different, instance-independent approach to estimate shadow prices; see Section 7).

We let p^* denote the vector of optimal shadow prices for the capacity constraint (12) in the problem with known types (11)–(13). Using the generalized imbalance condition, we show that these prices are uniquely determined; see Proposition D.2 in Appendix D.

Although $p^*(j) \neq p^N(j)$, for large N, the platform should be able to learn the type of a worker type early in her lifetime, leading to small $|p^*(j) - p^N(j)|$. This motivates an analog of (14):

$$\text{maximize}_{x_\pi \in \mathcal{X}^N} \sum_{i \in \mathcal{I}} \rho(i) \sum_{j \in \mathcal{J}} x_\pi(i,j)(A(i,j) - p^*(j)). \tag{15}$$

We develop a near-optimal algorithm for problem (15), such that (1) constraints on job capacities are not violated, and (2) complementary slackness conditions are satisfied, i.e., if $p^*(j) > 0$, then the job type j is fully utilized. We then show this leads to the upper bound in the main result.

4.2 Key idea 2: Meet required learning goals while minimizing regret

As noted in our discussion of the example in Figure 1, we must carefully define the *learning goals* of the algorithm: which worker types need to be distinguished from which others, and with what level of confidence? A key contribution of our work is to formalize the learning goals of our algorithm. In this section we define the learning goals of the algorithm, and outline the exploration phase that meets these goals.

Let the set of optimal job types for worker type i be defined by $\mathcal{J}(i) = \arg\max_{j \in \mathcal{J}} A(i,j) - p^*(j)$. (A standard duality argument demonstrates that in any optimal solution of the benchmark (11)–(13), a worker type i is assigned only to jobs in $\mathcal{J}(i)$.)

Recall that in the example in Figure 1, it is far more important not to misclassify a novice as an expert than to misclassify an expert as a novice. We formalize this distinction through the following definition.

Definition 4.1. We say that a type i needs to be *strongly distinguished* from a type i' if $\mathcal{J}(i) \setminus \mathcal{J}(i') \neq \emptyset$. For each worker type i, let $\text{Str}(i)$ be the set of all types i' from which i needs to be strongly distinguished, i.e., $\text{Str}(i) = \{i' : \mathcal{J}(i) \setminus \mathcal{J}(i') \neq \emptyset\}$.

In words, this means that i needs to be strongly distinguished from i' if it has at least one optimal job type that is not optimal for i', whereas it needs to be only *weakly distinguished* from i' if all optimal job types for i are also optimal for i'. This definition is most easily understood through the example in Figure 1 and our subsequent discussion. In particular, note that for that example, the benchmark shadow prices are $p^*(\text{easy}) = 0.1$ and $p^*(\text{hard}) = 0$, and thus $\mathcal{J}(\text{novice}) = \{\text{easy}\}$, while $\mathcal{J}(\text{expert}) = \{\text{easy}, \text{hard}\}$. Thus experts need to be strongly distinguished from novices, since hard jobs are optimal for experts but not for novices; on the other hand, novices need to be only weakly distinguished from experts, since easy jobs are optimal for experts as well.

In the exploration phase of our algorithm, our goal is to classify a worker's type as quickly as possible: the preceding definition is what we use to formalize the learning goals in this phase. In particular, consider making an error where the true type is i, but we misclassify it as i'. For any i'

[10]Further effort is needed to ensure the policy does not violate capacity constraints, and that complementary slackness holds.

not in $\mathrm{Str}(i)$, any probability of an error of $o(1)$ for a misclassification error is tolerable as N grows large (as in the example in Figure 1). We choose $\Theta(1/\log N)$ as the target error probability for this kind of error. On the other hand, for any $i' \in \mathrm{Str}(i)$, the optimal target error probability is much smaller. In particular, the optimal target error probability can be shown to be approximately $1/N$: if we choose a larger target, we will incur a relatively large expected regret during exploitation due to misclassification, if we choose a smaller target, the exploration phase is unnecessarily long, and we thus incur a relatively large regret in the exploration phase.

With the learning goals defined, the exploration phase of DEEM operates in one of two subphases: either "guessing" or "confirmation," as follows. First, we check whether the likelihood of the maximum likelihood estimator (MLE) of the worker type is sufficiently high. If this likelihood is low, we say the policy is in the "guessing" subphase of the exploration phase, and a job type is chosen at random for the next match. On the other hand, if this likelihood is high (in particular, greater than $\log N$ times the likelihood of any other worker type), then we say that the policy is in the "confirmation" subphase of the exploration phase: in this regime, the policy works to confirm the MLE as the correct type as quickly as possible. Specifically, in the confirmation subphase, the policy focuses only on *strongly distinguishing* the MLE from all other types in $\mathrm{Str}(i)$; the trade-off is that this must be done with minimum regret. We frame this as an optimization problem (see (16) below): essentially, the goal is to find a distribution over job types that minimizes the expected regret until the confirmation goals are met. In the confirmation subphase, the policy allocates the worker to jobs according to this distribution, until the type is confirmed.

4.3 Key idea 3: Optimally allocate in the exploitation phase while meeting capacity constraints

When the algorithm completes the exploration phase, it enters the exploitation phase; in this phase, the algorithm aims to match a worker to jobs that maximize the rate of payoff generation, given the confirmed type label. A naive approach would match a worker labeled type i to any job type in $\mathcal{J}(i)$, since these are the optimal job types for worker type i after externality adjustment.

This approach turns out to fail spectacularly and generically leads to $\Omega(1)$ regret (this occurs for any set of fixed shadow prices). To see why, we need the following fact.

FACT 1. *Under generalized imbalance, as long as there is at least one capacity constraint that is binding in some optimal solution x^* to the benchmark problem* (11)–(13) *with known types, there is at least one worker i such that $x^*(i,\cdot)$ is supported on multiple job types.*

This fact implies that appropriate *tie-breaking* between multiple optimal job types is *necessary during exploitation* for one or more worker types in order to achieve vanishing regret.

In order to implement appropriate tie-breaking, suppose that we assign jobs during the exploitation phase using the routing matrix x^* that solves the benchmark problem (11)–(13); in this case, each worker with confirmed type i is matched to job type j with probability $x^*(i,j)$. However, this naive approach needs further modification to overcome two issues. First, some capacity is being used in the exploration phase and the effective routing matrix during the exploration phase does not match x^*. Second, the exploration phase can end with an incorrectly classified worker type.

Our policy in the exploitation phase chooses a routing matrix y^* that resembles x^*, but addresses the two concerns raised in the preceding paragraph. Crucially, the chosen y^* should ensure that only job types in $\mathcal{J}(i)$ are assigned with positive probability, and satisfy the complementary slackness conditions. We show (in Proposition A.5, using Fact 1) that such a y^* indeed exists for an N large enough under the generalized imbalance condition, and we show how to compute it. Note that as y^* is a fixed routing matrix, it can be implemented in a decentralized manner.

We comment here that y^* is largely a theoretical device used to obtain the provable regret optimality of our policy. In our implementation of DEEM (see Section 7), we propose a far simpler solution: we use *dynamically updated shadow prices* to automatically achieve appropriate tie-breaking. The shadow prices respond in a "tâtonnement" manner based on the currently available supply of different job types: the price of job type j rises when the available supply falls. In particular, fluctuations in these shadow prices naturally lead to the necessary tie-breaking for efficient exploitation.

4.4 Formal definition: Decentralized Explore-then-Exploit for Matching (DEEM)

In this section we provide a formal definition of the policy π_N^*, based on the discussion above.

First, for each i define the maximal externality-adjusted utility $U(i) = \max_{j\in\mathcal{J}} A(i,j) - p^*(j)$. Then choose $\alpha(i)$ such that:

$$\alpha(i) \in \mathcal{A}(i) = \arg\min_{\alpha\in\Delta(\mathcal{J})} \frac{\sum_{j\in\mathcal{J}} \alpha_j\big(U(i) - [A(i,j) - p^*(j)]\big)}{\min_{i'\in\mathrm{Str}(i)} \sum_{j\in\mathcal{J}} \alpha_j \mathrm{KL}(i,i'|j)}, \tag{16}$$

where $\mathrm{KL}(i,i'|j)$ is the Kullback–Leibler divergence[11] between Bernoulli$(A(i,j))$ and Bernoulli$(A(i',j))$, and $\Delta(\mathcal{J})$ is the set of distributions over $\mathcal{J}$. The idea is that sampling job types from $\alpha(i)$ allows the policy to distinguish i simultaneously from all $i' \in \mathrm{Str}(i)$, while incurring the smallest possible externality-adjusted regret. If the optimization problem in (16) has multiple solutions, we pick the one that has the largest denominator (and hence the largest numerator as well), thus maximizing learning rate subject to optimality. Note that (16) can be written as a small linear program.[12]

For $m = 1,\cdots,N-1$, let the job type chosen at opportunity m be j_m and the outcome be X_m. For any $i \in \mathcal{I}$ and $j \in \mathcal{J}$, let $l(X,i,j) = A(i,j)\mathbf{1}_{\{X=1\}} + (1-A(i,j))\mathbf{1}_{\{X=0\}}$. Define $\lambda_0(i) = 1$, and for $k \geq 1$, let $\lambda_k(i) = \prod_{m=1}^{k} l(X_m,i,j_m)$ denote the likelihood of the observed history until the k-th job under worker type i. Let $\mathrm{MLE}_k = \arg\max_{i\in\mathcal{I}} \lambda_k(i)$ be the maximum likelihood estimate based on the history, and define $\Lambda_k(i,i') = \frac{\lambda_k(i)}{\lambda_k(i')}$, i.e., the ratio of the likelihoods of the history under type i vs. i'. For convenience, we refer to $\Lambda_k(i,i')$ as the *likelihood ratio.*

DEEM is defined as follows.

(1) **Phase 1: Exploration.** Suppose that $i = \mathrm{MLE}_k$.

(a) **Guessing subphase**. If $\min_{i'\neq i} \Lambda_k(i,i') < \log N$, choose the next job type uniformly at random in $\mathcal{J}$.

(b) **Confirmation subphase to strongly distinguish i from types in Str(i)**. If we have $\min_{i'\neq i} \Lambda_k(i,i') \geq \log N$ but $\min_{i'\in\mathrm{Str}(i)} \Lambda_k(i,i') < N$, draw the next job type i.i.d. from the distribution $\alpha(i)$.

(c) **Exit condition for the exploration phase.** If $\min_{i'\neq i} \Lambda_k(i,i') \geq \log N$ and $\min_{i'\in\mathrm{Str}(i)} \Lambda_k(i,i') \geq N$, then the worker is labeled as being of type i and the policy moves to the exploitation phase. (The worker is never returned to the exploration phase.)

(2) **Phase 2: Exploitation.** For every job opportunity, for a worker confirmed to be of type i, choose a job in $\mathcal{J}(i)$ with probability $y^*(i,j)$, where y^* is a routing matrix (specified in Proposition A.5 in Appendix A) such that system capacity constraints are not violated in steady state.

[11]The KL divergence between a Bernoulli(q) and a Bernoulli(q') distribution is defined as $q\log\frac{q}{q'} + (1-q)\log\frac{1-q}{1-q'}$.

[12]Denoting $\min_{i'\in\mathrm{Str}(i)} \sum_{j\in\mathcal{J}} \alpha_j \mathrm{KL}(i,i'|j)$ as h (where h is non-negative), the optimization problem is the same as minimizing $\sum_{j\in\mathcal{J}} \frac{\alpha_j}{h}\big(U(i) - [A(i,j) - p^*(j)]\big)$ subject to $\sum_{j\in\mathcal{J}} \frac{\alpha_j}{h}\mathrm{KL}(i,i'|j) \geq 1$, $\sum_j \frac{\alpha_j}{h} = \frac{1}{h}$, $\frac{1}{h} \geq 0$ and $\frac{\alpha_j}{h} \geq 0$ for all j. Now redefine $\frac{\alpha_j}{h} \triangleq \bar{\alpha}_j$ and $\frac{1}{h} \triangleq h'$ to obtain a linear program. Our tie-breaking rule amounts to the picking the optimum with the smallest value of h'.

5 MAIN RESULT

Our main result is the following theorem. In particular, we prove a lower bound on the regret of any policy, and show the sequence of policies π_N^* constructed in the preceding section (essentially) achieves this lower bound.

THEOREM 5.1. *Fix (ρ, μ, A) such that: (a) no two rows of A are identical; and (b) the generalized imbalance condition holds. Then there is a constant $C = C(\rho, \mu, A) \in [0, \infty)$ such that*

(1) (Lower bound) For any N and any WHO policy π that is feasible for (7)–(9),

$$V^* - W^N(x_\pi) \geq \frac{C \log N}{N}\Big(1 + \mathrm{o}(1)\Big) \text{ and} \tag{17}$$

(2) (Upper bound) The sequence of policies π_N^* is feasible for (7)–(9) *for each N, with:*

$$V^* - W^N(\pi_N^*) \leq \frac{C \log N}{N}\Big(1 + \mathrm{o}(1)\Big) + \mathrm{O}\Big(\frac{\log \log N}{N}\Big). \tag{18}$$

The constant C that appears in the theorem depends on the primitives of the problem, i.e., (ρ, μ, A); it is defined as follows:

$$C(i) = \min_{\alpha \in \Delta(\mathcal{J})} \frac{\sum_{j \in \mathcal{J}} \alpha_j \Big(U(i) - [A(i,j) - p^*(j)]\Big)}{\min_{i' \in \mathrm{Str}(i)} \sum_{j \in \mathcal{J}} \alpha_j \mathrm{KL}(i, i'|j)}\,; \qquad C = \sum_{i \in \mathcal{I}} \rho(i) C(i)\,. \tag{19}$$

(Note that $C(i)$ captures the regret per unit mass of service opportunities from workers of type i.) Informally, instances in which there is a conflict between exploration (i.e., learning worker type) and exploitation (i.e., maximizing short-term payoffs) have larger values of C. The case $C = 0$ corresponds to instances where the goals of learning and regret minimization are aligned; i.e., learning does not require regret of $\Omega(\log N/N)$. In this case, our result establishes that our chosen policies are nearly asymptotically optimal, to within $\mathrm{O}(\log \log N/N)$. On the other hand, the instances with $C > 0$ are those instances with a non-trivial tension between learning and short-term payoffs. For these instances, our result establishes that our chosen policies $(\pi_N^*)_{N \geq 1}$ achieve asymptotically optimal regret upto leading order in N.

The constant C is best understood in terms of the definition of α in the exploration phase (cf. (16)). Note that for a fixed α, for workers of true type i, the smallest log-likelihood ratio $\min_{i' \in \mathrm{Str}(i)} \log \Lambda_n(i, i')$ increases at an expected rate of $\min_{i' \in \mathrm{Str}(i)} \sum_{j \in \mathcal{J}} \alpha_j \mathrm{KL}(i, i'|j)$ during confirmation. Thus, when N is large, the time taken to confirm i against worker types in $\mathrm{Str}(i)$ is approximately $\log N/(\min_{i' \in \mathrm{Str}(i)} \sum_{j \in \mathcal{J}} \alpha_j \mathrm{KL}(i, i'|j))$. Hence, the externality-adjusted regret incurred until confirmation is complete, per unit mass of workers of type i, is approximately $\sum_{j \in \mathcal{J}} \alpha_j \Big(U(i) - [A(i,j) - p^*(j)]\Big) \log N/(\min_{i' \in \mathrm{Str}(i)} \sum_{j \in \mathcal{J}} \alpha_j \mathrm{KL}(i, i'|j))$. Optimizing over α results in an expected regret of nearly $C(i) \log N$ that must be incurred until the strong distinguishing goals are met for a unit mass of workers of type i. This translates to an expected regret of nearly $\hat{\rho}(i) C(i) \log N = \rho(i) C(i) \log N/N$ owing to workers of type i per time unit. This reasoning forms the basis of our lower bound, formalized in Proposition A.1 in Appendix A.

Now, a regret of $\Theta(\log N/N)$ is unavoidable when $C(i) > 0$ for some i. To develop some intuition for this case, consider the same example as before, but with a modified payoff matrix (see Figure 2). It can be shown that in this case, a regret of $\Omega(\log N/N)$ is unavoidable in the event that the true type of the worker is novice [Agrawal et al., 1989]. The problem is the following: to distinguish novices from experts, the policy must allocate workers to hard jobs. But hard jobs are strictly suboptimal for novices, and so if the true type of the worker is novice, some regret is unavoidable.

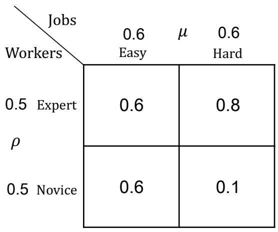

Fig. 2. An example where $\Omega(\log N/N)$ regret is unavoidable.

This discussion motivates the following definition.

Definition 5.2. Consider a worker type i. Suppose that there exists another type i' such that $A(i,j) = A(i',j)$ for all $j \in \mathcal{J}(i)$ and $\mathcal{J}(i) \cap \mathcal{J}(i') = \phi$. Then we say that the ordered pair (i,i') is a *difficult type pair.*

A similar definition also appears in [Agrawal et al., 1989]; the modification here is that the sets $\mathcal{J}(i)$ are defined with respect to externality-adjusted payoffs to account for capacity constraints.

The constant $C(i) > 0$ if and only if there is some other i' such that (i,i') is a difficult type pair. In general, if: (1) none of the job types in $\mathcal{J}(i)$ allow us to distinguish between i and i'; and (2) all the jobs in $\mathcal{J}(i)$ are strictly suboptimal for i', then any policy that achieves small regret must distinguish between i and i' and must assign the worker to jobs outside $\mathcal{J}(i)$ to make this distinction. This leads to a regret of $\Omega(\log N)$ per unit mass of workers of type i (over the lifetime of the workers).

On the other hand, if there is no difficult type pair, then there is no conflict between learning and regret minimization. Here, one can show that $C(i) = 0$ for each i, and this value is attained by some distribution $\alpha(i)$ that is supported on $\mathcal{J}(i)$. To see this note that if α is fully supported on $\mathcal{J}(i)$ (i.e., $\alpha_j > 0$ for all $j \in \mathcal{J}(i)$), then the numerator is 0; however if there is no type i' such that (i,i') is a difficult type pair, then the denominator is strictly positive, and thus $C(i) = 0$. In this case, $C = 0$ and our main result says that our algorithm achieves a regret of $O(\log\log N/N)$ asymptotically.[13]

6 PROOF SKETCH

The proof of Theorem 5.1 can be found in Appendix A. Here we present a sketch. The critical ingredient in the proof is the following relaxed optimization problem in which there are no capacity constraints, but capacity violations are charged with non-negative prices p^* from the optimization problem (11) with known worker types.

$$W^N_{p^*} \;=\; \max_{x\in\mathcal{X}^N} \sum_{i\in\mathcal{I}} \rho(i) \sum_{j\in\mathcal{J}} x(i,j)A(i,j) - \sum_{j\in\mathcal{J}} p^*(j)\Big[\sum_{i\in\mathcal{I}} \rho(i)x(i,j) - \mu(j)\Big]. \tag{20}$$

Lower bound on regret. If $C > 0$ (i.e., if there is at least one difficult pair of worker types; cf. Section 5), there is an upper bound on the performance of any policy in this problem, expressed relative to V^*. This result follows directly from [Agrawal et al., 1989]:

$$W^N_{p^*} \le V^* - \frac{C\log N}{N}(1 + o(1)),$$

where $C \ge 0$ is precisely the constant appearing in (19). By a standard duality argument, we know that $W^N \le W^N_{p^*}$, and hence this bound holds for W^N as well (see Proposition A.1), yielding the lower bound on regret on our original problem (7).

Upper bound on regret. There are two key steps in proving that π^*_N is feasible for problem (7)–(9), and $W^N(\pi^*_N) \ge V^* - C(\log N/N)(1 + o(1))$.

(1) First, we show that our policy π^*_N, with an arbitrary exploitation-phase routing matrix supported on $\mathcal{J}(i)$ for each $i \in \mathcal{I}$, achieves near optimal performance for the single multi-armed

[13] In fact, our proof demonstrates that this regret can be brought down to any $O(f_N/N)$ such that $f_N = o(1)$ by choosing a different threshold in the guessing phase.

bandit problem (20). Formally, if (with some abuse of notation) we let $W_{p^*}^N(\pi)$ denote the value attained by a policy π in problem (20), i.e.,

$$W_{p^*}^N(\pi) = \sum_{i \in \mathcal{I}} \rho(i) \sum_{j \in \mathcal{J}} x_\pi(i,j) A(i,j) - \sum_{j \in \mathcal{J}} p^*(j) \Big[\sum_{i \in \mathcal{I}} \rho(i) x_\pi(i,j) - \mu(j) \Big],$$

then

$$V^* - W_{p^*}^N(\pi_N^*) = C(\log N/N)(1 + \mathrm{o}(1)) + \mathrm{O}(\log\log N/N).$$

This is shown in[14] Proposition A.2. Thus we have $W_{p^*}^N(\pi_N^*) \geq W_{p^*}^N - \mathrm{o}(\log N/N)$, i.e., π_N^* is near-optimal in problem (20).

(2) In the next part of the proof, we show that we can design a routing matrix y^* (that depends on N) for the exploitation phase of the policy π_N^*, such that the following conditions are satisfied:

(a) (Complementary slackness) $\sum_{i \in \mathcal{I}} \rho(i) x_{\pi_N^*}(i,j) - \mu(j) = 0$ for all j such that $p^*(j) > 0$, and,

(b) (Feasibility) $\sum_{i \in \mathcal{I}} \rho(i) x_{\pi_N^*}(i,j) - \mu(j) \leq 0$ for all other $j \in \mathcal{J}$.

This is shown in Proposition A.5. We deduce that π_N^* with this choice of y^* in the exploitation phase is feasible for problem (7)–(9) and the complementarity slackness property implies that $W^N(\pi_N^*) = W_{p^*}^N(\pi_N^*)$, yielding our upper bound on regret.

Construction of y^*. At the end of the exploration phase of π_N^*, the correct label of the worker is learned with a confidence of at least $(1 - \mathrm{o}(1))$. This fact, coupled with the generalized imbalance condition (leading to flexibility in modifying x^*; cf. Fact 1), is sufficient to ensure an appropriate and feasible choice of $y^* = x^* + \mathrm{o}(1)$ will correct the deviations from x^* in terms of capacity utilizations of job types with $p^*(j) > 0$ arising because of the (short) exploration phase, and because of the infrequent cases in which exploitation is based on an incorrect worker label coming out of the exploration phase.

7 IMPLEMENTATION

In this section we describe some practical considerations for implementation of DEEM, and simulate its performance against other benchmark policies. For more details see Appendix B.

7.1 Practical considerations

Exploration with finite N. First, when N is relatively small (as is common in practice), we observe that we should adjust our notion of whether job type j helps to distinguish worker types i and i'. In particular, any job type j such that $A(i,j)$ and $A(i',j)$ are sufficiently close will not help in distinguishing types i and i' from each other within the fixed lifetime of the worker. Motivated by this observation, we replace $\mathrm{KL}(i,i'|j)$ by

$$\overline{\mathrm{KL}}(i,i'|j) = \mathrm{KL}(i,i'|j)\mathbb{I}_{\{\mathrm{KL}(i,i'|j) > \beta \log N/N\}}$$

in the definition of DEEM (cf. (16). The idea is that if $\mathrm{KL}(i,i'|j) < \beta \log N/N$, then one cannot strongly distinguish between i and i' using N/β jobs of type j; we use $\beta = 3$ in our simulations below. In this case we say that i and i' are *practically indistinguishable* using job type j; if this is true for every job type j, then we simply do not try to distinguish them at all.

Dynamic shadow prices. A key innovation in making DEEM practical is to use *dynamic* shadow prices, based on supply-demand imbalances in the market. In particular, suppose that the platform maintains a "queue" of jobs of each type. Then a reasonable approach is to set the shadow price on each job type via a decreasing function of the corresponding queue length, and use these shadow prices as an externality adjustment to payoffs.

[14] [Agrawal et al., 1989] proves this result for a similar policy.

If this function is appropriately chosen, then the prices obviate the need to explicitly compute y^* in our policy; instead the exploitation phase can be implemented by allocating optimally for each worker given the current queue-based prices (still a fully decentralized solution). The natural fluctuation in these prices ensures appropriate tie-breaking in allocation (cf. Fact 1). As an added benefit, since the solution to the benchmark (11)–(13) is no longer used, the resulting algorithm *does not need knowledge of arrival rates of workers or jobs.*

7.2 Simulations

Given an instance $(\hat{\rho}, N, \mu, A)$, we first describe our simulated marketplace.

Arrival process. Time is discrete. Fix a scaling constant τ. At the beginning of each time period t, a random $M_t(i)$ number of workers of type i and $L_t(j)$ jobs of type j arrive, such that $M_1,(i), M_2,(i),\cdots$ and $L_1,(i), L_2,(i),\cdots$ are i.i.d. sequences with $E(M_t(i)) = \tau\hat{\rho}(i)$, and $E(L_t(j)) = \tau\mu(\cdot)$. In our simulations, we generate $M_t(i)$ and $L_t(j)$ from a binomial distribution with the required means. Each worker stays in the system for N periods (each job takes one period).

Queues. As discussed above, arriving jobs accumulate in queues for the different types, each with a finite buffer B. We assume that if the buffer capacity B is exceeded for some job type, then the remaining jobs are lost. We use the queue-based dynamic shadow prices discussed above: if the queue length of job type j at any instant is $q(j)$, then we set the price of j at that instant to $p_q(j) = (B - q(j))/B$ (thus the price lies in $[0, 1]$). Note that $p_q(j)$ changes every time a job is assigned to a worker, or a new job arrives.

Matching process. In the beginning of each period, once all the new workers and jobs have arrived, the platform sequentially considers each worker in the platform,[15] and generates an assignment based on the history of the worker and the chosen policy. If a job of the required type is unavailable, then the worker remains unmatched. For each worker-job match, a random payoff is realized, drawn from the distribution specified by A, and the assignment-payoff tuple is added to the history of the worker.

Results. We considered instances with 3 types of workers and 3 types of jobs. We assumed that $N = 30$, and $\hat{\rho}(i) = 1$ for each i. We generated 350 instances where for each instance: (1) $\mu(j)$ is sampled from a uniform distribution on $[0.5, 1.5]$; and (2) each entry of the expected payoff matrix is sampled from a uniform distribution on $[0, 1]$. We chose the scaling constant $\tau = 30$, so that $E[M_t(i)] = 30$ for all i and $E[L_t(j)] = 30\mu(j)$ for the generated $\mu(j)$. We assumed that $M_t(i) = 30$ for all t, i.e., $M_t(i)$ is deterministic, and we generated $L_t(j)$ from a binomial distribution with mean $30\mu(j)$.

We implemented 4 policies, each using queue-length based prices, and compared them in these 350 instances: UCB [Auer et al., 2002], Thompson sampling (TS) [Agrawal and Goyal, 2011], Greedy, and a variant of DEEM that implements the modified learning goals above. UCB and TS are standard algorithms for the standard stochastic multi-armed bandit problem. Greedy simply chooses the job type that is myopically optimal, for the worker type that is the posterior mode (the prior distribution is proportional to the arrival rates of the workers, i.e., Uniform for our simulations). All algorithms, including DEEM, measure payoffs adjusted by the queue-based shadow prices described above; in this way, UCB, TS, and Greedy are effectively accounting for capacity constraints.

Figure 3 shows the cumulative distribution function – over the 350 instances – of the ratio of the payoff generation rate attained by a policy and the optimal payoff generation rate if the worker types are known, for the 5 candidate policies. The average of these ratios over the sample space for each policy is given in Table 1. As one can observe, DEEM substantially outperforms UCB, Greedy,

[15] This consists of the new workers and all the workers who have arrived in the past $N - 1$ periods, or from the beginning of time if $t < N$.

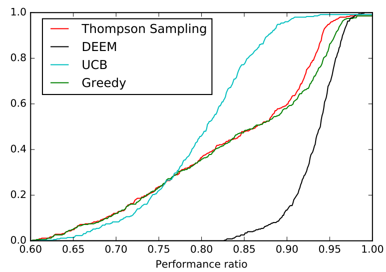

Fig. 3. The empirical CDF of the performance ratios of the different policies.

Policy	Average Performance Ratio
UCB	0.7973
TS	0.8301
Greedy	0.8355
DEEM	0.9330

Table 1. Average performance ratios of different policies across 350 instances.

and TS. This can be attributed to the careful design of the learning goals along with the learning policy (including the modifications for small N settings). Greedy and TS perform better than UCB on average, presumably benefiting from the knowledge of the expected payoff matrix A.

Out of 350 instances, 181 instances had a pair that cannot be practically distinguished using any job type. This suggests that although exact indistinguishability is almost never encountered, practical indistinguishability is encountered quite frequently and poses a concern. This verifies that the modifications that we made to our learning goals are important in practice.

8 CONCLUSION

This work suggests a novel and practical algorithm for learning while matching, applicable across a range of online matching platforms. Several directions of generalization remain open for future work. First, while we consider a finite-type model, a richer model of types would admit a wider range of applications; e.g., workers and jobs may be characterized by features in a vector-valued space, with compatibility determined by the inner product between feature vectors. Second, while our model includes only one-sided uncertainty, in general a market will include two-sided uncertainty (i.e., both supply and demand will exhibit type uncertainty). We expect that a similar approach using externality prices to first set learning objectives, and then achieve them while incurring minimum regret, should be applicable even in these more general settings.

We conclude by noting that our model ignores strategic behavior by participants. A simple extension might be to presume that workers are less likely to return after several bad experiences; this would dramatically alter the model, forcing the policy to become more conservative. The modeling and analysis of these and other strategic behaviors remain important challenges.

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APPENDICES

A PROOF OF THEOREM 5.1

For the rest of this section, let C be the quantity defined in (19). Recall problem 7. We will first show the following lower bound on the difference between V^* and W^N.

PROPOSITION A.1. *Suppose that $\{p^*(j)\}$ are the unique optimal prices in the matching problem with known worker types. Then,*

$$\limsup_{N\to\infty} \frac{N}{\log N}\left(V^* - W^N\right) \geq C.$$

PROOF. Consider the following relaxed problem:

$$W_{p^*}^N \quad = \quad \max_{x\in\mathcal{X}^N} \sum_{i\in\mathcal{I}} \rho(i) \sum_{j\in\mathcal{J}} x(i,j)A(i,j) - \sum_{j\in\mathcal{J}} p^*(j)[\sum_{i\in\mathcal{I}} \rho(i)x(i,j) - \mu(j)]. \tag{21}$$

By a standard duality argument, we know that $W_{p^*}^N \geq W^N$. The optimal policy in this problem is a solution to

$$\text{maximize}_{\pi\in\Pi^N} \sum_{i\in\mathcal{I},\, j\in\mathcal{J}} \rho(i)x_\pi(i,j)(A(i,j) - p^*(j))\,. \tag{22}$$

Then from Theorem 3.1 in [Agrawal et al., 1989], we know that

$$\limsup_{N\to\infty} \frac{N}{\log N}\left(V^* - W_{p^*}^N\right) \geq C.$$

The result then follows from the fact that $W^N \leq W_{p^*}^N$. □

Let $W_{p^*}^N(\pi^*)$ be the value attained by DEEM in optimization problem (20) (same as (21)), for any $y*$ in the exploitation phase such that $y^*(i,.)$ is supported on $\mathcal{J}(i)$. We will prove an upper bound on the difference between V^* and $W_{p^*}^N(\pi^*)$. Note that the difference in these values of the two problems is the same as the difference in

$$\sum_{i\in\mathcal{I}} \rho(i) \sum_{j\in\mathcal{J}} x_{\pi^*}(i,j)(A(i,j) - p^*(j)),$$

and $\sum_{i\in\mathcal{I}} \rho(i)U(i)$. Following is the result.

PROPOSITION A.2. *Consider the sequence of policies $(\pi^*(N))_{N\geq 1}$ such that the routing matrix y used in the exploitation phase satisfies $y(i,.) \in \Delta(\mathcal{J}(i))$. Then,*

$$\limsup_{N\to\infty} \frac{N}{\log N}\left(V^* - W_{p^*}^N(\pi^*)\right) \leq C.$$

Further, suppose that there are no difficult type pairs. Then,

$$\limsup_{N\to\infty} \frac{N}{\log\log N}\left(V^* - W_{p^*}^N(\pi^*)\right) \leq K$$

where $K = K(\rho,\mu,A) \in (0,\infty)$ is some constant.

In order to prove this Proposition, we need the following result that follows from Theorem 4.1 in [Agrawal et al., 1989].

Lemma A.3. *Let $X_1, X_2, \cdots$ be i.i.d. random variables where X_i is the outcome of choosing a job type $j \in \mathcal{J}$ according to a distribution $\alpha \in \Delta(\mathcal{J})$. Suppose $i \in \mathcal{I}$ and $\mathcal{B} \subseteq \mathcal{I} \setminus \{i\}$ are such that*

$$\sum_{j \in \mathcal{J}} \alpha_j \mathrm{KL}(i, i'|j) > 0$$

for each $i' \in \mathcal{B}$. Let $\Lambda_k^{\mathcal{B}}(i) = \min_{i' \in \mathcal{B}} \Lambda_k(i, i')$. Then,

(1)

$$\limsup_{N \to \infty} \frac{\mathrm{E}_i[\inf\{k \geq 0 | \Lambda_k^{\mathcal{B}}(i) \geq f(N)\}]}{\log f(N)} \leq \frac{1}{\min_{i' \in \mathcal{B}} \sum_{j \in \mathcal{J}} \alpha_j \mathrm{KL}(i, i'|j)},$$

(2)

$$\mathrm{P}_{i'}(\Lambda_k(i, i') \geq f(N) \text{ for some } k \leq N) \leq \frac{1}{f(N)}.$$

for any $f(N) = \omega(1)$.

Next, we also need the following result.

Lemma A.4. *Let $X_1^j, X_2^j, \cdots$ be i.i.d. random variables for each $j = 1, \cdots, K$, such that $|X_i^j| \leq M$ and $\mathrm{E}(X_i^j) = m^j > 0$. Let a, b and k^j be such that $a < k^j < b$ for each j. Let $S_n^j = k^j + \sum_{i=1}^n X_i^j$ and let $\mathcal{B} \subseteq \{1, \cdots, K\}$. Let E be the event:*

$$\{S_n^{j'} < a \text{ for some } j' \text{ before } S_n^j > b \text{ for all } j \in \mathcal{B}\}.$$

Let T be $\inf\{n : S_n^j < a$ for some $j\}$. Then

$$\mathrm{E}[T\mathbf{1}_E] \leq G$$

for some $0 < G < \infty$ that does not depend on a, b, or k^j for any j.

Proof. Define $k^j - a \triangleq z^j$. If we define $E_j = \{S_n^{j'} < a \text{ for some } n\}$, then we have $E \subseteq \cup_j E_j$, and thus we have $\mathrm{E}[T\mathbf{1}_E] \leq \sum_{j=1}^K \mathrm{E}[T\mathbf{1}_{E_j}]$. Now we have

$$\begin{aligned}
\mathrm{E}[T\mathbf{1}_{E_j}] &= \sum_{n=1}^{\infty} n\mathrm{P}(T = n) \\
&\leq \sum_{n=1}^{\infty} n\mathrm{P}(\sum_{i=1}^{n} X_i^j \leq -z_j) \\
&\leq \sum_{n=1}^{\infty} n \exp(\frac{-(nm_j + z_j)^2}{4nM^2}) \\
&= \sum_{n=1}^{\infty} n \exp(-\frac{nm_j^2}{4M^2} - \frac{m_j z_j}{2M^2} - \frac{z_j^2}{4nM^2}) \\
&\leq \sum_{n=1}^{\infty} n \exp(-\frac{nm_j^2}{4M^2}) = G(m^j, M) < \infty,
\end{aligned}$$

where the second inequality results from the Hoeffding bound. Taking $G = \sum_{j=1}^K G(m^j, M)$ proves the result.

□

Proof of Proposition A.2. Let X denote the type of the worker. Let $R(i)$ denote the expected total regret over the lifetime of a worker on the event $\{X = i\}$, defined as

$$R(i) = N \max_{j \in \mathcal{J}} [A(i,j) - p^*(j)] - N \sum_{j \in \mathcal{J}} x_{\pi^*}(i,j)[A(i,j) - p^*(j))].$$

Here $Nx_{\pi^*}(i,j)$ is the expected total number of times a job of type j is allotted to a worker of type i under the policy $\pi^*(N)$. We will refer to the above quantity as just regret. For the rest of the proof, all the expectations are on the event $\{X = i\}$. The proof will utilize the fact that the log likelihood ratio, $\log(\Lambda_k(i,i'))$, for any i and i' is a random walk, such that if α^k is the probability distribution over job types chosen at opportunity k, then

$$\log(\Lambda_{k+1}(i,i')) - \log(\Lambda_k(i,i')) = \log(\frac{p(X_k, i, j_k)}{p(X_k, i', j_k)}),$$

where the random variables $\{\log(\frac{p(X_k,i,j_k)}{p(X_k,i',j_k)})\}_k$ are independent random variables with a finite support (since X_k and j_k take finite values), and with mean $\sum_j \alpha_j^k \mathrm{KL}(i,i'|j)$. (Note here that if $\sum_j \alpha_j^k \mathrm{KL}(i,i'|j) = 0$ then since $\mathrm{KL}(i,i'|j) \geq 0$, it must be that $\mathrm{KL}(i,i'|j) = 0$ for all j such that $\alpha_j^k > 0$, and in this case we must have $A(i,j) = A(i,j')$ for all such j. Thus $\log(\frac{p(X_k,i,j_k)}{p(X_k,i',j_k)}) = 0$, i.e., the if the drift of the random walk is 0 at some k then the random walk has stopped.) Recall that the initial likelihoods in our policy are $\lambda_0(i) = 1$ for all i. Hence $\log(\Lambda_0(i,i')) = 0$.

Our goal is to compute an upper bound on $R(i)$. To do so we first compute the expected regret incurred till the end of the exploration phase in our algorithm. Denote this by $R_e(i)$. Below we will find an upper bound on this regret assuming that the worker performs an unbounded number of jobs. Clearly the same bound holds on the expected regret until the end of exploration phase if the worker leaves after N jobs.

Our strategy is as follows: we will decompose the regret till the end of exploration into the regret incurred till the first time one of the following two events occurs:

(1) Event A: $\min_{i' \neq i} \log \Lambda_k(i,i') \geq \log\log N$ (or $\min_{i' \neq i} \log \Lambda_k(i,i') \geq \log N$) and
(2) Event B: $\min_{i' \neq i} \log \Lambda_k(i,i') \leq -\log\log N$ (or $\min_{i' \neq i} \log \Lambda_k(i,i') \leq \frac{1}{\log N}$).

followed by the residual regret, which will depend on which event occurred first. Note that one of these two events will occur with probability 1.

We will compute two different upper bounds, depending on two different regimes of initial likelihoods of the different types (note that the likelihoods of the different types i under the observed history is a sufficient statistic at any opportunity under our policy). First, suppose that $\bar{R}(i)$ is highest expected regret incurred over all possible starting likelihoods that a) do not satisfy the conditions of both A and B and b) such that $\min_{i' \neq i} \log \Lambda_0(i,i') \geq 0$. Let L_1 be the set of starting likelihoods that satisfy these conditions. Next, suppose that $\tilde{R}(i)$ is the highest expected regret incurred, where the maximum is taken over all possible starting likelihoods that a) do not satisfy the conditions of both A and B and b) such that $\min_{i' \neq i} \log \Lambda_n(i,i') < 0$. Let L_2 be the set of likelihoods that satisfy these conditions. Clearly, $R_e(i) \leq \bar{R}(i)$.

Let $G(i)$ denote the maximum expected regret incurred by the algorithm till one of A or B occurs, where the maximum is taken over all possible starting likelihoods of the different types that do not satisfy the conditions of both A and B, i.e., $L_1 \cup L_2$. For convenience, we denote $A < B$ as the event that A occurs before B and vice versa (similarly for any two events). Thus we have

$$\bar{R}(i) \leq G(i) + \sup_{l_1 \in L_1} \mathrm{P}(A < B|l_1)\mathrm{E}(\text{Residual regret}|A, l_1) + \sup_{l_1 \in L_1} \mathrm{P}(B < A|l_1)\mathrm{E}(\text{Residual regret}|B, l_1)$$

and

$$\tilde{R}(i) \leq G(i) + \sup_{l_2 \in L_2} \mathrm{P}(A < B|l_2)\mathrm{E}(\text{Residual regret}|A, l_2) + \sup_{l_2 \in L_2} \mathrm{P}(B < A|l_2)\mathrm{E}(\text{Residual regret}|B, l_2).$$

First, let us find a bound on $G(i)$. This is easy, because, $G(i) \leq \mathrm{E}(\inf\{k > 0 : \min_{i \neq i'} \Lambda_k(i, i') \geq \log^2 N\}) = \mathrm{O}(\log\log N)$ from Lemma A.3 (since if neither condition A, nor B is satisfied, then the policy in the guessing phase, and thus all job types are utilized with positive probability, and hence the condition in the Lemma is satisfied). Also, from the second statement in Lemma A.3, since the likelihoods in L_1 are such that $\min_{i' \neq i} \Lambda_n(i, i') \geq 1$, we have that $\mathrm{P}(B < A|l_1) \leq \mathrm{P}(\text{B ever occurs till time N}) \leq 1/\log N$. Finally we have $\sup_{l_2 \in L_2} \mathrm{P}(B < A|l_2) = w < 1$. We thus have

$$\bar{R}(i) \leq \mathrm{O}(\log\log N) + \sup_{l_1 \in L_1} \mathrm{E}(\text{Residual regret}|A, l_1) + \frac{1}{\log N} \sup_{l_1 \in L_1} \mathrm{E}(\text{Residual regret}|B, l_1) \text{ and} \tag{23}$$

$$\tilde{R}(i) \leq \mathrm{O}(\log\log N) + \sup_{l_2 \in L_2} \mathrm{E}(\text{Residual regret}|A, l_2) + w \sup_{l_2 \in L_2} \mathrm{E}(\text{Residual regret}|B, l_2). \tag{24}$$

Next, consider $\sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|A, l_k)$. This depends on which of the two events happens next:

(1) Event A': $\min_{i' \neq i} \log \Lambda_k(i, i') < \log\log N$ (or $\min_{i' \neq i} \Lambda_k(i, i') < \log N$),
(2) Event A'': i gets confirmed, i.e., $\min_{i' \in \mathrm{Str}(i)} \log \Lambda_k(i, i') > \log N$ (or $\min_{i' \in \mathrm{Str}(i)} \Lambda_k(i, i') > N$).

Again conditional on A, one of the two events will occur with probability 1. We have

$$\begin{aligned}\sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|A, l_k) = & \sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|A, A' < A'', l_k)\mathrm{P}(A' < A''|A, l_k) \\ & + \sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|A, A' > A'', l_k)\mathrm{P}(A' > A''|A, l_k).\end{aligned}$$

Now from Lemma A.4 it follows that

$$\begin{aligned}&\mathrm{E}(\text{Residual regret}|A, A' < A'', l_k)\mathrm{P}(A' < A''|A, l_k) \\ &= \mathrm{E}(\text{Residual regret } \mathbb{I}_{\{A' < A''\}}|A, l_k) \leq M + \bar{R}(i) \sup_{l_k \in L_k} \mathrm{P}(A' < A''|A, l_k)\end{aligned}$$

for some constant M that does not depend on l_k or N. To see this, note that $A' < A''$ is the event that, starting from some values between $\log\log N$ and $\log N$, all the random walks $\Lambda_k(i, i')$ for each $i' \in Str(i)$ cross the threshold $\log N$, before any of the random walks $\Lambda_k(i, i')$ for $i' \neq i$ cross the (lower) threshold $\log\log N$. Now between these two thresholds, the job distribution $\alpha_k = \alpha(i)$ for all k. Hence the mean drift for any of the random walks $\Lambda_k(i, i')$ for each $i' \in Str(i)$ is strictly positive. Further, as we argued earlier, if the mean drift for any of these random walks is 0, then that random walk has stopped, and such random walks can be ignored. Thus the conditions of Lemma A.4 are satisfied, and hence $\mathrm{E}((\text{Time till } A')\, \mathbb{I}_{\{A' < A''\}}|A, l_k) = G < \infty$. Since the regret per unit time is bounded, the deduction follows. Moving on, we have

$$\begin{aligned}&\mathrm{E}(\text{Residual regret}|A, A'' < A', l_k) \\ &\leq \mathrm{E}(\inf\{k > 0 : \min_{i' \in Str(i)} \Lambda_k(i, i') \geq N\}) \sum_{j \in \mathcal{J}} \alpha_j \Big(U(i) - [A(i, j) - p^*(j)]\Big).\end{aligned}$$

Denoting $\sup_{l_k \in L_k} \mathrm{P}(A' < A''|A, l_k) = q_k$, we have

$$\sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|A, l_k) \leq \mathrm{O}(1) + q_k \bar{R}(i) + (1 - q_k)\mathrm{E}(\inf\{k > 0 : \min_{i' \in \mathrm{Str}(i)} \Lambda_k(i, i') \geq N\}) \tag{25}$$

Note that $q_k < 1$. Next, consider $\sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|B, l_k)$. This depends on which of the following two events occurs next:

(1) Event B': $\min_{i' \neq i} \log \Lambda_k(i, i') \geq -\log\log N$ (or $\min_{i' \neq i} \Lambda_n(i, i') \geq \frac{1}{\log N}$),
(2) Event B'': Some $i' \neq i$ gets confirmed, i.e., $\min_{i'' \in \mathrm{Str}(i')} \log \Lambda_k(i', i'') > \log N$ (or $\min_{i'' \in \mathrm{Str}(i')} \Lambda_k(i', i'') > N$).

Again conditional on B, one of the two events will occur with probability 1. Let $K(i)$ be the maximum expected time till either B or B'occurs given that B has occurred and the starting likelihoods were in L_k. Note that if $B'' < B'$ then the exploration phase ends and hence there is no residual regret (although if i' is such that $i \in \mathrm{Str}(i)$, then $\mathrm{P}(B'' < B'|B, L_k) \leq 1/N$ from the second statement in Lemma A.3).

$$\sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|B, l_k) \leq K(i) + \sup_{l_k \in L_k} \mathrm{P}(B' < B''|B, l_k)\tilde{R}(i)$$

Now first, we can show that if there is a type i' such that $i \in \mathcal{I} \setminus \mathrm{Str}(i')$, then $K(i) \leq \mathrm{O}(\log N)$. If there is no such type, then $K(i) = \mathrm{O}(1)$. Thus we have

$$\sup_{l_k \in L_k} \mathrm{E}(\text{Residual regret}|B, l_k) \leq \mathrm{O}(\log N) + \tilde{R}(i).$$

And thus we finally have

$$\begin{aligned}\bar{R}(i) &\leq \mathrm{O}(\log\log N) + q_1\bar{R}(i) + \frac{1}{\log N}\Big(\mathrm{O}(\log(N) + \tilde{R}(i)\Big)\\ &\quad + (1 - q_1)\mathrm{E}(\inf\{k > 0 : \min_{i' \in \mathrm{Str}(i)} \Lambda_k(i, i') \geq N\}) \sum_{j \in \mathcal{J}} \alpha_j\Big(U(i) - [A(i,j) - p^*(j)]\Big); \qquad (26)\\ \tilde{R}(i) &\leq \mathrm{O}(\log\log N) + q_2\bar{R}(i) + w\mathrm{O}(\log N) + w\tilde{R}(i)\\ &\quad + (1 - q_2)\mathrm{E}(\inf\{k > 0 : \min_{i' \in \mathrm{Str}(i)} \Lambda_k(i, i') \geq N\}) \sum_{j \in \mathcal{J}} \alpha_j\Big(U(i) - [A(i,j) - p^*(j)]\Big). \qquad (27)\end{aligned}$$

Combining the above two equations, we deduce that

$$\begin{aligned}R_e(i) \leq \bar{R}(i) &\leq \frac{1 - q_1}{1 - q_1 - q_2/\log N}\Big(\mathrm{O}(\log\log N) \qquad (28)\\ &\quad + \mathrm{E}[\inf\{k > 0 : \min_{i' \in Str(i)} \Lambda_n(i, i') \geq N\}] \sum_{j \in \mathcal{J}} \alpha_j\Big(U(i) - [A(i,j) - p^*(j)]\Big)\Big)\\ &= \mathrm{O}(\log\log N) + (1 + o(1))\mathrm{E}[\inf\{k > 0 : \min_{i' \in \mathrm{Str}(i)} \Lambda_n(i, i') \geq N\}] \sum_{j \in \mathcal{J}} \alpha_j\Big(U(i) - [A(i,j) - p^*(j)]\Big). \qquad (29)\end{aligned}$$

Now, we observed earlier that $\mathrm{P}(i' \text{ gets confirmed } \mid \{X = i\}) \leq 1/N$ if $i' \in \mathrm{Str}(i)$. Thus the regret in the exploitation phase is in the worst case of order $\mathrm{O}(N)$ with probability $1/N$ and 0 otherwise. Thus the total expected regret in the exploitation phase is $\mathrm{O}(1)$. Thus

$$R(i) \leq \mathrm{O}(\log\log N) + (1 + o(1))\mathrm{E}[\inf\{k > 0 : \min_{i' \in \mathrm{Str}(i)} \Lambda_k(i, i') \geq N\}] \sum_{j \in \mathcal{J}} \alpha_j\Big(U(i) - [A(i,j) - p^*(j)]\Big).$$

Thus Lemma A.3 implies the result (note that if there are no difficult type pairs, then $\sum_{j \in \mathcal{J}} \alpha_j\Big(U(i) - [A(i,j) - p^*(j)] = 0$).

□

Next, we prove that there is for a large enough N, one can choose a routing matrix y^* in the exploitation phase of DEEM that will ensure that matches optimize price-adjusted payoffs, and such that the capacity and complementary slackness conditions are satisfied.

Proposition A.5. *Suppose that the generalized imbalance condition is satisfied. Consider any optimal routing matrix x^* that is an optimal solution to problem* (11). *Then in the policy $\pi^*(N)$ for the N-job problem, for any N large enough, one can choose a routing matrix y^* such that $y^*(i,.) \in \Delta(\mathcal{J}(i))$ and that satisfies:*

(1) $\sum_{i\in\mathcal{I}} \rho(i)x_{\pi^*}(i,j) = \mu(j)$ for any j such that $\sum_{i\in\mathcal{I}} \rho(i)x^*(i,j) = \mu(j)$, and
(2) $\sum_{i\in\mathcal{I}} \rho(i)x_{\pi^*}(i,j) < \mu(j)$ for any other j.

We remark that the y^* we construct satisfies $\|y^* - x^*\| = o(1)$. In order to prove this proposition, we will need the following Lemma.

Lemma A.6. *Suppose that the generalized imbalance condition is satisfied. Consider any feasible routing matrix $[x(i,j)]_{\mathcal{I}\times\mathcal{J}}$. Consider any job j such that $\sum_{i\in\mathcal{I}} \rho(i)x(i,j) = \mu(j)$. Then there is a path on the complete bipartite graph between worker types $\mathcal{I}$ and job types $\mathcal{J}$ with the following properties:*

- *One end point is job j.*
- *The other end point is a job type whose capacity is under-utilized (it is permitted to be κ).*
- *For every job type on the path in between, they are operating at capacity/all jobs are being served. (All worker types are fully utilized by definition, since we formally consider an unassigned worker as being assigned to job type κ.)*
- *For every undirected edge on the path, there is a positive rate of jobs routed on that edge in x.*

Proof. Consider a bi-partite graph with jobs representing nodes on one side and workers on the other. There is an edge between a job j' and a worker i if $x(i,j) > 0$. Consider the connected component of job type j in this graph. Suppose it includes no job type that is underutilized. Then the arrival rate of jobs from the set of workers in the connected component exactly matches the total effective service rate of the sellers in connected component. But this is a contradiction since generalized imbalance holds. Hence there exists an underutilized job type j' that can be reached from j. Take any path from j to j'. Traverse it starting from j and terminate it the first time it hits any underutilized job type. □

Proof of Proposition A.5. Recall that for a given routing matrix $[y(i,j)]_{\mathcal{I}\times\mathcal{J}}$, $x_{\pi^*}(i,j)$, is the resulting fraction of jobs of type j directed to worker type i. In the course of this proof, we will suppress the subscript π^*. Clearly, there exist $\varepsilon_{i'}(i,j)$ for each $i \in \mathcal{I}, i' \in \mathcal{I} \cup \{0\}, j \in \mathcal{J}$ such that we have

$$x(i,j) = \varepsilon_0(i,j) + (1 - \varepsilon_i(i,j))y(i,j) + \sum_{i'\in\mathcal{I}\setminus\{i\}} \varepsilon_{i'}(i,j)y(i',j)\,. \tag{30}$$

The ε's depend on the guessing and confirmation phases but not on y. (In particular, ε_0 arises from the overall routing contribution of the guessing and confirmation phases, and ε_i's arise from the small likelihood that a worker who is confirmed as type i is actually some other type.) A key fact that we will use is that all ε's are uniformly bounded by $o(1)$.

Let $\mathcal{J}_{x^*} = \{s : \sum_{i\in\mathcal{I}} \rho(i)x^*(i,j) = \mu(j)\}$ and $\mathcal{J}_{\pi^*} = \{j : \sum_{i\in\mathcal{I}} \rho(i)x_{\pi^*}(i,j) = \mu(j)\}$. Now we want to find a y such that $y(i,\cdot) \in \Delta(\mathcal{J}(i))$ for all $i \in \mathcal{I}$ (call (i,j) a "permissible edge" in the bipartite graph between workers and jobs if $j \in \mathcal{J}(i)$), and such that:

- For each $j \in \mathcal{J}_{x^*}$ we also have $j \in \mathcal{J}_{\pi^*}$, i.e., $\mathcal{J}_{x^*} \subseteq \mathcal{J}_{\pi^*}$.
- $\|y - x^*\| = o(1)$.

Note that the two bullets together will imply the proposition, since $\|x - x^*\| = o(1)$ from Eq. (30), and this leads to $\sum_{i \in \mathcal{I}} \rho(i) x(i,j) = \sum_{i \in \mathcal{I}} \rho(i) x^*(i,j) + o(1) < \mu(j)$ for all $j \in \mathcal{J} \backslash \mathcal{J}_{x^*}$, for large enough N.

The requirement in the first bullet can be written as a set of linear equations using Eq. (30). Here we write y (and later also x^*) as a column vector with $|\mathcal{I}||\mathcal{J}|$ elements:

$$By + \hat{\varepsilon} = (\mu(j))_{j \in \mathcal{J}_{x^*}} .$$

Here we have $\|\hat{\varepsilon}\| = o(1)$ and matrix B can be written as $B = B_0 + B_\varepsilon$, where B_0 has 1's in columns corresponding to dimensions $(\cdot, s)$ and 0's everywhere else, and $\|B_\varepsilon\| = o(1)$. Expressing y as $y = x + z$, we are left with the following equation for z,

$$Bz = -(B_\varepsilon x^* + \hat{\varepsilon}) \tag{31}$$

using the fact that $B_0 x^* = (\mu(j))_{j \in \mathcal{J}_{x^*}}$ by definitions of B_0 and $\mathcal{J}_{x^*}$. We will look for a solution to this underdetermined set of equations with a specific structure: we want z to be a linear combination of flows along $|\mathcal{J}_{x^*}|$ paths coming from Lemma A.6, one path λ_j for each $j \in \mathcal{J}_{x^*}$. Each λ_j can be written as a column vector with +1's on the odd edges (including the edge incident on j) and −1's on the even edges. Let $\Lambda = [\lambda_j]_{j \in \mathcal{J}_{x^*}}$ be the path matrix. Then z with the desired structure can be expressed as $\Lambda\eta$, where η is the vector of flows along each of the paths. Now note that $Bz = (B_0 + B_\varepsilon)\Lambda\eta = (I + B_\varepsilon \Lambda)\eta$. Here we deduced $B_0 \Lambda = I$ from the fact that λ_j is a path which has j as one end point, and a worker or else a job not in $\mathcal{J}_{x^*}$ as the other end point. Our system of equations reduces to

$$(I + B_\varepsilon \Lambda)\eta = -(B_\varepsilon x^* + \hat{\varepsilon}) ,$$

Since $\|B_\varepsilon\| = o(1)$, the coefficient matrix is extremely well behaved being $o(1)$ different from the identity, and we deduce that this system of equations has a unique solution η^* that satisfies $\|\eta^*\| = o(1)$. This yields us $z^* = \Lambda\eta^*$ that is also of size $o(1)$, and supported on permissible edges since each of the paths is supported on permissible edges (Lemma A.6). Thus, we finally obtain $y^* = x^* + z^*$ possessing all the desired properties. Notice that the (permissible) edges on which y^* differs from x^* had strictly positive values in x^* by Lemma A.6, and hence this is also the case in y^* for large enough N.

□

Finally, we show that with the choice of y^* constructed in Proposition A.5 in the exploitation phase, the sequence of policies $(\pi^*(N))$ asymptotically achieve the required upper bound on regret.

Proposition A.7. *Suppose that the generalized imbalance condition is satisfied. Consider the sequence of policies $(\pi^*(N))_{N \geq 1}$, with the routing matrix y^* proposed in Proposition A.5. Let $W^N(\pi^*)$ be the value attained by this policy in optimization problem* (7). *Then*

$$\limsup_{N \to \infty} \frac{N}{\log N} \Big(V^* - W^N(\pi^*) \Big) \leq C.$$

Further, suppose that there are no difficult type pairs. Then,

$$\limsup_{N \to \infty} \frac{N}{\log \log N} \Big(V^* - W^N(\pi^*) \Big) \leq K$$

where $K = K(\rho, \mu, A) \in (0, \infty)$ is some constant.

Proof. From Proposition A.5 it follows that the policy π^* is feasible in problem 7, and further

$$W_{p^*}^N(\pi^*) = \sum_{i\in\mathcal{I}}\rho(i)\sum_{j\in\mathcal{J}}x_{\pi^*}(i,j)A(i,j) - \sum_{j\in\mathcal{J}}p^*(j)[\sum_{i\in\mathcal{I}}\rho(i)x_{\pi^*}(i,j)-\mu(j)]. \tag{32}$$

$$= \sum_{i\in\mathcal{I}}\rho(i)\sum_{j\in\mathcal{J}}x_{\pi^*}(i,j)A(i,j), \tag{33}$$

where the second equality follows from the fact that if $p^*(j) > 0$, then $\sum_{i\in\mathcal{I}}\rho(i)x^*(i,j)-\mu(j) = 0$ by complementary slackness, and hence from Proposition A.5 we obtain that $\sum_{i\in\mathcal{I}}\rho(i)x_{\pi^*}(i,j)-\mu(j) = 0$ as well for these j. Thus we have a policy π^* that is feasible, and that gives a rate $W_{p^*}^N(\pi^*)$ of accumulation of payoff in problem (7). Thus the result follows from Proposition A.2. □

B PRACTICAL IMPLEMENTATION OF DEEM

In this section, we describe a modified version of DEEM that incorporates certain changes that address practical considerations.

B.1 Practical values of N

First, we discuss the issues in the regime where N is small, which would be the case in many situations in practice. These issues are centered around the notion of distinguishability of two types of workers. Nominally, two worker types i and i' are distinguishable using job type j if $A(i,j) \neq A(i,j')$, since by repeatedly allotting job j and observing the outcomes, we can asymptotically distinguish between the two worker types. But when N is small, the degree of accuracy we can achieve in this distinction depends on how different $A(i,j)$ is from $A(i',j)$. For example, if $A(i,j) = 0.5$ and $A(i',j) = 0.6$, then $\mathrm{KL}(i,i'|j) \approx 0.0204$ and if $N = 30$, it would take approximately $\log 30/0.0204 \approx 166$ jobs to distinguish i and i' using j with a probability of error of $1/30$ (the requirement for distinguishing i strongly from i') – hence this distinction is impossible within 30 jobs. Thus, the practically relevant notion is distinguishability within $\sim N$ jobs: We say a worker type i is *practically indistinguishable* from a worker type i' using a job type j if $\mathrm{KL}(i,i'|j) \leq \gamma(N) = \beta\log N/N$. The motivation for this definition is that it is impossible to confirm that the type is i and not i' with a reasonably small probability of error - where we choose the strong distinguishing requirement of $1/N$ on the error probability - within a reasonable number of job allocations of type j - where we allow N/β job allocations - if $\mathrm{KL}(i,i'|j) \leq \gamma(N)$ holds.[16]

Practical indistinguishability affects the specification of our algorithm in the following ways. First, it could be that some type i is practically indistinguishable from a type i' no matter which job type is used for the distinction. In this case, it is hopeless to try to distinguish these types as a part of our learning goals. Next, practical indistinguishability is also a concern in the determination of the optimal policy in the confirmation phase. It could be that the only job type picked with positive probability by the sampling distribution $\alpha(i)$ (defined in (16)) to distinguish i from some other type i' that it needs to be strongly distinguished from, is such that i and i' are practically indistinguishable using this job type. In this case, although that job type may offer a non-zero learning rate, this rate may be so small that the distinguishability goals will be achieved too late relative to the lifetime of the worker. This is undesirable because there would not be enough opportunities left for exploitation where one can reap the benefits of having attained the learning goals. Thus it may be prudent to ensure that some other job type that makes this distinction is chosen by the confirmation policy (at the expense of potentially higher regret during learning).

[16]It takes approximately $\log N/\mathrm{KL}(i,i'|j)$ jobs for this distinction, and thus we require that $\log N/\mathrm{KL}(i,i'|j) \leq N/\beta$, resulting in $\mathrm{KL}(i,i'|j) \geq \beta\log N/N \triangleq \gamma(N)$.

To incorporate the two practical considerations above, we propose the following modifications to our policy.

(1) **Practically indistinguishable types:** Suppose that $\mathcal{S}(i)$ is the set of types that i is practically indistinguishable from using any job type, i.e., $\mathcal{S}(i) = \{i' \neq i \in \mathcal{I} : \mathrm{KL}(i, i'|j) < \gamma(N) \text{ for all } j \in \mathcal{J}\}$. Then, first remove $\mathcal{S}(i)$ from the set $\mathrm{Str}(i)$. Also, while implementing the algorithm, if MLE_n is worker type i, then the likelihoods under all worker types in $\mathcal{S}(i)$ are removed from consideration in deciding whether the conditions of either the guessing or the confirmation phase are satisfied. Essentially, we act as if types in $\mathcal{S}(i)$ do not exist in checking for these conditions.

(2) **Enforcing quicker learning:** Next, in the computation of the distribution $\alpha(i)$ to be used in the confirmation mode, (see equation (16)), define $\overline{\mathrm{KL}}(i, i'|j) = \mathrm{KL}(i, i'|j)\mathbb{I}_{\{\mathrm{KL}(i,i'|j)\geq\gamma(N)\}}$, and replace $\mathrm{KL}(i, i'|j)$ by $\overline{\mathrm{KL}}(i, i'|j)$ in the computation. The idea is to prevent the algorithm from relying on job types with a small learning rate (but with a low regret accumulation rate at the same time), which is achieved by replacing the learning rate they offer (i.e., the KL divergence) with 0 in computing the optimal $\alpha(i)$. For instance, this would be the case when there are practically difficult type pairs that are not (strictly) difficult. Note that for every i' in $\mathrm{Str}(i) \setminus \mathcal{S}(i)$ there is always a j such that $\mathrm{KL}(i, i'|j) > \gamma(N)$, and hence $\overline{\mathrm{KL}}(i, i'|j) > 0$, i.e., there exists an optimal policy $\alpha(i)$ that accomplishes the learning goals within a reasonable amount of time under the modified learning rates.

B.2 Using queue-length based prices

One of the central ideas in the design of our algorithm is that of employing appropriate externality adjustments to the payoffs so that the matching while learning problem can be decoupled into unconstrained per-worker problems. In principle, these externalities can be captured by the shadow prices in (7), but as discussed earlier, these are difficult to compute. Our results show that in the large N regime, the shadow prices $p^*(j)$ from the problem with known worker types approximately capture these externalities. But there are two practical considerations: 1) when N is small, we do not expect $p^*(j)$ to be a good proxy for the true shadow prices $p^N(j)$ and 2) it could be difficult to compute the routing policy y^* in the exploitation phase that ensures that the capacity constraints and complementarity conditions with respect to $p^*(j)$ are satisfied (in fact for a small N, it is not even clear if a y^* with the desired properties is feasible).

Moreover, in practice, both workers and jobs arrive in continuous time. In these situations, there is a practical alternative of using instantaneous *queue-length* based prices to capture the externalities. The idea is simple: we will assume that arriving jobs that remain unassigned accumulate in queues (one for each job type), and at each assignment opportunity the algorithm uses the instantaneous prices based on current queue lengths for each job type to adjust the payoffs (in the learning as well as the exploitation phase), where the price is a decreasing function of current queue length (as we described for our simulated marketplace in section 7). We now describe how such queue-length based prices can be incorporated in our policy.

(1) **Queue-length based prices in Learning and Exploitation:** In computing $\alpha(i)$ for each $i \in \mathcal{I}$ in the confirmation phase, replace $p^*(j)$ by the instantaneous queue-length based prices $p_q(j)$ in Eq. (16). Similarly, in the exploitation phase, instead of explicitly computing the routing matrix y^*, use queue-length based prices to decide assignments in the following manner. Define the sets $\mathcal{J}^*(i)$ as:

$$\mathcal{J}^*(i) = \arg\max_{j\in\mathcal{J}} A(i, j) - p_q(j).$$

If an assignment has to be made in the exploitation phase for some worker who has already been labeled as being of type i, then a job type $j^* \in \mathcal{J}^*$ is chosen (note that typically, $\mathcal{J}^*(i)$ will be a singleton).

(2) **Queue-length based prices to define Learning goals:** In determining the learning goals, determine the strong distinguishability requirements (see Definition 4.1) and hence the set $\mathrm{Str}(i)$ for each i based on the payoffs adjusted by queue-length based prices, instead of the prices from the problem with known worker types. To be more precise, instead of defining these sets based on $\mathcal{J}(i)$, define them based on $\mathcal{J}^*(i)$ defined above. Note that $\mathcal{J}^*(i)$ may change throughout the lifetime of a worker because of changing prices.

One would expect that the system would eventually reach a steady state, these prices will stabilize, and barring one caveat (see below), this will ensure that the resulting assignments made by the algorithm satisfy the following near-optimality conditions: 1) the per-worker multi-armed bandit problem with these price-adjusted payoffs is near-optimally solved and 2) the capacity constraints and the complementarity conditions with respect to these prices are satisfied.

The caveat is the following: we expect to see small fluctuations in the queue-length based prices around the stable values even at steady state. Although these fluctuations are essential for appropriate tie-breaking across multiple optimal job types in the exploitation phase, these fluctuations can result in major changes in the membership of $\mathcal{J}^*(i)$, and hence in the set $\mathrm{Str}(i)$ that determines the learning goals. Instead, for condition 1 described above to hold, these sets should be determined based on the mean values of the prices, ignoring the fluctuations altogether.

So we propose the following modification to point (2): we utilize an average of recent prices within some window, and modify the definition of $\mathcal{J}(i)$ to incorporate a small tolerance so that the set $\mathrm{Str}(i)$ remains unaffected by the fluctuations in the prices. To be precise, for a window size W, let $\bar{p}_q(i)$ be the unweighted average of the queue length based prices seen over the past W assignments system-wide (note again that $p_q(i)$ changes every time a job is assigned to a worker, and also when new jobs arrive). Next, for a tolerance $\varepsilon > 0$, we define

$$\mathcal{J}_\varepsilon^*(i) = \{j \in \mathcal{J} : A(i,j) - p(j) \geq \max_j [A(i,j) - p(j)] - \varepsilon\}.$$

Then the strong distinguishability requirements (see Definition 4.1) and hence the set $\mathrm{Str}(i)$ for each i is defined based on $\mathcal{J}_\varepsilon^*(i)$. In our simulations, we picked $W = 900$ and $\varepsilon = 0.05$.

C PRACTICAL IMPLEMENTATION OF OTHER POLICIES

(1) **UCB:** The upper confidence bound (UCB) algorithm is a popular multi-armed bandit algorithm [Audibert and Munos, 2011, Auer et al., 2002, Bubeck and Cesa-Bianchi, 2012], that embodies the well known approach of "optimism in the face of uncertainty" to solving these problems. In its classical implementation, one keeps track of high-probability confidence intervals for the expected payoffs of each arm, and at each step chooses an arm that has the highest upper confidence bound, i.e., the highest upper boundary of the confidence interval. To be precise, if $\bar{r}_j(t)$ is the average reward seen for some arm j, that has been pulled n_j times until time t, then the upper confidence bound for the mean reward for this arm is given by:

$$u_j(t) = \bar{r}_j(t) + \sqrt{2 \log t / n_j}.$$

The algorithm chooses arm $j = \arg\max_j \mu_j(t)$. In our context, the arms are the job types, and if k jobs have already been allotted to a worker, and $\bar{r}_j(k)$ is the average payoff obtained from past assignments of job j, and n_j is the number of these assignments, we will define

$$u_j(k) = \bar{r}_j(k) + \sqrt{2 \log k / n_j} - p_q(j),$$

where $p_q(j)$ is the current queue length based price for job j. The algorithm then chooses job type $j = \arg\max_{j \in \mathcal{J}} \mu_j$ to be assigned to the worker next. Note that this algorithm does not require the knowledge of the instance primitives $(\hat{\rho}, N, \mu, A)$.

(2) **Thompson sampling (TS):** Thompson sampling is another popular multi-armed bandit algorithm [Audibert and Munos, 2011, Bubeck and Cesa-Bianchi, 2012] employing a Bayesian approach to the problem of arm selection. The description of the algorithm is simple: starting with a prior, at every step select an arm with a probability equal to the posterior probability of that arm being optimal. These posterior probabilities are updated based on observations made at each step. One can incorporate information about correlation between the rewards of the different arms in computing these posteriors, which makes it a versatile algorithm that exploits the reward structure in multiple settings. It is known to give asymptotically tight regret guarantees in many multi-armed bandit problems of interest [Agrawal and Goyal, 2011, Kaufmann et al., 2012, Russo and Van Roy, 2014].

In our simulations, TS is implemented as follows. The prior probability of the worker being of type i is $\frac{\hat{\rho}(i)}{\sum_{i'} \hat{\rho}(i')}$. With this as the prior, depending on each worker's history, a posterior distribution of the type of the worker is computed using the knowledge of the expected payoff matrix A. Then a worker type is sampled from this distribution. Suppose this type is i, then a job type $j^* = \arg\max_j A(i,j) - p_q(j)$ is assigned to the worker. In contrast to the UCB algorithm, Thompson sampling does utilize the knowledge of the expected payoff matrix A as well as the arrival rates $\hat{\rho}$ (the latter to construct a starting prior, and the former for the posterior updates).

(3) **Greedy:** The greedy policy is simple: as in Thompson sampling, we keep track of the posterior distribution of the type for each worker, but at each assignment opportunity, instead of sampling a type from the posterior, we take the type i with the highest posterior probability and attempt to assign a job type $j^* = \arg\max_j A(i,j) - p_q(j)$. In the setting with no capacity constraints, one can show that if all the entries of the expected payoff matrix are distinct from each other (which also means that there are no difficult type pairs; see definition 5.2), then this policy achieves a regret of $O(1/N)$ asymptotically.[17] This is likely to be the case for payoff matrices encountered in practice. But in the simulations we demonstrate that for a small N, in the light of our discussion in section B.1, using this policy could prove to be disastrous.

D OTHER PROOFS

D.1 Sufficiency of worker-history-only policies

We show that there is a worker-history-only (WHO) policy that achieves a rate of payoff accumulation that is arbitrarily close to the maximum possible. We will think of N as being fixed throughout this section.

Suppose the system starts at time $t = 1$ with no workers already present before the period.[18] Arrivals thereafter occur as described in Section 3. Consider any (arbitrary, time varying) policy π and let $x_{\pi,t}(i,j)$ denote the derived quantity representing the fraction of workers of type i who are assigned jobs on type j in period t under π. Then the largest possible rate of payoff accumulation

[17] The probability that the true worker type is not identified as the MLE at opportunity t decays as $\exp(-\eta t)$, where η is an instance dependent constant. Thus the total expected regret over the lifetime of a worker is bounded as O(1) in expectation.

[18] The analysis would be very similar and produce the same results if the starting state is an arbitrary one with a bounded mass of workers already present.

under policy π over long horizons is

$$\overline{V}(\pi) = \text{limsup}_{T\to\infty} V_T(\pi) \tag{34}$$

$$\text{where } V_T(\pi) = \frac{1}{T}\sum_{t=1}^{T}\sum_{i\in\mathcal{I}}\rho(i)\sum_{j\in\mathcal{J}} x_{\pi,t}(i,j)A(i,j)\,. \tag{35}$$

Note that we have ignored the effect of less than $\rho(i)$ workers of type i being present for the first N periods, but this does not change the limiting value $\overline{V}$. Also, note that randomization in π cannot increase the achievable value of $\overline{V}$, since one can always do as well by picking the most favorable sample path.

Claim D.1. *Fix any policy π and any $\varepsilon > 0$. Then there is a worker-history-only (WHO) policy that achieves a steady state rate of payoff accumulation exceeding $\overline{V}(\pi) - \varepsilon$.*

Proof. We suppress dependence on π. By definition of $\overline{V}$, we know that there exists an increasing sequence of times $T_1, T_2, \ldots$ such that $V_{T_i} > \overline{V} - \varepsilon/2$ for all $i = 1, 2, \ldots$. We will construct a suitable WHO policy by using a sufficiently large time in this sequence. Let $\nu_t(H_k)$ be the measure of workers in the system with history H_k just before the start of time t, and abusing notation, let $\nu_t(H_k)_j$ be the measure of such workers who are assigned to job type j at time t. Since the policy cannot assign more jobs than have arrived in any period, we have

$$\sum_{k=1}^{N}\sum_{H_k}\nu_t(H_k)_j \le \mu(j) \qquad \text{for all } t \ge 1\,. \tag{36}$$

Fix T, which we think of as a large member of the sequence above. The average measure of workers with history H_k who are present is

$$\bar{\nu}(H_k) = \frac{1}{T}\sum_{t=1}^{T}\nu_t(H_k) \qquad \text{for all } H_k \text{ and } k = 1, 2, \ldots, N\,. \tag{37}$$

The average measure of such workers who are assigned job j is similarly defined and denoted by $\bar{\nu}(H_k, j)$. We immediately have that

$$\sum_{k=1}^{N}\sum_{H_k}\bar{\nu}(H_k)_j \le \mu(j)\,, \tag{38}$$

by averaging Eq. (36) over times until T. Now, consider a worker with history H_k assigned a job of type j. Using the known A matrix and arrival rates ρ, we can infer the posterior distribution of the worker type based on H_k, and hence, the likelihood of the job of type j being successfully completed. Let $p(H_k, j)$ denote the probability of success. Then the distribution of H_{k+1} for the worker is simply given by

$$H_{k+1} = \Big(H_k, \big(j, \text{Bernoulli}(p(H_k, j))\big)\Big)\,.$$

Barring the edge effect at time T caused by workers whose history was H_k at time T, this allows us to uniquely determine $\bar{\nu}(H_{k+1})$ based on $\bar{\nu}(H_k, j)$'s. In particular, for any $\delta_1 > 0$, if $T \ge \max_{i\in\mathcal{I}} \rho(i)/(N\delta_1)$ we have that

$$\begin{aligned}
\bar{\nu}(H_k, (j,1)) &\overset{\delta_1}{\approx} \bar{\nu}(H_k)_j p(H_k, j) \\
\bar{\nu}(H_k, (j,0)) &\overset{\delta_1}{\approx} \bar{\nu}(H_k)_j \big(1 - p(H_k, j)\big)\,.
\end{aligned} \tag{39}$$

Here, $a \overset{\delta}{\approx} b$ represents the bound $|a - b| \le \delta$. Note that we have

$$V_T = \sum_{k=1}^{N} \sum_{H_k} \bar{\nu}(H_k)_j p(H_k, j) . \tag{40}$$

We are now ready to define our WHO policy $\underline{\pi}$. For every H_k such that $\bar{\nu}(H_k) \ge \delta_2$, this policy will attempt to assign a fraction $\bar{\nu}(H_k)_j / \bar{\nu}(H_k)$ of workers with history H_k to jobs of type j. Ignore capacity constraints for the present. (We will find that the capacity constraints will be almost satisfied.) Leave the choice of δ_2 for later; we will choose δ_2 small and then choose $0 < \delta_1 < \delta_2/N$ so as to achieve the desired value of δ_3 below. Workers with *rare* histories, i.e., histories such that $\bar{\nu}(H_k) < \delta_2$, will not be assigned jobs under $\underline{\pi}$. Note that the definition of rare histories refers to frequency of occurrence under π. Then, this uniquely specifies $\underline{\pi}$ as well as the steady state mix of workers before any time t. In particular, the steady state mass $\underline{\nu}(H_k)$ under $\underline{\pi}$ of workers with history H_k that are not rare is bounded as

$$(1 - \delta_1/\delta_2)^{k-1} \bar{\nu}(H_k) \le \underline{\nu}(H_k) \le (1 + \delta_1/\delta_2)^{k-1} \bar{\nu}(H_k) \tag{41}$$

using Eq. (39), and the fact that all subhistories of H_k are also not rare. It follows that

$$\underline{\nu}(H_k) \overset{\delta_3}{\approx} \bar{\nu}(H_k) \quad \text{where } \delta_3 = \max(\exp(N\delta_1/\delta_2) - 1, \delta_2) , \tag{42}$$

for all histories (including rare histories), using $k \le N$ and $\nu(H_k) \le 1$. Violation of the j-capacity constraint under $\underline{\pi}$ is given by

$$\left(\sum_{k=1}^{N} \sum_{H_k} \underline{\nu}(H_k)_j - \mu(j) \right)_+ \le \left(\sum_{k=1}^{N} \sum_{H_k} \bar{\nu}(H_k)_j - \mu(j) \right)_+ + 2^N |\mathcal{J}|^{N-1} \delta_3 = 2^N |\mathcal{J}|^{N-1} \delta_3$$

using Eq. (42) and Eq. (38), and the fact that there are $\sum_{k \le N} (2|\mathcal{J}|)^{N-1} \le 2^N |\mathcal{J}|^{N-1}$ possible histories. It follows that the sum of capacity constraint violations across $j \in \mathcal{J}$ is bounded by $(2|\mathcal{J}|)^N \delta_3$. Pick an arbitrary set of workers to go unmatched to get rid of any capacity violations (this can be done while remaining within the class of WHO policies). In worst case, this will cause payoff loss of 1 for each period remaining in the worker's lifetime. Thus, the loss caused by the need to remedy capacity violations is bounded by $\delta_4 = N(2|\mathcal{J}|)^N \delta_3$ per period.

Ignoring capacity violations, the steady state rate of accumulation of payoff under $\underline{\pi}$ is

$$\sum_{k=1}^{N} \sum_{H_k} \underline{\nu}(H_k)_j p(H_k, j) \overset{\delta_5}{\approx} \sum_{k=1}^{N} \sum_{H_k} \bar{\nu}(H_k)_j p(H_k, j) = V_T(\pi)$$
$$\text{where } \delta_5 = 2^N |\mathcal{J}|^{N-1} \delta_3 < \delta_4 . \tag{43}$$

again using Eq. (42) and the fact that there are $\sum_{k \le N} (2|\mathcal{J}|)^{N-1} \le 2^N |\mathcal{J}|^{N-1}$ possible histories.

Let $V(\underline{\pi})$ denote the true steady state rate of accumulation of payoff under $\underline{\pi}$ when capacity constraints are considered. Combining the above, we deduce that $V(\underline{\pi}) \ge V_T(\pi) - 2\delta_4$. The time T will be chosen as a member of the sequence defined at the beginning of the proof, ensuring $V_T(\pi) \ge \overline{V}(\pi) - \varepsilon/2$; hence it will suffice to show $V(\underline{\pi}) \ge V_T(\pi) - \varepsilon/2$. Hence, it suffices to have $\delta_4 = \varepsilon/4$, which can achieved using $\delta_3 = \delta_2 = \varepsilon/(4N(2|\mathcal{J}|)^N)$ and $\delta_1 = \delta_3 \log(1 + \delta_3)/N$ and T a member of the sequence satisfying $T \ge \max_{i \in \mathcal{I}} \rho(i)/(N\delta_1)$. This yields the required bound of $V(\underline{\pi}) \ge \overline{V}(\pi) - \varepsilon$. $\square$

D.2 Uniqueness of prices under generalized imbalance

Proposition D.2. *Under the generalized imbalance condition, the job shadow prices p^* are uniquely determined.*

Proof of Proposition D.2. The dual to problem 11 can be written as

$$\begin{aligned} &\text{minimize} \sum_{j\in\mathcal{J}} \mu(j)p(j) + \sum_{i\in\mathcal{I}} \rho(i)v(i) \\ &\text{subject to} \\ &p(j) + v(i) \geq A(i,j) \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, \\ &p(j) \geq 0 \quad \forall j \in \mathcal{J}, \\ &v(i) \geq 0 \quad \forall i \in \mathcal{I}. \end{aligned}$$

The dual variables are (P, V) where "job prices" $P = (p(j))_{j\in\mathcal{J}}$ and "worker values" $V = (v(i))_{i\in\mathcal{I}}$. We will prove the result by contradiction. Suppose there are multiple dual optima. Let D be the set of dual optima. Let $\mathcal{J}'$ be the set of jobs such that the prices of those jobs take multiple values in D. Formally,

$$\mathcal{J}' = \{j \in \mathcal{J} : p(j) \text{ takes multiple values in } D\}. \tag{44}$$

Similarly, let $\mathcal{I}'$ be the set of workers such that the prices of those workers take multiple values in D. Formally,

$$\mathcal{I}' = \{i \in \mathcal{I} : v(i) \text{ takes multiple values in } D\}. \tag{45}$$

For each $j \in \mathcal{J}'$, we immediately deduce that there exists a dual optimum with $p(j) > 0$, and hence the capacity constraint of job type j is tight in all primal optima. Similarly, we deduce that for each $i \in \mathcal{I}'$, worker type i is assigned a job in all periods, i.e., $\sum_{j\in\mathcal{J}} x(i,j) = 1$. By assumption, we have

$$\sum_{i\in\mathcal{I}'} \rho(i) \neq \sum_{j\in\mathcal{J}'} \mu(j).$$

Suppose the left hand side is larger than the right (the complementary case can be dealt with similarly). Take any primal optimum x^*. The jobs in $\mathcal{J}'$ do not have enough capacity to serve all workers in $\mathcal{I}'$, hence there must be some worker $i \in \mathcal{I}'$ and a job $s \notin \mathcal{J}'$ such that $x^*(i,s) > 0$. Since $s \notin \mathcal{J}'$, we must have that $p(j)$ has a unique optimum value in D. Call this value $p^*(j)$. Let the largest and smallest values of $v(i)$ in D be $v^{\max}(i)$ and $v^{\min}(i)$. By complementary slackness, we know that

$$\begin{aligned} v^{\max}(i) + p^*(j) &= A(i,j) = v^{\min}(i) + p^*(j) \\ \Rightarrow v^{\max}(i) &= v^{\min}(i). \end{aligned}$$

But since $i \in \mathcal{I}'$ we must have $v^{\max}(i) > v^{\min}(i)$. Thus we have obtained a contradiction.

□

The proof of the next proposition shows that a very simple "learn then exploit" strategy achieves a regret of $O(\log N/N)$. This follows from the fact that, under an identifiability condition, the sequence of sets $(\mathcal{X}^N)$ converges to the set $\mathcal{D}$ in an appropriately defined distance.

Proposition D.3. *Suppose that no two rows in A are identical. Then* $\sup_{x\in\mathcal{D}} \inf_{y\in\mathcal{X}^N} \|x - y\| = O\left(\frac{\log N}{N}\right)$

Proof. It is clear that $x^N \subseteq \mathcal{D}$. We will find an inner approximation $\tilde{\mathcal{X}}^N$ to $\mathcal{X}^N$ such that $\tilde{\mathcal{X}}^N \subseteq \mathcal{X}^N$, and $\tilde{\mathcal{X}}^N$ converges to $\mathcal{D}$ in an appropriate sense as N goes to infinity. To define this approximation, suppose that in the learning problem corresponding to a fixed N, one starts off with a exploration phase of a fixed length $\mathrm{O}(\log N)$, where each job j is presented to the worker O_s number of times (where $O_s = \mathrm{O}(\log N)$, fixed a priori), so that after this phase, the type of the worker becomes known with a probability of error at most $\mathrm{O}(1/N)$. This will then allow us to relate the problem to the problem in which the user type is known.
Suppose after this phase, the probability that a worker of type i is correctly identified is $p(i)$ and the probability that she is mis-identified as some other type i' is $p(i,i')$. Note that since no two rows in A are identical, $p(i,i') = \mathrm{O}(1/N)$ for all $i \neq i'$. Let $d(i,j)$ denote the expected number of times a worker that has been identified as being of type i (correctly or incorrectly) is directed towards job j after the exploration phase, i.e., from job $O_s + 1$ till the N^{th} job. Let $\bar{d}(i,j) = d(i,j)/N$. Then we can see that, one can attain all x in the following set:

$$\tilde{\mathcal{X}}^N = \Big\{ x \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|} : \bar{d}(i,j) \geq 0;\ \sum_{j\in\mathcal{J}} x(ib,s) = 1; \tag{46}$$

$$x(i,j) = \frac{O_s}{N} + p(i)\bar{d}(i,j) + \sum_{i'\neq i} p(i,i')\bar{d}(i',s) \Big\} \tag{47}$$

Now since $\bar{d}(i,j) \leq 1$, and since $p(i,i') \leq \mathrm{O}(1/N)$, we can express the above set as:

$$\tilde{\mathcal{X}}^N = \left\{ x \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|} : \bar{d}(i,j) \geq 0;\ \sum_{j\in\mathcal{J}} x(i,j) = 1;\ x(i,j) = \bar{d}(i,j) + \mathrm{O}\Big(\frac{\log N}{N}\Big) \right\} \tag{48}$$

This in turn is the same as:

$$\tilde{\mathcal{X}}^N = \left\{ x \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|} : x(i,j) \geq \mathrm{O}\Big(\frac{\log N}{N}\Big);\ \sum_{j\in\mathcal{J}} x(i,j) = 1 \right\} \tag{49}$$

Note that by construction, $\tilde{\mathcal{X}}^N \subseteq x^N$. But we can now see that $\tilde{\mathcal{X}}^N$ converges to $\mathcal{D}$ in the sense that

$$\sup_{x\in\mathcal{D}} \inf_{y\in\tilde{\mathcal{X}}^N} \|x-y\| = \mathrm{O}\Big(\frac{\log N}{N}\Big),$$

and hence,

$$\sup_{x\in\mathcal{D}} \inf_{y\in\mathcal{X}^N} \|x-y\| = \mathrm{O}\Big(\frac{\log N}{N}\Big)$$

as well. □

Proposition D.4. *The set $\mathcal{X}^N$ is a convex polytope.*

Proof. For the purpose of this proof, let

$$\overline{\mathcal{X}}^N = \{Nx : x \in \mathcal{X}^N\}.$$

We will show that $\overline{\mathcal{X}}^N$ is a polytope, from which the result will follow. We will prove this using an induction argument. We will represent each point in $\overline{\mathcal{X}}^N$ as a $|\mathcal{I}| \times |\mathcal{J}|$ matrix $(x(i,j))_{|\mathcal{I}|\times|\mathcal{J}|}$. Let worker types in $\mathcal{I}$ be labeled as $i_1,\ldots,i_{|\mathcal{I}|}$ and let job types in $\mathcal{J}$ be labeled as $j_1,\ldots,j_{|\mathcal{J}|}$.

Now clearly, $\overline{\mathcal{X}}^0 = \{(0)_{|\mathcal{I}|\times|\mathcal{J}|}\}$ which is a convex polytope. We will show that if $\overline{\mathcal{X}}^N$ is a convex polytope, then $\overline{\mathcal{X}}^{(N+1)}$ is one as well, and hence the result will follow. To do so, we decompose the assignment problem with $(N+1)$ jobs, into the first job and the remaining N jobs.

A policy in the $(N+1)$- jobs problem is a choice of a randomization over the jobs in $\mathcal{J}$ for the first job, and depending on whether a reward was obtained or not with the chosen job, a choice of a point in $\overline{\mathcal{X}}^N$ to be achieved for the remaining N jobs. Each such policy gives a point in the $\overline{\mathcal{X}}^{(N+1)}$. Suppose that $\eta_1 \in \Delta(\mathcal{J})$ is the randomization chosen for job 1, and let $R(j,1) \in \overline{\mathcal{X}}^N$ and $R(j,0) \in \overline{\mathcal{X}}^N$ be the points chosen to be achieved from job 2 onwards depending on the job j that was chosen, and whether a reward was obtained or not, i.e.. $R(.,.)$ is a mapping from $\mathcal{J} \times \{0,1\}$ to the set $\overline{\mathcal{X}}^N$. Then this policy achieves the following point in the $(N+1)$- jobs problem:

$$\begin{bmatrix} \eta_1(j_1) & \eta_1(j_2) & \dots & \eta_1(j_{|\mathcal{J}|}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1(j_1) & \eta_1(j_2) & \dots & \eta_1(j_{|\mathcal{J}|}) \end{bmatrix} + \sum_{j\in\mathcal{J}} \eta_1(j)\Big(\mathrm{Diag}[A(.,j)]R(j,1) + \mathrm{Diag}[\bar{A}(.,j)]R(j,0)\Big),$$

where

$$\mathrm{Diag}[A(.,j)] = \begin{bmatrix} A(i_1,j) & 0 & \dots & 0 \\ 0 & A(i_2,j) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & A(i_{|\mathcal{I}|},j) \end{bmatrix}$$

and

$$\mathrm{Diag}[\bar{A}(.,j)] = \begin{bmatrix} 1-A(i_1,j) & 0 & \dots & 0 \\ 0 & 1-A(i_2,j) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 1-A(i_{|\mathcal{I}|},j) \end{bmatrix}.$$

And thus we have

$$\overline{\mathcal{X}}^{(N+1)} = \left\{ \begin{bmatrix} \eta_1(j_1) & \eta_1(j_2) & \dots & \eta_1(j_{|\mathcal{J}|}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1(j_1) & \eta_1(j_2) & \dots & \eta_1(j_{|\mathcal{J}|}) \end{bmatrix} + \sum_{j\in\mathcal{J}} \eta_1(j)\Big(\mathrm{Diag}[A(.,j)]R(j,1) + \mathrm{Diag}[\bar{A}(.,j)]R(j,0)\Big) : \eta_1 \in \Delta(\mathcal{J}),\ R(.,.) \in \overline{\mathcal{X}}^N \right\}.$$

Let $\mathbf{1}_s$ be the $|\mathcal{I}| \times |\mathcal{J}|$ matrix with ones along column corresponding to job type j and all other entries 0. Then the set

$$\mathcal{J}(s) = \left\{ \mathbf{1}_s + \mathrm{Diag}[A(.,j)]R(j,1) + \mathrm{Diag}[\bar{A}(.,j)]R(j,0) : R(s,.) \in \overline{\mathcal{X}}^N \right\},$$

is a convex polytope, being a linear combination of two convex polytopes, followed by an affine shift. It is easy to see that $\overline{\mathcal{X}}^{(N+1)}$ is just a convex combination of the polytopes $\mathcal{J}(s)$ for $j \in \mathcal{J}$, and hence $\overline{\mathcal{X}}^{(N+1)}$ is a convex polytope as well. □