The size of the core in assignment markets*

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Revised: September 2016

(Initial version: October 2015)

Abstract

We consider a two-sided assignment market with agent types and a stochastic structure, similar to models used in empirical studies. We characterize the size of the core in such markets. Each agent has a randomly drawn productivity with respect to each type of agent on the other side. The value generated from a match between a pair of agents is the sum of the two productivity terms, each of which depends only on the type (but not the identity) of one of the agents, and a third deterministic term driven by the pair of types. We prove, under reasonable assumptions, that keeping the number of agent types fixed, the relative size of the core vanishes rapidly as the number of agents grows. Numerical experiments confirm that the core is typically small. Our results provide justification for the typical assumption of a unique core outcome in such markets, that is close to a limit point. Further, our results suggest that, given market composition, the wages are almost uniquely determined in equilibrium.

Keywords: Assignment markets, matching, transferable utility, core, convergence, uniqueness of equilibrium, random market.

*A preliminary version of this work appeared in SODA 2015 Kanoria et al. (2015). That publication is an extended abstract containing only a subset of the current theoretical results, no proofs, and no numerical results.

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1 Introduction

We study bilateral matching markets, such as marriage, labor, and housing markets, where participants form partnerships for mutual benefit. The two classic models of matching markets are the *non-transferable* utility (NTU) model of Gale and Shapley (1962), where payments are not allowed between the agents; and the Shapley–Shubik–Becker *transferable* utility (TU) model in Shapley and Shubik (1971) and Becker (1973), where monetary transfers are allowed between arbitrary sets of agents. For each of these models the natural solution concept is that of a *stable* outcome, in which there is no pair of agents who would prefer being matched with each other over their current outcome.

In the TU matching market setting, it is well known that the notion of a stable outcome coincides with that of a competitive equilibrium. Furthermore, a stable outcome is guaranteed to exist in any two-sided market, but is generically not unique (Shapley and Shubik 1971). Therefore, a fundamental question that arises is how big is the core (i.e., the set of stable matchings) in a given market. This work seeks to characterize the size of the core as a function of market characteristics.[1]

The motivation for our study of core size in TU matching markets is twofold. First, it is of interest to know whether basic market primitives, i.e., the number of agents and the values of possible matches, are sufficient to determine the outcome of the market, or whether there is significant ambiguity arising from which equilibrium the market is in. As an example, can a labor market support higher wages for labor without adding jobs or improving productivity, just by moving to a different equilibrium? A small core would suggest a negative answer to this question. Matching platforms such as Upwork, AirBnB, TaskRabbit and others commonly have a "suggested wage/rate" feature, which is in the nature of a suggestion; i.e., it is not binding (AirBnB 2015). Such platforms would then not be able to use such a feature to select between equilibria, since the equilibrium is unique.[2] Second, the concept of stability is widely used as a

[1]A small core has been found in special cases of the TU setting as in Gretsky et al. (1992, 1999), Hassidim and Romm (2015), which we discuss below. In the NTU setting, real markets have almost always been found to contain a nearly unique stable outcome; see, e.g., Roth and Peranson (1999). A body of theory explains this phenomenon (Immorlica and Mahdian 2005, Kojima and Pathak 2009, Ashlagi et al. 2016, Holzman and Samet 2013, Azevedo and Leshno 2016).

[2]Our results extend to the case where transfers are taxed by the platform; cf. Remark 1.

starting point in theoretical and empirical studies in matching markets, which in general require or assume a nearly unique stable matching in order to facilitate predictions, comparative statics, and so on. For instance, several papers (Choo and Siow 2006, Galichon and Salanié 2010) make a continuum limit assumption in TU markets, and in the continuum limit there is a unique equilibrium. However, there is insufficient theoretical basis to justify such an assumption on the size of the core. We ask when such an assumption is justified. The core size and other features of real NTU markets can be studied as the data on ordinal preferences is often available. In TU matching markets this becomes a harder task given that preferences (i.e., the value generated by a pair/match) are difficult to observe, which has hindered empirical studies of features like core size in TU markets, providing additional motivation for a theoretical study of core size.

Our main contribution is to bound the size of the core as a function of the market primitives. In particular, we characterize the rate at which the core shrinks as a function that depends primarily on the size of the market. We find that the size of the core in TU matching markets is typically small. This suggests that online matching platforms have limited ability to redistribute welfare across agents, without changing the rules of the market. In addition, our findings justify the continuum limit assumption that is typically made in such markets. Under additional (mild) assumptions, we find that the core rapidly approaches the unique equilibrium in the continuum limit.

In particular, we consider the traditional assignment game model of Shapley and Shubik (1971), consisting of "workers" and "firms", that can each match with at most one agent on the other side. To model the different skills of the workers and the different requirements of the firms, we assume that there are K types of workers and Q types of firms. Matching worker i with firm j generates a value Φ_{ij} (this can be divided between i and j in an arbitrary manner since transfers are allowed), which we model as a sum of two terms: a term $u(\cdot,\cdot)$ that depends only on the *types* of i and j, and a term $\psi_{i,j}$ that represents the "idiosyncratic" contributions of worker i to firm j. In our model the $u(\cdot,\cdot)$ is assumed to be fixed, but the ψ_{ij} is the sum of two random variables, the "productivity" of worker i with respect to the *type* of firm j and, symmetrically, the "productivity" of firm j with respect to the *type* of worker i. These productivities are assumed to be independently drawn from any distribution (satisfying some mild assumptions) for each (agent, type) pair. In addition to its plausibility, such a generative model for the value of a match has been used in empirical studies of marriage markets, starting with Choo and Siow (2006) (see also Chiappori et al. 2015, Galichon and Salanié 2010).

We study the size of the set of stable outcomes for a random market constructed in this way. Shapley and Shubik (1971) showed that the set of stable outcomes (which is the same as the core) has a lattice structure, and thus has two extreme stable matchings: the worker optimal stable match, where each worker earns the maximum possible and each firm the minimum possible in any stable matching; and the firm optimal stable match, which is the symmetric counterpart. Also, in all stable outcomes, the matching between workers and firms must be such that social welfare is maximized. Given these structural properties, our metric for the relative size of the core is quite natural: we consider the difference between the maximum and minimum total utility of workers (equivalently, firms) in the core, scaled by the (maximum) social welfare. We show that if the number of types is fixed, then the core is small and provide bounds on the size of the core under three different sets of assumptions.

First, we consider productivities drawn from a general distribution satisfying mild conditions.[3] Our first main result (Theorem 1) establishes a small core under some reasonable assumptions on market structure: specifically, the expected core size is $O^*(1/\sqrt[\ell]{n})$ in a market with n agents, and at most $\ell = \max(K, Q)$ types of agents on each side (with ℓ fixed).[4] We show that this bound is essentially tight by constructing a sequence of markets such that the core size is[5] $\Omega(1/\sqrt[\ell]{n})$. Thus the core shrinks with market size, and this shrinking is faster when there are fewer types of agents. Additionally, we obtain a tighter upper bound in the special case with just one type of employer and more employers than workers (Theorem 2). Our upper bound in this case improves sharply as the number of additional employers m increases; we establish a bound of $O^*(1/(n^{1/\ell}m^{1-1/\ell}))$, where ℓ is the number of worker types.

Second, we again consider a fixed set of types, but assume that productivities are drawn

[3] Namely, we require the support of the distribution to be a (possibly unbounded) interval, and the density to be continuous and positive everywhere in the support. These requirements are satisfied by the normal, Gumbel, uniform, and exponential distributions, among others.

[4] We write $f(n) = O^*(g(n))$ if there exists $r < \infty$ such that $f(n) \leq r(\log n)^r g(n)$ for all n. In words, this corresponds to the big-O notation where poly-logarithmic factors are also suppressed.

[5] Similarly, we write $f(n) = \Omega(g(n))$ if $\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| > 0$. This corresponds to the standard big-O notation.

from a distribution with unbounded support (Theorem 3), and obtain stronger results. We show an $O^*(1/n)$ bound on core size in this case. We also establish that the core solutions in the finite market converge to the unique equilibrium in the continuum limit market, bounding the distance between any core solution and the limit equilibrium by $O^*(1/\sqrt{n})$. In particular, this bounds the convergence rate of the (scaled) number of matched pairs belonging to each pair of types, which is the empirically observed quantity in many settings (transfers are often not observed, e.g., Choo and Siow 2006), and hence bounds the error that occurs when match utilities are estimated in empirical studies based on a continuum limit assumption. Both our bounds are again tight.

To supplement our theoretical findings, we conduct computational experiments (Section 5). We run simulations with a variety of distributions for the idiosyncratic productivity terms and also go beyond our theoretical development by allowing the number of agent types to grow. In a broad range of settings, we find that the core is small, even in relatively small markets. The only exception we find is the case of productivities following a Pareto distribution, and a large number of agent types. Overall, our results strongly suggest that the core is small in practically relevant settings. Our experiments also show that the (scaled) number of matched pairs belonging to a each pair of types rapidly converge to the appropriate limiting value.

To conclude, we briefly describe the theoretical component of our work. Our model has the following property (here, think of $u(\cdot, \cdot)$ as being formally incorporated in the worker productivity): there is a "price" associated with each (worker type, firm type) pair, such that for every matched pair of agents of these types, the utility of each agent is her productivity (with respect to the type on the other side), which is "corrected" additively on both sides of the market (in opposite directions) by the price. Each of our bounds on core size is proved by showing uniform bounds on variation in type-pair prices across core allocations. A key component of our analysis is to relate the combinatorial structure of the core to order statistics of certain independent identically distributed (i.i.d.) random variables (r.v.s). These r.v.s are one-dimensional projections of point processes in (particular subregions of) the unit hypercube, where the point processes correspond to the market realization. An analytical challenge that we face is that the relevant projections as well as the relevant order statistics are themselves a random function of the market realization. We overcome this via appropriate union bounds. In our proof of Theorem 3 we further use an order-theoretic approach to control the set of candidate core solutions (this also yields a limiting characterization of core solutions as market size grows). Our analysis sheds light on how market structure affects the core and its size.

Related Literature Most of the related literature focuses on the NTU model of Gale and Shapley (1962). Building on that model, a number of papers establish a small core under various assumptions such as short preference lists or many unmatched agents (Immorlica and Mahdian 2005, Kojima and Pathak 2009, Kojima et al. 2013, Menzel 2015, Peski 2015), and strongly correlated preferences (Holzman and Samet 2013, Azevedo and Leshno 2016).[6] In a recent paper Ashlagi et al. (2016) show that in a random NTU matching market with long lists and uncorrelated preferences, even a slight imbalance results in a significant advantage for the short side of the market and that there is approximately a unique stable matching.[7] Further, Ashlagi et al. (2016) find the near uniqueness of the stable matching to be robust to varying correlations in preferences and other features, suggesting that a small core may be generic in NTU matching markets.

There is an extensive literature on large assignment games that extends the many structural properties established by Shapley and Shubik for finite assignment games to a setting in which the agents form a continuum; see, for example, Gretsky et al. (1992, 1999). Those papers also show convergence of large finite markets to the continuum limit, including that the core shrinks to a point. However, unlike in our model, they model the productivity of each partnership as a deterministic function of the pair of types, with the only randomness being in the number of agents of each type. The work on assignment games that is most closely related to our work is a recent preprint of Hassidim and Romm (2015): in their model, all workers (firms) are a priori identical, and the value of matching worker i to firm j is a random draw from

[6] Azevedo and Leshno (2016) consider a fixed number of schools and a growing number of students. One can think of each school as being replaced with a linearly growing number of agents (equal to the number of seats in the school) having identical preferences (identical to those of the school), without affecting the core. Hence, we see Azevedo and Leshno (2016) as implicitly considering preferences that are strongly correlated in a particular way.

[7] Our results on a small core in TU markets also hold under a similar "generalized imbalanced" requirement, which is generically satisfied; cf. Section 3.

a bounded distribution, independently for every pair (i, j). For such a model, they establish an approximate "law of one price," i.e., that workers are paid approximately identical salaries in any core allocation, and that the long side gets almost none of the surplus in unbalanced markets. By contrast, we work with multiple types of workers and firms, and the value of a match depends on the types of each agent, and random variables that depend on the identity of one of the agents and the *type* (but not the identity) of the other agent. We believe that our model better captures features of real markets, as it allows for correlations in preferences across types.

There has been recent work in the operations literature characterizing equilibria in matching markets. Nguyen (2015) considers bargaining in a network en route to formation of coalitions (a generalization of matching) and characterizes the stationary equilibria of the game. Alaei et al. (2016) generalize the Shapley–Shubik model to the case of utilities that are not necessarily quasilinear in payments. The paper characterizes equilibria, and provides an algorithm for efficiently computing the extreme equilibria. This work leads to a mechanism for ad auctions that has good properties even when there is inconsistency in click-through rate estimates. Overall, the goal of this line of work, including the present paper, is to obtain a refined understanding of equilibria to enable the design of better marketplaces for matching.

The rest of the paper is organized as follows. We present our model in Section 2 and a statement of our results in Section 3. An overview of the proof of our main result for general distributions (Theorem 1) is in Section 4 (the proof is deferred to the appendices). We defer the proof of Theorems 2 and 3 to the appendices. Numerical experiments complementing our theoretical findings are described in Section 5. We conclude with a discussion in Section 6.

2 Model Formulation

We consider a two-sided transferable utility matching market with a finite number of agents. The sides of the market are represented by labor ($\mathcal{L}$) and employers ($\mathcal{E}$). Let $n_{\mathcal{L}}$ be the number of workers in $\mathcal{L}$ and $n_{\mathcal{E}}$ be the number of employers in $\mathcal{E}$; we let $n := |\mathcal{L}| + |\mathcal{E}|$ denote the size of the market, i.e., the total number of agents. If $n_{\mathcal{L}} = n_{\mathcal{E}}$ we say that the market is *balanced.*

Otherwise, we say that the market is *unbalanced*. A *matching* is a mapping M from $\mathcal{L} \cup \mathcal{E}$ to itself such that for every $i \in \mathcal{L}$, $M(i) \in \mathcal{E} \cup \{i\}$, and for every $j \in \mathcal{E}$, $M(j) \in \mathcal{L} \cup \{j\}$, and for every $i, j \in \mathcal{L} \cup \mathcal{E}$, $M(i) = j$ implies $M(j) = i$. That is, each agent is either matched to one agent on the other side, or remains unmatched. If $M(i) = j \neq i$, we say $(i, j) \in M$. We assume that the underlying graph is complete; that is, all pairs of agents can potentially be matched.

Each side of the market is partitioned into a finite number of *types*. We define $\mathcal{T}_{\mathcal{L}} := \{1, \ldots, K\}$ and $\mathcal{T}_{\mathcal{E}} := \{1', \ldots, Q'\}$ to be the set of types on the labor and employer sides respectively. Let $\mathcal{T} = \mathcal{T}_{\mathcal{L}} \times \mathcal{T}_{\mathcal{E}}$ denote the set of pairs of types. For a given type $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$, we denote by n_t the number of agents of type t. Finally, $\tau(a)$ denotes the type of agent $a \in \mathcal{L} \cup \mathcal{E}$; given a type t and an agent a, we say that $a \in t$ if $\tau(a) = t$. In what follows, we typically use i to denote an individual agent in $\mathcal{L}$, and j to denote an individual agent in $\mathcal{E}$.

The value of the match between i and j is denoted $\Phi(i, j)$. An *outcome* is a pair (M, γ), where M is a matching between agents in $\mathcal{L}$ and $\mathcal{E}$, and γ is a payoff vector such that $\gamma_i + \gamma_j = \Phi(i, j)$ for every pair of matched agents, and $\gamma_i = 0$ for every unmatched agent. That is, the vector γ indicates how the value of a match is divided between the agents involved in the match. In this paper we shall be concerned with outcomes that are in the *core*, i.e., outcomes such that no coalition of players can produce greater value among themselves than the sum of their utilities. Shapley and Shubik (1971) show that for this matching market model, an outcome (M, γ) is in the core if and only if it is *stable*, a seemingly weaker condition. An outcome is said to be *stable* if no agent prefers not to participate in the matching (because of a negative payoff), and if there is no *blocking pair* of agents who can both do better by matching with each other (because the value they generate by matching with each other exceeds the sum of their current payoffs). Thus, the stability condition can be mathematically described as $\gamma_i + \gamma_j \geq \Phi(i, j)$ for all $i \in \mathcal{L}$ and $j \in \mathcal{E}$, and further $\gamma_i \geq 0$ for all $i \in \mathcal{L} \cup \mathcal{E}$. Furthermore, it is known that the matching M in any stable outcome must be utility-maximizing; that is, $M \in \operatorname{argmax}_{M' \in \mathcal{M}} \sum_{(i,j) \in M'} \Phi(i, j)$, where $\mathcal{M}$ is the set of all possible matchings. For a given matching M, we refer to $\sum_{(i,j) \in M} \Phi(i, j)$ as the *weight* of the matching.[8]

[8] These results are formalized by Shapley and Shubik (1971). They show that the set of stable outcome utilities is the set of optima of the dual to the maximum-weight matching linear program, implying in particular that the matching M in a stable outcome must be a maximum-weight matching.

2.1 Structure of $\Phi(i,j)$

We now impose some structure on the value of matching two agents. Recall that each agent in our model is associated with a type. These types are used to group agents with similar characteristics. As an example, consider a labor market consisting of Ph.D. candidates (workers) and firms. Suppose that the type of a worker is the university and program he is graduating from, while the type of a firm corresponds to the industry it belongs to (e.g., tech, consulting, finance).[9] It is then natural to think that some portion of Φ will be determined by the types of the agents alone; in other words, matching two agents of a certain type will induce some baseline value. For instance, depending on the curricula of the programs, students at one program might be better trained for the tech industry than for consulting, while the opposite might hold for students at another program. However, each worker also has her own characteristics, which might make her more or less suitable for a particular industry, relative to other workers from the same program. Similarly, a particular firm may be more or less attractive to workers from a particular program, relative to other firms in the same industry.[10]

To capture both the correlation in preferences between agents of the same type as well as individual variations, we model the value $\Phi(i,j)$ of matching i and j as the sum of two components: a type-type utility term $u(\tau(i),\tau(j))$, and a match-specific term $\psi_{i,j}^{\tau(i),\tau(j)}$. The utility term $u(\tau(i),\tau(j))$ depends only on the agents' types, and allows us to express how well suited, in general, a worker of type $\tau(i)$ is to work in a firm of type $\tau(j)$. The match-specific term $\psi_{i,j}^{\tau(i),\tau(j)}$ potentially depends on both the identity of the agents as well as their types, and captures how useful are i's individual skills to perform job j. We further assume that $\Phi(i,j)$ is additively separable as follows.

Assumption (Separability)**.** $\Phi(i,j) = u(\tau(i),\tau(j)) + \epsilon_j^{\tau(i)} + \eta_i^{\tau(j)}$.

The separability assumption states that the match-specific component, $\psi_{i,j}^{\tau(i),\tau(j)}$, is further additively separable into two terms, $\epsilon_j^{\tau(i)}$ and $\eta_i^{\tau(j)}$; *each term depends on the identity of one*

[9]More generally, several other relevant characteristics may be included to define a type, such as location and past experience.

[10]An alternative way of thinking about types is in the context of an empirical model. There, types represent the observable characteristics of the agents (age, sex, location, education level). In addition, agents also have some unobservable characteristics, captured by allowing for idiosyncratic variations.

agent and only the type of the other agent. In particular, for any fixed employer j and two distinct workers $i, i' \in \mathcal{L}$ we have $\epsilon_j^{\tau(i)} = \epsilon_j^{\tau(i')}$ whenever $\tau(i) = \tau(i')$, as the term ϵ depends only on the *type* of the agents in $\mathcal{L}$. Analogously, the term η depends on the *individual* worker $i \in \mathcal{L}$ but only on the *type* of the firm $j \in \mathcal{E}$.

To illustrate the rationale behind the separability assumption, consider our previous example of a market consisting of Ph.D. candidates and firms. Students in a particular program have their own idiosyncratic skills; these are captured by the term $\eta_i^{\tau(j)}$, which indicates how an individual student i is valued by a certain type of industry $\tau(j)$, relative to other students of type $t(i)$. Note that all industries of type $\tau(j)$ value candidate i equally. Similarly, the productivity terms associated with firms ($\epsilon_j^{\tau(i)}$) capture how attractive that firm is to students emerging from a given program, relative to the other firms of type $\tau(j)$. Throughout the paper, we refer to the term $u(\tau(i), \tau(j))$ as the *type-type compatibility*, and to the terms η and ϵ as the *idiosyncratic productivity* terms.

Unless otherwise stated, we model the term $u(\tau(i), \tau(j))$ as a fixed constant, whereas the ϵ and η terms are modeled as random variables, independent across agent type pairs. We assume that the distributions of ϵ and η terms are supported on a (possibly unbounded) interval, with positive and continuous density everywhere in the support. The continuum limit of such a model was first introduced by Choo and Siow (2006), to empirically estimate certain structural features of marriage markets. The model has since been employed in other contexts in the empirical literature in matching markets (Galichon and Salanié 2010, Chiappori et al. 2015, Fox 2008). This model is attractive for both theoretical and empirical work, as it allows for reasonable correlation in agents' preferences, and also heterogeneity and idiosyncratic variation via the random productivities, while still remaining tractable due to a fixed number of types.[11]

2.2 Preliminaries

We now state some preliminary observations on the structure of the core under the separability assumption. We start by showing that the payoffs can be expressed more conveniently. For each

[11] These features have also been important in facilitating identification (Choo and Siow 2006, Chiappori et al. 2015). A researcher is typically able to observe only the cross-section marriage/matching distribution, namely, the number of type t agents matched to type t' agents on the other side of the market.

$i \in \mathcal{L}$ and each type $q \in \mathcal{T}_\mathcal{E}$, let $\tilde{\eta}_i^q = u(\tau(i), q) + \eta_i^q$.

Denote by $M(t)$ the set of agents who are matched to an agent of type t. In addition, let U denote the set of unmatched agents. It is easy to check that the weight of a matching depends only on the *type* of partner that each agent is matched to. This implies that the maximum-weight matching is typically not unique under the separability assumption. However, since the idiosyncratic productivities are random with non-atomic distributions, the following holds.

Observation 1. *With probability 1, the sets U and $M(t)$ for $t \in \mathcal{T}_\mathcal{L} \cup \mathcal{T}_\mathcal{E}$ are the same under all maximum weight matchings. We call this the* type-matching.

We take this as the definition of the sets $M(t)$ and U for our purposes, since in any stable outcome, the matching is a maximum weight matching; cf. footnote 8.

Proposition 1. *Any core solution (M, γ) corresponds to a vector $\alpha = \{\alpha_{kq}\} \in \mathbb{R}^{K \times Q'}$ such that the payoffs can be expressed as:*

- $\gamma_i = \tilde{\eta}_i^q - \alpha_{kq}$ *for all $i \in \mathcal{L}$ such that $\tau(i) = k$ and $i \in M(q)$.*
- $\gamma_j = \epsilon_j^k + \alpha_{kq}$ *for all $j \in \mathcal{E}$ such that $\tau(j) = q$ and $j \in M(k)$.*
- $\gamma_i = 0$ *for all $i \in U$.*

Proposition 1 follows directly from stability. This proposition formalizes the existence of a single "price" for every type-pair (k, q) that is common across all matched pairs of agents with those types. In particular, if worker i and firm j of types k and q respectively are matched, the worker payoff is the type-type compatibility plus her own productivity with type q minus the price α_{kq}; on the other hand, the firm's payoff is its productivity with type k plus α_{kq}. Note that this implies that if two firms j' and j'' are of the same type and are matched to the same type, the difference in their utilities must be equal to the difference in their productivities with respect to that type.

Based on Proposition 1, any core solution can be expressed in terms of the maximum weight matching M and the vector α. The following proposition states necessary and sufficient conditions for (M, α) to be a core outcome. (The maximum over an empty set is defined as $-\infty$.)

Proposition 2. *The following conditions are necessary and sufficient for* (M, α) *to be a core solution:*

(ST) For every pair of types (k, q), $(k', q') \in \mathcal{T}$*:*

$$\min_{i \in k' \cap M(q')} \tilde{\eta}_i^{q'} - \tilde{\eta}_i^q + \min_{j \in q \cap M(k)} \epsilon_j^k - \epsilon_j^{k'} \geq \alpha_{k'q'} - \alpha_{kq} \geq \max_{i \in k \cap M(q)} \tilde{\eta}_i^{q'} - \tilde{\eta}_i^q + \max_{j \in q' \cap M(k')} \epsilon_j^k - \epsilon_j^{k'}.$$

(IR) For every pair of types $(k, q) \in \mathcal{T}$*:*

$$\min_{j \in q \cap M(k)} \epsilon_j^k \geq -\alpha_{kq} \geq \max_{j \in q \cap U} \epsilon_j^k,$$

$$\text{and} \quad \min_{i \in k \cap M(q)} \tilde{\eta}_i^q \geq \alpha_{kq} \geq \max_{i \in k \cap U} \tilde{\eta}_i^q.$$

We refer the reader to Chiappori et al. (2015, Proposition 1) for a proof. The first set of conditions (ST) follow from re-expressing the condition imposing the non-existence of a blocking pair of matched agents; i.e., for every $(i, j) \in M$, we have $\gamma_i + \gamma_{j'} \leq \Phi(i, j)$ for all $j \in \mathcal{E}$ and $\gamma_{i'} + \gamma_j \leq \Phi(i, j)$ for all $i \in \mathcal{L}$. In particular, if the condition for pairs (k, q) and (k', q') fails to hold, it means that we can find either (1) a worker i of type k who is matched to a firm in q ($i \in k \cap M(q)$) and a firm j' of type q' matched to an agent in k' such that i and j' would rather be matched together, or, (2) a worker i' of type k' who is matched to a firm in q' and a firm j in q matched to an agent in k such that i' and j prefer to be matched together. By considering the difference between two prices, these conditions state how much a type-pair price can vary relative to another type-pair price; therefore, these conditions impose bounds on the *relative variation* of prices.

The second conditions (IR) follow from combining two facts. The payoffs of matched agents are non-negative, as otherwise they would rather be unmatched (see Proposition 1); this implies the left inequalities. The non-existence of a blocking pair involving an unmatched agent, implies the right inequalities. In particular, note that the right inequality for a type-pair price α_{kq} is meaningful only if an unmatched agent of type k or q exists. The (IR) conditions constitute bounds on the *absolute variation* of type-pair prices. Note that Proposition 2 highlights the lattice structure of the set of core solutions (Shapley and Shubik 1971).

We conclude with a definition of the *size of the core*, denoted by $\mathcal{C}$. We define $\mathcal{C}$ as the difference between the maximum and minimum total payoff of workers (or firms) among core outcomes, scaled by the overall social welfare (the total weight of M) in any core outcome. This can be equivalently stated in terms of the vector α. For each pair of types $(k, q) \in \mathcal{T}$, let $\alpha_{kq}^{\max}$ and $\alpha_{kq}^{\min}$ be the maximum and minimum possible values of α_{kq} in the core. Defining $\alpha^{\max} = (\alpha_{kq}^{\max})_{k \in \mathcal{T}_{\mathcal{L}}, q \in \mathcal{T}_{\mathcal{E}}}$, note that $(M, \alpha^{\max})$ is in the core and constitutes the firm-optimal stable outcome. Similarly, $(M, \alpha^{\min})$ is the worker-optimal stable outcome, where the definition of $\alpha^{\min}$ is analogous to that of $\alpha^{\max}$. The size of the core is defined in terms of $\alpha^{\max}$ and $\alpha^{\min}$ as follows.

Definition 1 (Size of the core)**.** *Let M be the unique maximum weight type-matching. For each pair of types $(k, q) \in \mathcal{T}$, let $N(k, q)$ denote the number of matches between agents of type k and agents of type q. Then, the size of the core is denoted by $\mathcal{C}$ and is defined as*

$$\mathcal{C} = \frac{\sum_k \sum_q N(k, q) |\alpha_{kq}^{\max} - \alpha_{kq}^{\min}|}{\text{weight}(M)}.$$

It is worth noting that $\mathcal{C}$ is always between 0 and 1. This is because weight(M) is the total surplus produced by the match, while $|\alpha_{kq}^{\max} - \alpha_{kq}^{\min}|$ captures how much the surplus kept by one side can vary, which in turn is scaled by the number of matches involving agents of such a type-pair. The stability conditions imply that, for each match, the variation in surplus kept by each side must always be less than the value of the match.

We conclude this section by noting that the results in the paper can be further generalized.

Remark 1. *All of our analysis and bounds extend to a model with taxation of transfers (Jaffe and Kominers 2014), which captures commissions charged by matching platforms. Think of the utility of each agent in a matched pair (i, j) as arising from both the sum of a base utility from participating in the match and the amount received/paid in a transfer payment between the matched partners. Suppose the base utility is $-c(\tau(i), \tau(j))$ for the worker, whereas the remaining match value, i.e., $\Phi(i, j) + c(\tau(i), \tau(j))$, accrues as the base utility to the firm (thus, all the stochastic variation due to idiosyncratic productivities is in the base utility that accrues to the firm; the base utility/cost of the worker depends only on the pair of types). Suppose that the* *matching platform collects a fraction $\lambda \in [0,1)$ of the transferred amount as taxes/commission. Following Jaffe and Kominers (2014), we assume that the base utility is (weakly) negative for workers, i.e., $c(k,q) \geq 0$ for all $k \in \mathcal{T}_{\mathcal{L}}, q \in \mathcal{T}_{\mathcal{E}}$, capturing the base cost to a worker of type k of matching to a firm of type q. This ensures that transfers are only from firms to workers, making this a so-called "wage market". Consider any such market $\mathcal{M}_{\text{tax}}$. There is a one-to-one correspondence between the core solutions in $\mathcal{M}_{\text{tax}}$ and those in a market $\tilde{\mathcal{M}}$ with no taxes, the same set of agents, and modified values from matching pairs of agents*[12]

$$\tilde{\Phi}(i,j) = \Phi(i,j) - \frac{\lambda c(\tau(i), \tau(j))}{1-\lambda} \qquad \forall\, i \in \mathcal{L}, j \in \mathcal{E}\,.$$

The correspondence between core solutions of $\mathcal{M}_{\text{tax}}$ and $\tilde{\mathcal{M}}$ is as follows: the matching is identical in both markets, firm payoffs are identical in both markets, and the payoffs of workers are a factor $(1-\lambda)$ smaller in $\mathcal{M}_{\text{tax}}$ relative to $\tilde{\mathcal{M}}$. It follows that the size of the core[13] *in $\mathcal{M}_{\text{tax}}$ is identical to that in $\tilde{\mathcal{M}}$.*

3 Results

We now turn our attention to the main objective of the paper, which is to understand how the size of the core scales as the market grows. Given the stochastic nature of our model, the size of the core $\mathcal{C}$ is itself a random variable. Therefore, the rest of this section is devoted to studying how the expected/typical value of $\mathcal{C}$ depends on the characteristics of the market. Throughout this section we keep the number of agent types as well as $u(\,\cdot\,,\,\cdot\,)$ fixed and allow the number of agents to grow.

In markets with a finite number of agents there is always a finite number of stability constraints. Thus, it is generically possible to marginally modify some payoffs in a core solution without violating stability and, therefore, the size of the core is strictly positive with probability one (Shapley and Shubik 1971).[14] However, as the size of the market increases (whereas

[12] The modification can be absorbed in the $u(\tau(i), \tau(j))$ term, leaving the idiosyncratic productivities unaffected.

[13] One may define the size of the core in $\mathcal{M}_{\text{tax}}$ as the difference between the maximum and minimum total payoff of firms in core outcomes (this difference exceeds the corresponding difference for workers), divided by the weight of M (all stable matchings once again live on M, and weight(M) is again the total payoff of the workers, employers, and the platform combined).

[14] In the continuum limit markets as used in the empirical studies (e.g., Choo and Siow 2006, whose model is

the number of agent types stays the same), the number of stability constraints also increases, limiting the possible perturbations to the payoffs (cf. Proposition 2). Hence, one would expect the set of core vectors α to shrink as the market size increases. In this section, we characterize the *rate* at which the size of the core shrinks as the size of the market increases, as a function of the market primitives. In particular, in Section 3.1, we present results for the general case of productivities drawn independently from any bounded or unbounded distribution satisfying mild conditions. In Section 3.2, we present stronger results for distributions with (two-sided) unbounded support, including convergence of the core to a particular limiting point.

3.1 Idiosyncratic productivities with a general distribution

In this section, we consider general distributions F for the idiosyncratic productivities, that satisfy the condition that F is supported on an interval of the form $[C_l, C_u]$ or $(-\infty, C_u]$ or $[C_l, \infty)$ for some finite C_l, C_u, and its density f is positive and continuous everywhere on the support.

One type on each side of the market. We start by considering the simple case of markets with one type on each side, that is, $K = Q = 1$. Given that there is only one type of agent on each side, the deterministic type-type utility term $u = u(\tau(i), \tau(j))$ will be the same for all matches, regardless of the identity of the agents. The value of a match between agents $i \in \mathcal{L}$ and $j \in \mathcal{E}$ is $\Phi(i, j) = u + \eta_i + \epsilon_j$, where one may think of η_i as representing the quality of worker i and ϵ_j as representing the quality of firm j. Suppose $u > 0$ is a fixed constant.

Definition. *We say $f(n) = O^*(g(n))$ if there exists $C < \infty$ such that $f(n) \leq (\log n)^C g(n)$ for all $n \geq 2$.*

Definition. *A sequence of events $\mathcal{E}_n$ occurs with high probability if* $\lim_{n\to\infty} \Pr(\mathcal{E}_n) = 1$.

Remark 2. *The size of the core depends on the number of agents on each side of the market.*

very similar to ours), there is a unique core solution. Real-world markets, on the other hand, are finite and thus always have multiple core solutions. Therefore, bounding the size of the core as a function of the market size and other market primitives may provide some justification for the continuum market assumption.

- *In a balanced market, i.e., when $n_{\mathcal{L}} = n_{\mathcal{E}}$, the core is large if F is supported on positive values. For instance, if F is Uniform$(0,1)$, the above market has $\mathcal{C} \geq u/(u+2)$ with probability 1. In particular, we have $E[\mathcal{C}] = \Omega(1)$.*

- *In any unbalanced market, i.e., $n_{\mathcal{L}} \neq n_{\mathcal{E}}$, there exists $f(n) = O^*(1/n)$ such that, with high probability, we have $\mathcal{C} \leq f(n)$. Also, $E[\mathcal{C}] = \Omega(1/n)$.*

In a balanced market, all agents will be matched in a stable solution and, by Proposition 1, we can describe the size of the core in terms of a single parameter α; by Proposition 2, the core consists of all $\alpha \in [-\min_j \epsilon_j, u + \min_i \eta_i]$. In other words, the value u that is part of $\Phi(i,j)$ for each (i,j) can be split in an arbitrary fashion between employers and workers. On the other hand, in any unbalanced market, i.e., $n_{\mathcal{L}} \neq n_{\mathcal{E}}$, the core is small and rapidly shrinks with market size. It turns out that the short side of the market has a significant advantage: if there are fewer workers than firms, the price α is always negative in the core, and is bounded as $\min_{j\in M} \epsilon_j \geq -\alpha \geq \max_{j\in U} \epsilon_j$. This observation agrees with some of the existing results for the non-transferable utility setting (Ashlagi et al. 2016).

Multiple types on both sides of the market. We now consider the general case of K types of labor and Q types of employers. The following condition generalizes the imbalance condition to the case of multiple types. The idea is to get rid of the cases that, for certain values of deterministic type-type utilities $u(\cdot,\cdot)$, may resemble a balanced problem.[15]

Assumption 1 (Generalized Imbalance)**.** *For every pair of subsets of types $\mathcal{S} \subseteq \mathcal{T}_{\mathcal{L}}$ and $\mathcal{S}' \subseteq \mathcal{T}_{\mathcal{E}}$, we have $\sum_{t\in\mathcal{S}} n_t \neq \sum_{t\in\mathcal{S}'} n_t$. In words, there is no subset of types for which the induced submarket is balanced.*

We highlight that in our setting with fixed K and Q and growing n, "most" markets satisfy Assumption 1.[16]

Throughout this section we allow the number of agents to grow, while keeping the number of types fixed. We limit the way in which the market grows by assuming that there is at least

[15] Whenever Assumption 1 is not satisfied, one can construct a market for which $\mathrm{E}[\mathcal{C}] = \Omega(1)$, by extending the reasoning leading to Remark 2.

[16] Consider possible vectors $\mathcal{N} = \{(n_t)_{t\in\mathcal{T}_{\mathcal{L}}\cup\mathcal{T}_{\mathcal{E}}} : \sum_t n_t = n\}$ describing the number of agents of each type. Then $O(1/n)$ fraction of these vectors violate Assumption 1.

a linear number of agents of each type.

Assumption 2. *There exists $C > 0$ such that for all types $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$, we have $n_t \geq Cn$.*

Under Assumption 2, each type has a comparable number of agents and no type vanishes as the market increases. We now present our main theorem.

Theorem 1. *Consider $K \geq 1$ types of labor, and $Q \geq 1$ types of employers. Let the idiosyncratic productivities be drawn i.i.d. from any fixed distribution F that is supported on an interval of the form $[C_l, C_u]$ or $(-\infty, C_u]$ or $[C_l, \infty)$ or $(-\infty, \infty)$ where $C_l > -\infty$ and $C_u < \infty$, and whose density f is strictly positive and continuous everywhere on the support.*[17,18] *Under Assumption 1 and Assumption 2, for a market with n agents we have that*

- *(Upper Bound) There exists $f(n) = O^*\left(n^{-1/\max(K,Q)}\right)$ such that, with high probability, we have*

$$\max_{\substack{(k,q)\in\mathcal{T}\\ N(k,q)>0}} |\alpha_{kq}^{\max} - \alpha_{kq}^{\min}| \leq f(n).$$

  Also, we have $E[\mathcal{C}] \leq f(n)$.

- *(Lower Bound) There exists a sequence of markets with K types of labor and Q types of employers such that $E[\mathcal{C}] = \Omega\left(n^{-1/\max(K,Q)}\right)$.*

In words, our main result says that under reasonable conditions, with high probability, the variation in the type-pair prices is uniformly bounded by $O^*\left(n^{-1/\max(K,Q)}\right)$, which vanishes as n grows. The same bound holds for $\mathrm{E}[\mathcal{C}]$. In addition, this bound is tight in the worst case. Thus, the core size shrinks to zero as the market grows larger, at a rate that is faster (in the worst case) if there are fewer types of agents. The proof of Theorem 1 is sketched in Section 4.1; the complete proof is in Appendices C and D.

As the type-matching is the same across all core solutions, the bound on the type-pair prices implies that the maximum difference between the utilities of *any* agent in any two core solutions

[17] Thus, allowed distributions include typical continuous distributions such as Gumbel, Gaussian, Pareto, and uniform.

[18] We remark that we in fact only need a weaker condition on f, namely, that it is bounded below by a function that is positive and continuous everywhere in the support, and is non-atomic. Also, F can be different for different type pairs.

vanishes at a rate of $n^{-1/\max(K,Q)}$, as the size of the market increases. Therefore, the fraction of welfare that can flexibly move from workers to firms (and vice versa) in core solutions vanishes. Overall, Theorem 1 shows that the vector of prices that support a competitive equilibrium (a core solution) is approximately unique, provided the market is big enough.[19] As a result, the payoff of an agent is roughly the same in all stable solutions.

The upper bound in Theorem 1 can be improved if further constraints are imposed on the number of types and the imbalance. As an illuminating example, we show that for multiple worker types, a single employer type, and a type with more employers than workers, the size of the core can be bounded above by a function that depends on both the size of the market and the size of the imbalance in the market.

Theorem 2. *Let idiosyncratic productivities be drawn i.i.d. from any fixed distribution F that is supported on an interval of the form $[0, C_u]$ or $[0, \infty)$ where $C_u < \infty$, and whose density f is strictly positive and continuous everywhere on the support.*[20] *In addition, suppose that $u(k,1) \geq 0$ for all $k \in \mathcal{T}_{\mathcal{L}}$. Consider the setting in which $K \geq 2$, $Q = 1$, $n_{\mathcal{E}} > n_{\mathcal{L}}$, and let $m = n_{\mathcal{E}} - n_{\mathcal{L}}$. Under Assumption 2, we have, with high probability, that $\mathcal{C} \leq O^*\left(n^{-\frac{1}{K}} m^{-\frac{K-1}{K}}\right)$. Also, $E[\mathcal{C}] \leq O^*\left(n^{-\frac{1}{K}} m^{-\frac{K-1}{K}}\right)$.*

For $m = O^*(1)$, the bound in Theorem 2 matches that in Theorem 1. However, the bound here becomes tighter as the imbalance m grows. In fact, for $m = \Theta(n)$, the core size is bounded as $O^*(1/n)$. It is noteworthy that, under a linear imbalance, the scaling behavior does not depend on the number of worker types.

A formal proof of Theorem 2 can be found in Appendix E. The idea behind the proof is to use the unmatched agents and condition (IR) in Proposition 2 (for the employers) to control absolute variation in one of the α's, and this control improves as m grows. We separately control the relative variation of the α's in the core using condition (ST) in Proposition 2 under Assumption 2. Combining these we obtain the stated bound on $\mathcal{C}$.

Our results and proofs extend immediately to productivities drawn independently from an

[19] This is similar in spirit to the result obtained by Hassidim and Romm (2015), albeit in a different model where a single marketwide price emerges.

[20] A positive lower limit on the support of F can be absorbed into the $u(\cdot,\cdot)$'s, and hence this case is covered.

arbitrary non-atomic distribution supported on a closed interval, with positive density everywhere in the support, where the density is bounded below by some $\epsilon > 0$.

3.2 Idiosyncratic productivities with unbounded support

For our second main result, we require the terms $\epsilon_j^{\tau(i)}, \eta_i^{\tau(j)}$ to be drawn independently from a distribution having unbounded support, with positive density everywhere (for instance, Gumbel and Gaussian distributions will be covered by our analysis here). Again, the distribution may depend on $\tau(i)$ and $\tau(j)$, but to ease the burden of notation, we assume that all these terms are drawn i.i.d. from a single distribution whose density $f : (-\infty, \infty) \to \mathbb{R}_+$ is positive everywhere and continuous.[21] Let $F : (-\infty, \infty) \to [0, 1]$ be the corresponding cumulative distribution. Recall that $N(k, q)$ is the number of agents of type k matched with agents of type q, under the unique core type-matching M.

In the following theorem, we fix the fraction of agents of each type and scale the number of agents.[22] We show that the core prices and the fraction of matches corresponding to a type-pair, each converge to a unique limit, and we further provide a bound on the rate of convergence. The second part of the theorem leverages these limit characterizations to bound the size of the core.

Theorem 3. *Fix K and Q and the distribution F. Also fix the fraction $\rho_k > 0$ of each agent type k, where k can be a type of worker or a type of employer. Consider a market with n agents that includes $n\rho_k$ agents of each type*[23] *k, with idiosyncratic productivities drawn i.i.d. from the distribution F. We obtain the following bounds as a function of n.*

- **Limit characterization of α.** *There exists $\alpha^* = \alpha^*(K, Q, \rho, F)$ and corresponding $\left(\nu^*_{kq}\right)_{(k,q)\in\mathcal{T}}$ such that as $n \to \infty$, we have that both $\alpha^{\max}$ and $\alpha^{\min}$ converge (in probability and almost surely) to α^*, and $N(k, q)/n \to \nu^*_{kq}$ for all $(k, q) \in \mathcal{T}$. In fact, with high*

[21] In fact, it is enough for f to be bounded below by a function that is positive everywhere and continuous.

[22] This scaling is a bit stronger than the one in Assumption 2, as we require the fraction to remain constant throughout. Note that this is necessary to ensure convergence to a limit point.

[23] In fact, it is sufficient that the fraction of agents of type k converges to ρ_k as $n \to \infty$. If we have $\max_{k\in\mathcal{T}_\mathcal{L}\cup\mathcal{T}_\mathcal{E}} |\rho_k - (\#\text{ agents of type k})/n| \le \delta_n$ for some $\delta_n = o(1)$, then we obtain bounds of $O^*(\max(1/\sqrt{n}, \delta_n))$ in the first part of the theorem, while Eq. (3) remains unchanged.

probability, all core outcomes (M, α) *satisfy*

$$\|\alpha - \alpha^*\| \leq O^*(1/\sqrt{n})\,, \tag{1}$$

$$\left|N(k,q)/n - \nu^*_{kq}\right| \leq O^*(1/\sqrt{n}) \qquad \text{for all } (k,q) \in \mathcal{T}\,. \tag{2}$$

- **Size of the core.** *With high probability, the size of the core is bounded as*

$$\mathcal{C} \leq O^*(1/n)\,. \tag{3}$$

See Appendix F for a statement of Theorem 3 with additional technical details and a proof. The first part of our result (the limit characterization of α) is analogous to the one obtained in Azevedo and Leshno (2016) for NTU markets. Roughly, the limit point α^* is the unique price vector that clears the limit market, in the sense that for each type pair (k, q), the "demand" of worker type k for employer type q is equal to the "supply" of employer type q for worker type k (both demand and supply take a value of ν^*_{kq}). Our analysis takes advantage of the fact that the limit market satisfies a strong gross substitutes condition on both sides (leading to uniqueness of the equilibrium price, and allowing it to be computed via tatonnement), whereas the finite n market also satisfies a weak gross substitutes condition on both sides. We use an order-theoretic approach along with convergence of the empirical distribution of (type, productivity vector) among agents to the limiting distribution, to relate the finite n market to the limit market. This connection leads to the bounds that we obtain.

Theorem 3 has several noteworthy features:

1. The scaling of the bounds with n does not depend on the number of types K and Q. The stronger assumption of F having full support on $(-\infty, \infty)$ yields a stronger upper bound on core size than that in Theorem 1, without the need for Assumption 1 (generalized imbalance).[24]

[24] The generalized imbalance assumption guarantees that at least one agent is unmatched, which is necessary for a small core to arise. Here, as F is unbounded on both sides, some agents will remain unmatched with high probability for a big enough market.

2. The bound on core size (the second part of the theorem) is tight if F has a first moment, as demonstrated by Example 1 (Appendix A). The idea is that $1/n$ is the typical (expected) gap between consecutive order statistics of the idiosyncratic productivities.

3. The first part of the theorem states that the core converges to a limit point (α^*, ν^*), and provides a bound of $O^*(1/\sqrt{n})$ on the rate of convergence. Again, Example 1 (Appendix A) shows that this is tight. The idea is that the actual fraction of agents of a particular type who satisfy some fixed conditions on their idiosyncratic productivities has stochastic variations of order $1/\sqrt{n}$.

   Past works that establish a small core and show convergence to a limit point, e.g., Gretsky et al. (1999), Kamecke (1992), are superficially similar. However, the novel feature of (random) idiosyncratic productivity terms in our model (as well as bounds on the rates of convergence) makes Theorem 3 significantly more powerful than previous results.

4. Theorem 3 provides a bound on the estimation error in empirical studies of matching. Often in empirical studies, only $N(k, q)$'s are observed (no cardinal utilities or transfers are observed), and the goal is to estimate type-type utilities (the $u(k, q)$'s), under suitable assumptions on the idiosyncratic productivities, such as that they follow a standard Gumbel distribution. The estimation is typically performed using a continuum limit assumption, as a result of which there is a tractable one-to-one mapping between observed quantities $(N(k, q))_{(k,q)\in\mathcal{T}}$, $(\rho_k)_{k\in\mathcal{T}_\mathcal{L}\cup\mathcal{T}_\mathcal{E}}$, and the estimated quantities $(u(k, q))_{(k,q)\in\mathcal{T}}$; cf. Choo and Siow (2006, Eq. (11)). Moreover, this mapping is well behaved (e.g., it is Lipschitz continuous in both directions at interior points), and so we can immediately use Eq. (2) to obtain a bound of $O^*(1/\sqrt{n})$ on the error in the estimate of $(u(k, q))_{(k,q)\in\mathcal{T}}$.

4 Overview of the proof of Theorem 1

We now present an overview of our proof of Theorem 1. We first discuss the key steps in establishing the upper bound (the complete proof can be found in Appendix C), and then sketch the proof of the lower bound in Section 4.2 (completed in Appendix D). Throughout this section, we assume that there is a unique maximum weight type-matching as per Observation 1;

i.e., unique $M(t)$ and U. Recall that $N(k,q)$ is defined as the number of matches between agents of type k and agents of type q in M.

We start by constructing a graph associated with matching M as follows. Let $G(M)$ be the bipartite graph whose vertex sets are the types in $\mathcal{L}$ and $\mathcal{E}$, and such that there is an edge between types $k \in \mathcal{T}_{\mathcal{L}}$, $q \in \mathcal{T}_{\mathcal{E}}$ if and only if there is an agent of type k matched to an agent of type q in M, i.e., $N(k,q) > 0$. We say that $G(M)$ is the *type-adjacency graph* associated with matching M. The following lemma states a key fact regarding the structure of $G(M)$.

Lemma 1. *Let $G(M)$ be the associated type-adjacency graph. Suppose we* mark *the vertex in $G(M)$ corresponding to type t if and only if at least one agent of type t is unmatched under M. Then, under Assumption 1 (generalized imbalance), every connected component in $G(M)$ must contain a marked vertex.*

4.1 Overview of the upper bound proof

The intuition for the result is as follows. We start by bounding the type-pair prices associated with types with at least one unmatched agent. By applying the bounds given by the (IR) conditions in Proposition 2, we show that the difference between the maximum and minimum α corresponding to a type-pair price associated with at least one such type is small with high probability. Using these bounds, we move on to bounding the prices associated with types with all agents matched. We use the (ST) conditions in Proposition 2, which bound the relative variation of a type-pair price with respect to other type-pair prices. That such a relative variation is also small with high probability, taken together with the absolute variation on the types with unmatched agents, implies the result.

We first establish the result for the case where F is Uniform$[0,1]$. The two main challenges in the proof of this case are as follows. First, we must express the bounds in Proposition 2 as a function of the market primitives, i.e., market size and number of types. To do so, we re-interpret the stability conditions as geometric conditions over appropriately defined random regions in unit hypercubes, as explained in Section 4.1.1. Second, we need to relate the bounds given by the (IR) constraints to those given by the (ST) constraints, to obtain uniform absolute bounds on the variation of *all* type-pair prices. We address this issue in Section 4.1.2, by exploiting the combinatorial structure of the problem. Finally, we extend the proof to deal with general distributions. We show that under appropriate scaling and translation, the bounds obtained for the uniform case will also hold (within a constant factor) for any general distribution with positive density in an interval. This is shown in Section 4.1.3

We provide a detailed overview of the proof next.

4.1.1 Geometric interpretation of the stability conditions

We now briefly describe the geometric interpretation of the (IR) and (ST) conditions. This will allow us to re-express the constraints in Proposition 2 as bounds that depend on the market primitives.

The intuition is as follows. Once we focus on a single type t, the random productivities of an agent of type t can be described by a $D(t)$-dimensional vector within the $[0,1]^{D(t)}$-hypercube, where $D(t)$ is the dimension of the productivity vector of agents of type t (i.e., $D(t) = K$ if $t \in \mathcal{T}_{\mathcal{E}}$ and $D(t) = Q$ if $t \in \mathcal{T}_{\mathcal{L}}$). Furthermore, the location of these points can be described by a point process in $[0,1]^{D(t)}$. Every such hypercube can be partitioned into (at most) $D(t) + 1$ random regions: one region corresponding to agents matched to each of the $D(t)$ types on the other side, and pne region corresponding to unmatched agents. For every α vector in the core, the regions are separated by hyperplanes as per the (ST) and (IR) conditions. Since there is a unique type-matching with probability one (by Observation 1), each agent (point) must fall into the same region for all core α's. Thus, the stability conditions can be interpreted as geometric conditions in the unitary hypercube, namely, how far we can move the boundaries of each random region without changing the members of each region. Possible fluctuations in the boundaries are bounded by the distance between consecutive order statistics of the projections of points distributed independently in (sub-regions of) the hypercube. We show that the event where all such distances are bounded by functions in $O(f(n))$ occurs with high probability, and thus achieves a bound that depends only on the market primitives. Next, we formally define some of the regions, random sets, and random variables that will be useful in our analysis,[25] and we relate such definitions to the stability constraints as defined in Proposition 2.

[25]Remaining definitions can be found in Appendix B.

Consider a type $t \in \mathcal{T}_{\mathcal{E}}$. For each employer j with $\tau(j) = t$, there is a vector of idiosyncratic productivities ϵ_j distributed uniformly in $[0,1]^K$ and independently across employers. In this subsection we consider these productivities for a given t. We suppress t in the definitions to simplify notation (so n here corresponds to n_t, and so on). Analogous definitions can be made for $t \in \mathcal{T}_{\mathcal{L}}$.

Consider n i.i.d. points $(\epsilon_j)_{j=1}^n$, distributed uniformly in the $[0,1]^K$-hypercube. Here $\epsilon_j = (\epsilon_j^1, \epsilon_j^2, \ldots, \epsilon_j^K)$. Let $\mathcal{T}_{\mathcal{L}} = \{1, 2, \ldots, K\}$ denote the set of dimension indices. Intuitively, for a fixed type $t \in \mathcal{T}_{\mathcal{E}}$ with unmatched agents, one can bound α_{kt} by using condition (IR) in Proposition 2: $\min_{j \in t \cap M(k)} \epsilon_j^k \geq -\alpha_{kt} \geq \max_{j \in t \cap U} \epsilon_j^k$. To apply this bound, we care just about the projection onto the k-th coordinate of the points ϵ_j with $j \in M(k) \cup U$. The main analytical challenge we face is that these relevant subregions (containing sets such as $M(k) \cup U$) are themselves a random function of the market realization, as both $M(k)$ and U are themselves random sets. We overcome this difficulty by appropriately defining the region $\tilde{\mathcal{R}}^k(\delta) := \tilde{\mathcal{R}}^k(t, \delta)$ (throughout the rest of this section, the type t is suppressed in the definition of the associated regions, sets, and random variables[26]). Specifically, for $\delta \in (0, 1/2]$ and $k \in \mathcal{K}$, define

$$\tilde{\mathcal{R}}^k(\delta) = \left\{x \in [0,1]^K : x^{k'} \leq \delta \quad \forall k' \in \mathcal{K}, k' \neq k\right\}. \tag{4}$$

By choosing k and δ appropriately, we can guarantee that $\tilde{\mathcal{R}}^k(\delta)$ only contains points corresponding to agents in $M(k) \cup U$. (Intuitively, we can choose δ so that the other types are not attractive enough with such a low productivity.) Once we have done that, it should be easy to see that $\min_{j \in t \cap M(k)} \epsilon_j^k - \max_{j \in t \cap U} \epsilon_j^k$ is upper-bounded by the maximum distance between two consecutive points in $\tilde{\mathcal{R}}^k(\delta)$, when projected onto their k-th coordinate (the corner cases of all points being in $M(k)$, or in U, turn out to be easy to handle). This becomes precise once we

[26] While for our purposes we consider the hypercube associated to a given type t, the results in this section are intended more generally as results for random point processes in a hypercube. Thus, types are omitted from the definitions.

introduce the set $\tilde{\mathcal{V}}^k(\delta)$ and the random variable $\tilde{V}^k(\delta)$, defined as follows:

$$\tilde{\mathcal{V}}^k(\delta) = \{x : x = \epsilon_j^k \text{ for } \{j : \epsilon_j \in \tilde{\mathcal{R}}^k(\delta)\}\} \tag{5}$$

$$\text{and} \quad \tilde{V}^k(\delta) = \max\left(\text{Difference between consecutive values in } \tilde{\mathcal{V}}^k(\delta) \cup \{0,1\}\right). \tag{6}$$

Thus, $\tilde{\mathcal{V}}^k(\delta) \subset [0,1]$ is the set of values of the k-th coordinate of the points lying in $\mathcal{R}^k(\delta)$, and $\tilde{V}^k(\delta) \in \mathbb{R}$ is the maximum difference between consecutive values in $\tilde{\mathcal{V}}^k(\delta) \cup \{0,1\}$. (As an example, if $\tilde{\mathcal{V}}^k(\delta) = \{0.3, 0.4, 0.8\}$, the differences between consecutive values in $\tilde{\mathcal{V}}^k(\delta) \cup \{0,1\}$ are $0.3, 0.1, 0.4, 0.2$, resulting in $\tilde{V}^k(\delta) = 0.4$. Note that $\tilde{\mathcal{V}}^k(\delta)$ is a random and finite set, and $\tilde{V}^k(\delta)$ is a random variable.) Therefore, the variation in α_{kt} is bounded by $\tilde{V}^k(\delta)$. One of our intermediate results is to show that with high probability $\{\max_k \tilde{V}^k(\delta) \leq f_1(n,K)\}$, for some $f_1(n,K) = O^*(1/n^{1/K})$.

To complete the proof overview, note that once the absolute variation of, say, $\alpha_{tt'}$ has been bounded, a bound on the absolute variation of $\alpha_{tt''}$ can be obtained by bounding the relative variation in prices (i.e., the variation in $\alpha_{tt'} - \alpha_{tt''}$) using the (ST) conditions; cf. Section 4.1.2. Therefore, we need to define additional regions, sets, and variables that will allow us to apply the (ST) conditions. Such regions will be analogous to those defined above. However, instead of being a function of a single parameter k, they will be defined as functions of two parameters k_1, k_2; this difference is due to the fact that, unlike the (IR) constraints that focus on a single type-pair, the (ST) constraints bound the variation between two type-pairs. Since the relationship between these random regions and the α's is more involved, their definition and analysis is relegated to Appendix B. However, analogously to the case explained above, we still care about the difference between (appropriately defined) consecutive order statistics in these regions.

4.1.2 Using the combinatorial structure of the problem to relate the bounds given by (IR) and (ST) constraints

We consider some suitably defined "typical" events as discussed in Section 4.1.1, which occur with high probability in the markets being considered. When these events occur, the possible fluctuations in the boundaries of the random regions (see Section 4.1.1) are bounded by functions in $O(f(n))$, where $f(n) = O^*\left(\frac{1}{n^{1/\max(K,Q)}}\right)$, and $f(n)$ agrees with that in the statement of Theorem 1. Under these events, we show that the variation in the type-pair prices is uniformly bounded as follows:

$$\max_{(k,q)\in\mathcal{T}_{\mathcal{L}}\times\mathcal{T}_{\mathcal{E}},N(k,q)>0} |\alpha_{kq}^{\max} - \alpha_{kq}^{\min}| \leq f(n).$$

This bound on the type-pair prices, together with the fact that the defined events occur with a high probability, imply our result.

We divide the proof of the bound on the type-pair prices into two steps. First, note that types with at least one unmatched agent are "easier" to bound; we use the (IR) conditions in Proposition 2 to bound the absolute variation of prices α associated with such types. In particular, for each type t with at least one matched and one unmatched agent, we use the (IR) conditions to show that $\max_{t':\ N(t,t')>0}\left(\alpha_{t,t'}^{\max} - \alpha_{t,t'}^{\min}\right) \leq O^*\left(\frac{1}{n^{1/\max(K,Q)}}\right)$. This is done in Lemma C.2.

It remains to show that the bound holds for prices between types such that all agents are matched. Unfortunately, as the (IR) conditions in Proposition 2 cannot be applied to such types, proving the result is not straightforward. Only the (ST) conditions can be used, which provide only relative bounds on the values of the associated type-prices.

To prove the bounds on the variations of the prices associated with fully matched types, we use the graph $G(M)$ as defined above. Given a type $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$, let the *distance* $d(t)$ be defined as the minimum distance in $G(M)$ from t to *any* marked vertex. By Lemma 1, every unmarked vertex t must be at a finite distance from a marked one. Furthermore, $\max_{t\in\mathcal{T}_{\mathcal{L}}\cup\mathcal{T}_{\mathcal{E}}} d(t) \leq K+Q$, regardless of the realization of the graph. Our argument to control the variation in the α's is by induction on $d(t)$ starting with $d(t) = 0$. The induction base has already been established, as we showed that the variation in *all* the relevant α's associated with marked types (these types have distance zero) is bounded. Note that these types are those with at least one matched and one unmatched agent.

In the inductive step, we assume that the bound holds for every α associated with a type whose distance is d or less; i.e, for every $(t,t') \in \mathcal{T}_{\mathcal{L}} \times \mathcal{T}_{\mathcal{E}}$ such that $\min(d(t), d(t')) \leq d$, we have

$\alpha^{\max}_{t,t'} - \alpha^{\min}_{t,t'} \leq O^*\left(\frac{1}{n^{1/\max(K,Q)}}\right)$. Then, we use the inductive hypothesis to show that the result must also hold for all types whose distance is $d+1$. By the definition of distance, for every type t such that $d(t) = d+1$, there must exist a type t^* such that $d(t^*) = d$ and $N(t,t^*) > 0$. Therefore, by our inductive hypothesis, we must have $\alpha^{\max}_{t,t^*} - \alpha^{\min}_{t,t^*} \leq O^*\left(\frac{1}{n^{1/\max(K,Q)}}\right)$. Separately, we control the relative variation of the α's associated with type t in the core using the (ST) conditions; i.e., we show that $\alpha_{t,t_1} - \alpha_{t,t_2}$ for types t_1, t_2 with matches to type t can vary only within a range bounded by $O^*\left(\frac{1}{n^{1/\max(K,Q)}}\right)$. Combining, we deduce the desired bound on the variation in all α's associated with type t. This is formally achieved in Lemma C.5.

4.1.3 Extension to general distributions

So far we have given an overview of the proof for the case in which F is Uniform$(0,1)$. We now argue that the same bounds can be obtained for any distribution satisfying the conditions in Theorem 1, via appropriate scalings and translations of productivities and base utilities.

We first show that in any core solution, each type-pair price must lie in a bounded interval whose extremes are independent of n.

Lemma 2. *Fix K, Q, u's, and F. Then under Assumption 2, there exists $U = U(K,Q,(u_{kq})_{k,q},F) < \infty$ such that, with high probability, any core outcome (M,α) satisfies*

$$-U \leq \alpha_{kq} \leq U \qquad \text{for all } k,q\,. \tag{7}$$

Lemma 2 is proved in Appendix G.

Recall that in our proof of Theorem 1 for the Uniform$(0,1)$ case, we were able to bound the variation in type-pair prices by bounding the distances between relevant consecutive order statistics of the productivities and their linear combinations. In particular, we followed an inductive approach where we first focused on agent types with an unmatched agent and bounded all the prices corresponding to that type, and then we proceeded inductively to bound the prices corresponding to types where all agents are matched. Note that when F is an unbounded distribution, the distance between consecutive order statistics might be larger than any

$O^*(n^{1/\max(K,Q)})$ function. However, using Lemma 2, it follows that the relevant interval where realized values matter is

$$I = \begin{cases} [\max(C_l, -U), \min(C_u, U)] & \text{for } \epsilon_i^k\text{'s} \\ \left[\max(C_l, -U + \min_{k,q} u(k,q)), \min(C_u, U - \max_{k,q} u(k,q))\right] & \text{for } \eta_j^q\text{'s.} \end{cases} \tag{8}$$

In other words, if we fix a type $t \in \mathcal{T_E}$, then we care about the consecutive order statistics of the ϵ's that occur in the I^K-hypercube, and analogously for $t \in \mathcal{T_L}$. Now $f_{\min} = \inf_{x \in I} f(x) > 0$; by assumption, f is positive and continuous, and I is a compact set. If we suitably translate the productivity (achieved via an equal and opposite change in u) and then scale utilities down by $|I|$, we can map I to $[0, 1]$, and the density in $[0, 1]$ is now lower-bounded by $f_{\min}|I|$. Thus, if the productivities of type $t \in \mathcal{T_E}$ are being considered, we apply this translation to all $k \in \mathcal{T_L}$, and then scale all utilities down by $|I|$. As a result, we obtain that the density of the productivity vector $(\epsilon_i^k)_{k \in \mathcal{T_L}}$ for $\tau(i) = t$ is lower-bounded uniformly by $(f_{\min}|I|)^K$ everywhere in $[0, 1]^K$, and that this is now the relevant region where realized values matter. Hence, our uniform bounds on the gaps between consecutive order statistics for the case where F is Uniform$(0, 1)$ suffice.[27] The corresponding bound on order statistics for the original problem is simply $|I|$ times (a constant factor) larger.

4.2 Proof of the lower bound

Our lower bound follows from the following proposition, proved in Appendix D.

Proposition 3. *Consider a sequence of markets (indexed by $\tilde{n}$) with $|\mathcal{T_L}| = K$ types of labor, with $\tilde{n}$ workers of each type, and with a single type $1'$ of employer, and $(K-1)\tilde{n}+1$ employers of this type. (Note that these markets satisfy Assumptions 1 and 2.) Set $u(k_*, 1') = 0$ for some $k_* \in \mathcal{L}$, and $u(k, 1') = 3$ for all $k \in \mathcal{L} \backslash k_*$. For this market, we have $E[\mathcal{C}] = \Omega^*(1/(n^{1/K}))$.*

The rough intuition for our construction in Proposition 3 is as follows. For our choice of u's it is not hard to see that all workers of types different from k_* are always matched in the core. One

[27] Formally, we divide realized productivity vectors into those that are considered and those that are not. The density of considered realizations is $(f_{\min}|I|)^K$ in $[0, 1]^K$ and 0 elsewhere; overall, a fraction $(f_{\min}|I|)^K > 0$ of realizations are considered and hence, w.h.p., $\Omega(n)$ workers of type k are considered.

employer j_* is matched to a worker of type k_*. Suppose vector $(\alpha_k)_{k\in\mathcal{T}_\mathcal{L}}$ is in the core. Given that all types $k \neq k_*$ are a priori symmetric, we would expect that the α_k's for $k \neq k_*$ are close to each other (we formalize using Lemma E.4 that they are typically no more than $\delta \sim 1/\sqrt{\tilde{n}}$ apart). Assuming this is the case, we can order employers based on $X_j = \max_{k\neq k_*} \epsilon_j^k - \epsilon_j^{k_*}$, and j_* should usually be the employer with the smallest X_j, since this employer has the largest productivity with respect to k_*, relative to the other types. Now, the X_j's are i.i.d., and a short calculation establishes that the distance between the first- and second-order statistics of $(X_j)_{j\in\mathcal{E}}$ is $\Theta(1/n^{1/K})$. This "large" gap between the first two order statistics allows $(\alpha_{k_*}, (\alpha_k + \theta)_{k\neq k_*})$ to remain within the core for a range of values of $\theta \in \mathbb{R}$ that has an expected length of $\Theta(1/n^{1/2})$ for $K = 2$ and $\Theta(1/n^{1/K}) - O(\delta) = \Theta(1/n^{1/K})$ for $K > 2$, which leads to the stated lower bound on $\mathcal{C}$.

We remark that the key quantity here, namely, the gap between the first two order statistics of $(X_j)_{j\in\mathcal{E}}$, is determined by the tail behavior of the distribution (both the left and right tails) of the ϵ's, along with the number of types K. See Section 6 for further discussion.

Note that the construction above can easily be adapted to accommodate $Q \leq K$ types of firms.[28] If $Q > K$, we simply swap the roles of workers and firms in our construction, leading to $\mathrm{E}[\mathcal{C}] = \Omega^*(1/(n^{1/Q}))$, as needed. Thus, the lower bound in Theorem 1 follows from Proposition 3.

5 Numerical Experiments

In the previous sections, we provided asymptotic bounds on the size of the core. The purpose of this section is twofold. First, in Section 5.1, we provide simulation results to illustrate our theoretical results, and study the core behavior in markets with a small number of agents. Second, in Section 5.2, we explore what happens when the number of types is allowed to grow. Our simulation results reveal that the core is indeed small across a wide range of settings, including relatively small markets, reinforcing our main conclusion, namely, that typical assignment markets are likely to have a small core.

[28]Let each worker type $q \neq 1$ have $\tilde{n}$ agents each and $u(\cdot, q) = -2$. These workers are always unmatched, leaving the core unaffected.

In our numerical experiments, we find that the core size typically appears to decay at a rate between $\Theta(1/\sqrt{n})$ and $\Theta(1/n)$ as the number of agents increases (with at least two types on each side); in particular, the decay rate is observed to be roughly independent of the number of types K and faster than the worst-case rate captured by Theorem 1. Note that a core size of at least $1/n$ is expected simply from the fact that the distance between an arbitrary pair of consecutive order statistics for n random variables drawn i.i.d. from some distribution is $\Omega(1/n)$ in expectation, combined with Proposition 2. We find also that, in all the cases we studied, the core remains small with an increasing number of types, except when idiosyncratic productivities follow a Pareto distribution, in which case when there is a very large number of types the core size may no longer be small; see Figure 3.

5.1 Simulation of theoretical results

Our main results state that, if the number of types is fixed and bounded by K, then the core becomes small as the market size grows. In particular, the core size in a market with n agents and at most K types on each side is bounded as $O^*(1/n^{1/K})$ (Theorem 1). In addition, if the distribution of the idiosyncratic terms is supported on the interval $(-\infty, +\infty)$, the size of the core is bounded as $1/n$ and the core solutions converge to a limit point (Theorem 3). We now numerically investigate finite markets, including relatively small markets.

5.1.1 Illustration of Theorem 1

In Theorem 1 we show that our bound is tight in the worst case by carefully constructing a specific family of instances whose expected core size is $\Omega^*(1/n^{1/K})$. However, such markets are unlikely to occur in practice. We are now interested in understanding the size of the core in more "typical" settings. To that end, we define a *typical market* with a distribution over the number of agents of each type as follows.

Definition 2 (Typical market)**.** *A typical market is a market of random size defined by the quadruple* $(n_{\mathcal{L}}, n_{\mathcal{E}}, [p_1, \dots, p_K], [r_1, \dots, r_Q])$*, where* $[p_1, \dots, p_K]$ *(resp.* $[r_1, \dots, r_Q]$*) is a list of positive reals of length* K *(resp.* Q*) such that* $\sum_{i=1}^{K} p_i = 1$ *and* $\sum_{i=1}^{Q} r_i = 1$*, and such that the* *number of agents of type $k \in K$ (resp. $q \in Q$) is given by an independent random draw from a Poisson distribution with mean $p_k n_{\mathcal{L}}$ (resp. $r_q n_{\mathcal{E}}$).*

We highlight that the above definition of a typical market is valid regardless of the distribution used to determine the idiosyncratic productivities. If $n_{\mathcal{L}} = n_{\mathcal{E}}$ we say that the market is *balanced*. Otherwise, we say that the market is *unbalanced*. Note that even a balanced typical market is very unlikely to have exactly the same number of agents on each side; the likelihood of exact balance is $O(1/\sqrt{n})$ and such realizations have a negligible impact on our numerical results since we are focusing on the *median* core size across realizations.

We tested the intuition provided by Theorem 1 by simulating typical markets with a varying number of agents, number of types, and type-type utilities. We consider three types of distributions for the idiosyncratic productivities: uniform, exponential (right-unbounded with a lighter tail), and Pareto (right-unbounded with a heavier tail).[29] For each set of parameters, we ran 100 trials and reported the median core size over those markets.

To illustrate our findings, in Figure 1 we show the median core size as a function of the number of agents for balanced markets with a fixed number of types (2 and 5 types on each side), the same expected number of agents of each type, and all type-type utilities equal to zero. The idiosyncratic terms were drawn independently from, respectively, a uniform(0,1) distribution, an exponential distribution with mean 1, and a Pareto distribution with shape parameter $\alpha = 2$ (the choice of scale parameter has no impact on core size). We also added the functions $1/n$ and $1/\sqrt{n}$ to the graph, and used logarithmic scales on both axes to facilitate the comparison with polynomials of n. In the figure, one observes that the dependence of decay rate on the number of types is weak, and the decay rate appears to be between $1/\sqrt{n}$ and $1/n$.

We also simulated the effect of imbalance and changing type-type utilities.[30] In our experiments, the core size tends to slightly decrease as the imbalance increases (see Theorem 2). However, the difference in the core size between balanced and unbalanced typical markets was not large and the core size was found to always decay at a rate of between $1/\sqrt{n}$ and $1/n$. Further, consistent with our theoretical insights, we find that the values of the type-type utilities

[29] The Pareto distribution has two parameters: the shape parameter α and the minimum point in the support $x_{\min}$. Given those parameters, the CDF at $x \geq x_{\min}$ is given by $1 - \left(\frac{x_{\min}}{x}\right)^{\alpha}$.

[30] These results are not explicitly reported for the sake of brevity.

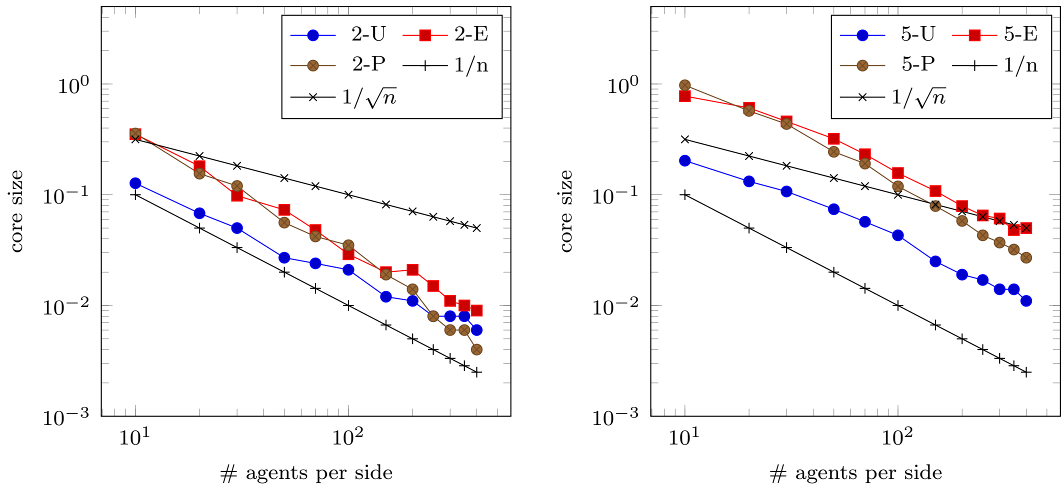

Figure 1: (*Left*) Median core size as a function of number of agents, for balanced markets with two types of agents on each side and both the type-type productivities and idiosyncratic terms drawn from uniform (U), exponential (E), and Pareto (P) distributions, respectively. The graph uses logarithmic scales on both axes. The functions $1/n$ and $1/\sqrt{n}$ are added for comparison. (*Right*) Core size as a function of number of agents, for balanced markets with five types of agents on each side.

do not play a significant role in the size of the core.

5.1.2 Illustration of Theorem 3

We now numerically illustrate our result in Theorem 3, using a standard Gumbel as the distribution for the idiosyncratic productivities. This case is particularly important: most empirical studies using this model assume a fixed number of types and idiosyncratic terms drawn from a Gumbel distribution (see, e.g., Choo and Siow (2006)). For these experiments, we fixed the fraction ρ_t of agents of each type $t \in \mathcal{T_L} \cup \mathcal{T_E}$ and the type-type utilities, and increased the number of agents on each side of the market. For each set of parameters, we ran 100 trials and reported the average distance to the limiting type-type prices α^*, the average distance to the limiting (scaled) number of matches of each type-pair ν^*, and the median core size. As expected, the observed decrease in the core size occurs at a rate of approximately $1/n$, even when n is relatively small. We also found that the average distances from the realized α's and ν's to α^* and ν^*, respectively, decrease at a rate of $1/\sqrt{n}$, as predicted. Also, as predicted by Theorem 3, we found that the values of the type-type utilities do not alter the rate of convergence, and this holds even for small markets (results for different type-type utilities are not reported).

We consider markets with two and five types on each side. Agents on each side are evenly distributed across types, but there is an imbalance across sides (we considered $1.2n$ workers and n firms). We also set the type-type utilities to be equal to 3.0, for each type pair. In this setting, all agents on each side are ex-ante symmetric and thus, in the limit, the proportion of agents of type k that are matched to agents of type q, ν^*_{kq}, is the same for all type-pairs. Similarly, the limiting prices α^*_{kq} are equal across all type-pairs. In Figure 2 we illustrate this, by plotting the median of $\text{Average}_{k,q}(|\alpha_{kq} - \alpha^*_{kq}|)$, the median of $\text{Average}_{k,q}(|N(k,q)/n - \nu^*_{kq}|)$, and the median size of the core as n increases. We observe that our asymptotic bounds capture the behavior of the core even in small markets.

5.2 Beyond our theory: Increasing the number of types

Our theoretical results assume that the maximum number of types (K) on each side remains fixed as the number of agents (n) increases. To conclude this section, we now numerically analyze what happens if we allow the number of types to increase.

In particular, we fix the number of agents n, and study the behavior of the core when the number of types K increases. We again show results for the case where the type-type utilities are drawn independently from the same distribution as the idiosyncratic terms. (We found that the core size is not sensitive to the distribution of the type-type utility term.) Clearly, the size of the core (see Definition 1) is bounded by 1. Therefore, it should not be surprising that the core size as a function of the number of types eventually flattens out. The question is, how much can the core grow if we allow the number of types to grow, and how many types of agents are needed before the core stops growing?

If the random productivities are drawn from a uniform distribution, then allowing the number of types to increase as n increases does not have a significant effect and the core remains small. This is illustrated by Figure 3, where we consider balanced typical markets with 200 and 400 agents on each side, respectively, and plot the median core size as a function of the number of types for uniform, exponential, and Pareto distributions. Observe that under both uniform and exponential distributions, the curve flattens out when K is approximately 30 in each case. On the other hand, the effect of increasing the number of types for Pareto distributions is significant,

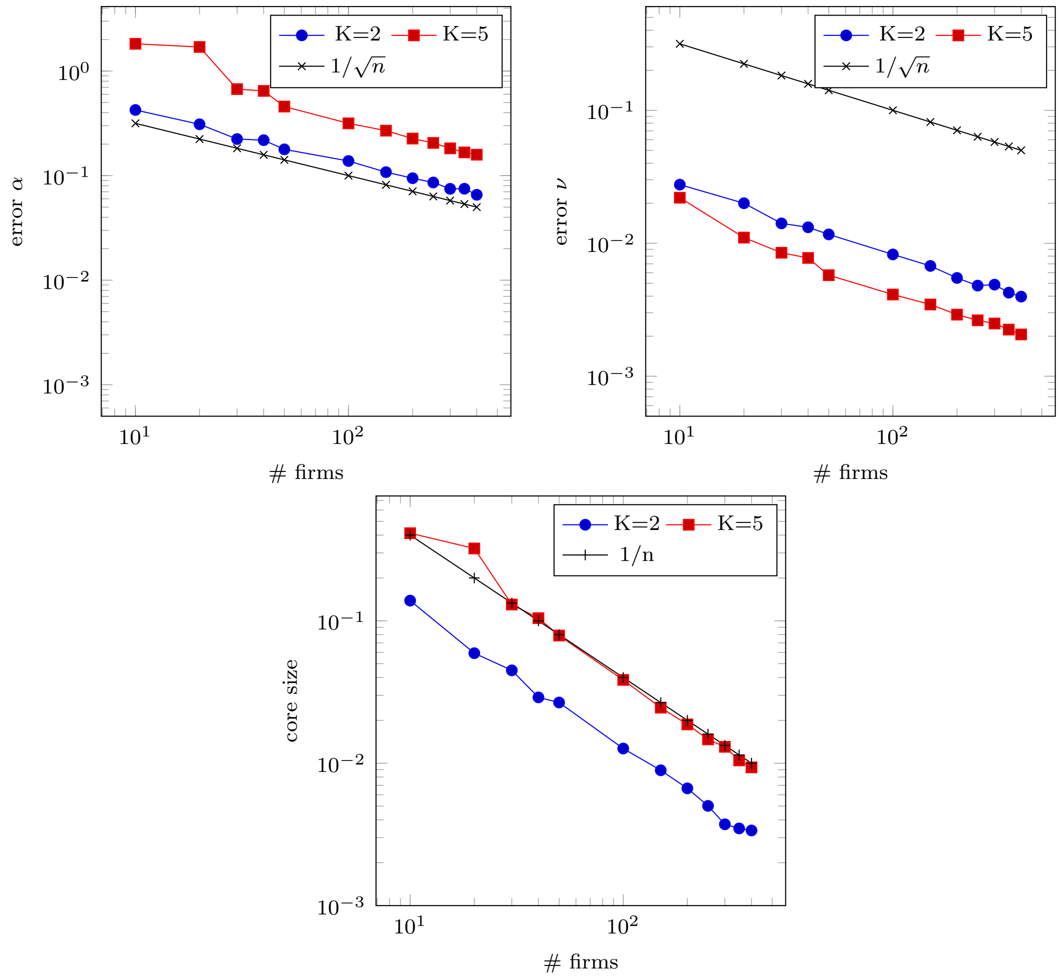

Figure 2: (*Left top*) Median distance to α^* as a function of the number of workers for markets with two and five types of agents on each side, idiosyncratic terms drawn from standard Gumbel distribution, type-pair utilities set to zero, and 1.2 times as many workers as firms. For each simulation, we compute the distance to α^* as $\text{Average}_{(k,q)}|\alpha_{k,q} - \alpha^*|$. When $K = 2$, we have $\alpha^* = 0.132$ and, when $K = 5$, $\alpha^* = 1.026$. (*Right top*) Median distance to ν^*, where for each simulation, we compute the distance to ν^* as $\text{Average}_{(k,q)}|N(k,q)/n - \nu^*|$. When $K = 2$, we have $\nu^* = 0.106$ and, when $K = 5$, we have $\nu^* = 0.018$. (*Bottom*) Median core size as a function of the number of workers. All graphs use logarithmic scales on both axes. The functions $1/\sqrt{n}$ and $1/n$ are added respectively for comparison.

and a "large" core may arise as a result (with core size exceeding 0.1), albeit only when there is a large number of types (e.g., more than ~ 30 types with 200 agents on each side). As seen in Figure 3, the size of the core does not flatten out even with 100 types per side in this case.[31] However, such a large number of types would not be expected to occur in practice, and we

[31] Our intuition for the underlying reason for a large core is as follows: when the number of types exceeds $\sqrt{n}$, then most of the matched pairs consist of agents who are especially well suited to each other in terms of their productivities for each other's type. This leaves significant leeway for each pair of agents to negotiate the division of the surplus they generate, even while keeping other agents' payoffs fixed and retaining stability.

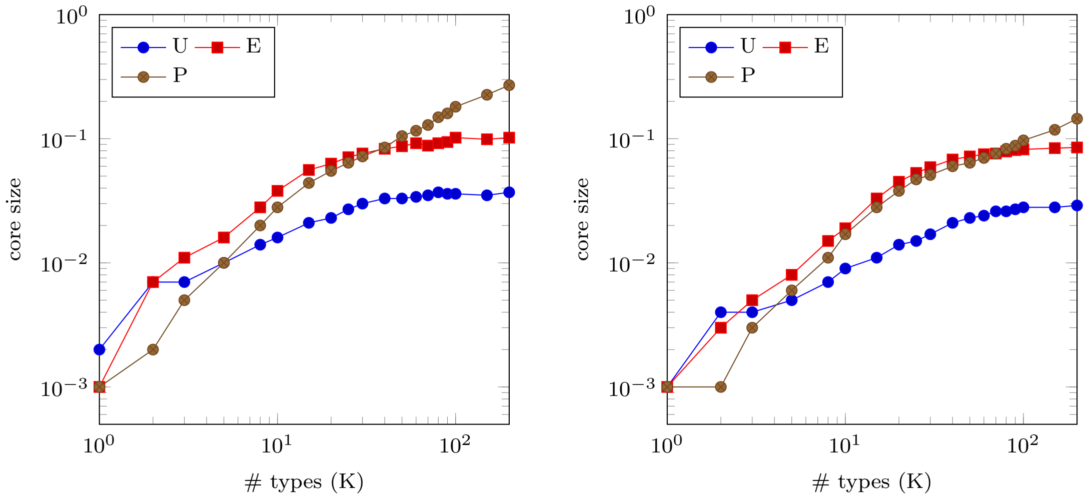

Figure 3: (*Left*) Median core size as a function of number of types, for balanced typical markets with 200 agents on each side and idiosyncratic terms drawn from a uniform distribution (U), an exponential distribution (E) and a Pareto distribution (P) with shape parameter $\alpha = 2$. The graph uses logarithmic scales on both axes. (*Right*) Median core size as a function of number of types, for typical markets with 400 agents on each side.

consider our findings here as a confirmation that the core is indeed small under plausible values for n, K, and Q.

6 Discussion

This paper quantifies the size of the core in matching markets with transfers, as a function of market characteristics. We considered a model of an assignment market with a fixed number of types of workers and firms. We modeled the value of a match between a pair of agents as a sum of a deterministic term determined by the pair of types, and a random component that is the sum of two terms, each depending on the identity of one of the agents and the type of the other. Assuming a fixed number of types, we showed that the size of the core is bounded as $O^*(1/n^{1/\ell})$, where each side of the market contains no more than ℓ types, when the random terms are drawn from a distribution whose density is strictly positive and continuous everywhere in the (possibly unbounded) support. This bound holds in the worst case over the number of agents. These results imply that the vector of prices that support a competitive equilibrium (a core solution) is approximately uniquely determined, and thus the payoff of an agent is roughly the same in all stable solutions.

We provide additional results for the practically relevant case of distributions with support $(-\infty, \infty)$ (including Gumbel and normal): we show a tighter bound of $O^*(1/n)$ on the size of the core, as well as convergence to a unique limit of both the core prices and the fraction of matches of each type-pair, and bounds of $O^*(1/\sqrt{n})$ on these rates of convergence.

Using numerical experiments, we show that our small core finding holds across a range of practically relevant situations.

It would also be interesting to extend our results to many-to-one markets, where employers can each have more than one opening. We expect that our results regarding the core (also our proofs) extend to the case where each employer's capacity is bounded by a constant, and each employer's utility is additive across matches.

Another interesting direction is to investigate the role of a marketplace operator (e.g., AirBnB, Upwork) in determining the wage/price levels in a decentralized market. Our work suggests that price recommendations alone may not be a good tool to control prices, since prices are uniquely determined in equilibrium. However, there are nontrivial search costs in many such marketplaces. Hence, search engine design and other aspects of the search/recommendation environment provided by a platform can play a role in modulating prices.

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A Examples supporting Section 3.2

We present an example showing that the bounds in Theorem 3 are tight.

Example 1. *Suppose there is one type of worker k and one type of employer q, i.e., $K = Q = 1$ and $u(k,q) = 0$. Fix arbitrary F that has a first moment and with density f being positive and continuous everywhere. Suppose $\rho_k = 1/2$, implying $\rho_q = 1 - \rho_k = 1/2$. Then $\alpha_* = 0$ by symmetry, and the limiting core outcome is one in which workers with $\eta_i > 0$ match (worker i*

earns utility η_i), with employers with $\epsilon_j > 0$ (employer j earns utility ϵ_j). The expected number of workers with $\eta_i > 0$ is $(1-F(0))n/2$ and this is also the expected number of employers with $\epsilon_j > 0$. However, in the n-th market, the number of workers with $\eta_i > 0$ deviates from its expectation by order $\sqrt{n}$, and similarly for the number of firms with $\epsilon_j > 0$, and further, the difference between these two numbers is $\Delta \sim \sqrt{n}$. Core α's are those which lead to exactly the same number of workers with $\eta_i - \alpha > 0$ as the number of employers with $\epsilon_i + \alpha > 0$. Consequently, core α's differ from $\alpha^* = 0$ by about $\Delta/(nf(0)) \sim 1/\sqrt{n}$ (in particular, $E[|\alpha|] = 1/\sqrt{n}$ for any core α), and the number of matched pairs scaled by n deviates by order $1/\sqrt{n}$ from its limiting value of $1/4$, i.e., $E[|N(1,1)/n - 1/4|] = \Theta(1/\sqrt{n})$. The (expected) size of the core is[32] of order $1/n$ since all core α's lie between the same pair of order statistics of η_j's and $-\epsilon_i$'s. (We use that F has a first moment to obtain that $\mathrm{weight}(M) = \Theta(n)$ *whp.)*

B Additional definitions and results on point processes in the unit hypercube

B.1 Additional definitions

Analogously to the definitions of the regions in Section 4.1.1, we define the regions $\mathcal{R}^k(t)$ (for appropriate k) and $\mathcal{R}^{k_1,k_2}(t,\delta)$ (for appropriate k_1, k_2) which allow us to apply the conditions (IR) and (ST) respectively. For consistency, the type t is suppressed in the definition of the associated regions, sets and random variables (e.g. $\mathcal{R}^k := \mathcal{R}^k(t)$). Let

$$\mathcal{R}^k = \{x \in [0,1]^K : x^k \geq x^{k'} \quad \forall k' \neq k, k' \in \mathcal{K}\} \tag{9}$$

For $k_1, k_2 \in \mathcal{K}, k_1 \neq k_2$ and for $\delta \in [0, 1/2]$, define the region

$$\mathcal{R}^{k_1,k_2}(\delta) = \{x \in [0,1]^K : x^{k_1} \geq x^k \quad \forall k \notin \{k_1, k_2\}, k \in \mathcal{K}, x^{k_1} \geq \delta\}. \tag{10}$$

The relationship between these random regions and the α's is more involved, so the expla-

[32]Showing this rigorously requires a small (straightforward) argument to get around the dependence between the realization of productivities, and the relevant order statistics.

nation is delayed to the proofs. However, and analogously to the case explained about, we still care about the difference between (appropriately defined) consecutive order statistics in these regions. To that end, we define:

$$\mathcal{V}^k = \{x : x = \epsilon_j^k \text{ for } \{j : \epsilon_j \in \mathcal{R}^k\}\}\,, \tag{11}$$

$$\text{and} \qquad V^k = \max\big(\text{Difference between consecutive values in } \mathcal{V}^k \cup \{0,1\}\big)\,; \tag{12}$$

as well as,

$$\mathcal{V}^{k_1,k_2}(\delta) = \{x : x = \epsilon_j^{k_1} - \epsilon_j^{k_2} \text{ for } \{j : \epsilon_j \in \mathcal{R}^{k_1,k_2}(\delta)\}\}\,, \tag{13}$$

$$\text{and} \qquad V^{k_1,k_2}(\delta) = \max\big(\text{Difference between consecutive values in } \mathcal{V}^{k_1,k_2}(\delta) \cup \{-1+\delta,1\}\big)\,. \tag{14}$$

Thus, $\mathcal{V}^k \subset [0,1]$ is the set of values of the k-th coordinate of the points lying in $\mathcal{R}^k$, and $V^k \in \mathbb{R}$ is the maximum difference between consecutive values in $\mathcal{V}^k \cup \{0,1\}$. Analogously, $\mathcal{V}^{k_1,k_2} \subset [-1+\delta, 1]$ is the set of values of the difference between the k_1-th and k_2-th coordinate of points lying in $\mathcal{R}^{k_1,k_2}$, and $V^{k_1,k_2} \in \mathbb{R}$ is the maximum difference between consecutive values in $\mathcal{V}^{k_1,k_2} \cup \{-1+\delta, 1\}$.

Using the above notation, we now define two events that will help us prove the results by providing us with a bound on the maximum difference between consecutive values in the relevant regions. Specifically, the events are defined as follows:

$$\mathcal{B}_1(t,\delta) = \Big\{ \max\Big(\max_{k\in\mathcal{K}} V^k(t), \max_{(k_1,k_2)\in\mathcal{K}^{(2)}} V^{k_1,k_2}(t,\delta)\Big) \le f_1(n_t,K) \Big\}, \tag{15}$$

for some $f_1(n_t,K) = O^*(1/n_t^{1/K})$ defined in Lemma B.1, $\delta \in [0,1/2]$ and where $\mathcal{K}^{(2)} = \{(k_1,k_2) : k_1,k_2 \in \mathcal{K}, k_1 \neq k_2\}$. (If $K = 1$, then $\mathcal{K}^{(2)}$ is the empty set $\emptyset$ in which case we follow the convention that $\max_\emptyset[\,\cdot\,] = -\infty$.). In addition,

$$\mathcal{B}_2(t,\delta) = \Big\{ \max_{k\in\mathcal{T}_\mathcal{L}} \tilde{V}^k(t,\delta) \le f_2(n_t)/\delta^{K-1} \Big\} \tag{16}$$

for some $f_2(n_t) = O^*(1/n_t)$ defined in Lemma B.2, $\delta \in (0,1]$ and $\tilde{V}^k(t,\delta)$ as defined in Eq. (6).

The proof of all lemmas auxiliary to the proof of Theorem 1 assume that these events (or some subset of them) occur. As shown by the next result, that assumption does not pose a problem as these events simultaneously occur with high probability.

Theorem 4. *There exists $\hat{C} = \hat{C}(K,Q) < \infty$ such that, for any $\delta = \delta(n) \in (0,1/2]$, the event $\bigcap_{t\in\mathcal{T}_{\mathcal{L}}\cup\mathcal{T}_{\mathcal{E}}} \big(\mathcal{B}_1(t,\delta) \cap \mathcal{B}_2(t,\delta)\big)$ occurs with probability at least $1-\hat{C}/n$.*

B.2 Results

Consider the K dimensional unit hypercube $[0,1]^K$, and the Poisson process of uniform rate n in this hypercube, leading to N points $(\epsilon_i)_{i=1}^N$. (Note that $\mathrm{E}[N] = n$.) Here $\epsilon_i = (\epsilon_i^1, \epsilon_i^2, \ldots, \epsilon_i^K)$. Let $\mathcal{K} = \{1,2,\ldots,K\}$ denote the set of dimension indices.

Let $\mathcal{R}^k$ be the region defined by Eq. (9), and let $\mathcal{V}^k$ and V^k be as defined by Eqs. (11) and (12) respectively. Similarly, let $\mathcal{R}^{k_1,k_2}(\delta)$ be the region defined by Eq. (10), and let $\mathcal{V}^{k_1,k_2}(\delta)$ and $V^{k_1,k_2}(\delta)$ be as defined by Eqs. (13) and (14) respectively.

The following lemma, key to our proof of Theorem 1, says that with high probability, all the (V^k)'s and the (V^{k_1,k_2})'s are no larger than a (deterministic) function[33] of n that scales as $O^*(1/n^{1/K})$.

Lemma B.1. *Let $\mathcal{R}^k$ be the region defined by Eq. (9), and let $\mathcal{V}^k$ and V^k be as defined by Eqs. (11) and (12) respectively. Similarly, let $\mathcal{R}^{k_1,k_2}(\delta)$ be the region defined by Eq. (10), and let $\mathcal{V}^{k_1,k_2}(\delta)$ and $V^{k_1,k_2}(\delta)$ be as defined by Eqs. (13) and (14) respectively. Fix $K \geq 1$. Then there exists $f(n,K) = O^*(1/n^{1/K})$ such that for any $\delta = \delta(n) \in [0,1/2]$ the following holds: Let*

$$\mathcal{B}_1 = \left\{ \max\left(\max_{k\in\mathcal{K}} V^k, \max_{(k_1,k_2)\in\mathcal{K}^{(2)}} V^{k_1,k_2}(\delta) \right) \leq f(n,K) \right\}, \tag{17}$$

where $\mathcal{K}^{(2)} = \{(k_1,k_2) : k_1,k_2 \in \mathcal{K}, k_1 \neq k_2\}$. (If $K=1$, then $\mathcal{K}^{(2)}$ is the empty set $\emptyset$ in which

[33] In fact, our proof of Lemma B.1 identifies a bound of $(C \log n/n)^{1/K}$ where $C = 6K(K-1)$, for sufficiently large n.

case we follow the convention that $\max_{\emptyset}[\,\cdot\,] = -\infty$.) *We have*

$$\Pr(\mathcal{B}_1) \geq 1 - 1/n\,.$$

Proof. Let $m = \lfloor 1/(C \log n/n)^{1/K} \rfloor$ for some $C < \infty$ that we will choose later, and let $\Delta = 1/m$. Note that

$$\Delta \geq (C \log n/n)^{1/K}\,. \tag{18}$$

In our analysis of V^k (resp. V^{k_1,k_2}), we will divide the interval $[0,1]$ (resp. $[-1+\delta, 1]$) into subintervals of size Δ each, and show that with large probability, each subinterval contains at least one value of $\epsilon_i \in \mathcal{R}^k$ (resp. $\epsilon_i^{k_1} - \epsilon_i^{k_2}$ for $\{i : \epsilon_i \in \mathcal{R}^{k_1,k_2}\}$). We will find that the density of points in $\mathcal{V}^k$ (resp. $\mathcal{V}^{k_1,k_2}$) is smallest near 0 (resp. $-1+\delta$), but even for the interval $[0,\Delta]$ (resp. $[-1+\delta, -1+\delta+\Delta]$), the number of points is Poisson with parameter $\Theta(n\Delta^K) = \Theta(\log n)$, allowing us to obtain the desired result for appropriately chosen C.

We first present our formal argument leading to a bound on V^k, followed by a similar argument leading to a bound on V^{k_1,k_2}. Let

$$\mathcal{B}^k \equiv \bigcap_{i=0}^{m-1} \{\, [i\Delta, (i+1)\Delta] \cap \mathcal{V}^k \neq \emptyset \,\}\,, \tag{19}$$

where $\emptyset$ is the empty set. Clearly, $\mathcal{B}^k \Rightarrow V^k \leq 2\Delta$. We now show that for any $k \in \mathcal{K}$, we have $\Pr\big(\overline{\mathcal{B}^k}\big) \leq 1/n^{K+2}$, for appropriately chosen C. Define

$$h^j(x,\theta) = \begin{cases} x^j & \text{for } x \in [\theta, 1] \\ 0 & \text{otherwise}\,. \end{cases} \tag{20}$$

It is easy to see that $\mathcal{V}^k$ follows a Poisson process with density $nh^{K-1}(\,\cdot\,,0)$. The number of points in interval $[i\Delta, (i+1)\Delta]$ is hence Poisson with parameter

$$n \int_{i\Delta}^{(i+1)\Delta} h^{K-1}(x)\,\mathrm{d}x = ((i+1)^K - i^K)n\Delta^K/K \geq n\Delta^K/K \geq C \log n/K\,,$$

where we used the lower bound on Δ in (18). It follows that

$$\Pr([i\Delta, (i+1)\Delta] \cap \mathcal{V}^k = \emptyset) \le \exp(-C \log n/K) = 1/n^{C/K} \le 1/n^3\,,$$

for $C \ge 3K$. We deduce by union bound over $i = 0, 1, \dots, m-1$ and De Morgan's law on (19) that

$$\Pr\big(\overline{\mathcal{B}^k}\big) \le m/n^3 \le n^{1/K}/n^3 \le 1/n^2\,.$$

Using union bound over k we deduce that

$$\Pr\Big(\cup_k \overline{\mathcal{B}^k}\Big) \le K/n^2 \tag{21}$$

We now present a similar argument to control V^{k_1,k_2} when $K \ge 2$. Let $m' = (1-\delta)/\Delta$. (To simplify notation we assume m' is an integer. The case when it is not an integer can be easily handled as well.) Let

$$\mathcal{B}^{k_1,k_2} \equiv \bigcap_{i=-m'}^{m-1} \{\, [i\Delta, (i+1)\Delta] \cap \mathcal{V}^{k_1,k_2} \neq \emptyset \,\}\,, \tag{22}$$

where $\emptyset$ is the empty set. Clearly, $\mathcal{B}^{k_1,k_2} \Rightarrow V^{k_1,k_2} \le 2\Delta$. We now show that for any $k_1 \neq k_2$, we have $\Pr\big(\overline{\mathcal{B}^{k_1,k_2}}\big) \le K(K-1)/n^2$, for appropriately chosen C. It is easy to see that the two-dimensional projection $(x,y) = (\epsilon_i^{k_1}, \epsilon_i^{k_2})$ of points in $\mathcal{R}^{k_1,k_2}$ follows a two-dimensional Poisson process with density $h^{K-2}(x)\mathbb{I}(y \in [0,1])$, cf. (20). We deduce that values in $\mathcal{V}^k$ follow a one-dimensional Poisson process with density ng for $g = h^{K-2}(\,\cdot\,,\delta) * \mathbb{I}(\in [-1,0])$, where $*$ is the convolution operator. A short calculation yields

$$g(x) = \begin{cases} \big[(x+1)^{K-1} - \delta^{K-1}\big]/(K-1) & \text{for } x \in [-1+\delta, 0) \\ \big[1 - \delta^{K-1}\big]/(K-1) & \text{for } x \in [0, \delta) \\ (1 - x^{K-1})/(K-1) & \text{for } x \in [\delta, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The number of points in interval $[i\Delta, (i+1)\Delta]$ is Poisson with parameter

$$n \int_{i\Delta}^{(i+1)\Delta} g(x)\, \mathrm{d}x \,.$$

Below we bound the value of this parameter for different cases on i, obtaining a bound of $(K+3)\log n$ in each case, for large enough C.

For $-m' \le i < 0$, the smallest parameter occurs for $i = -m'$, since $g(x)$ is monotone increasing in $[-1+\delta, 0]$. Thus, the Poisson parameter is lower bounded by its value for $i = -m'$, which is

$$\begin{aligned} & n\big[\big((\delta+\Delta)^K - \delta^K\big)/K - \delta^{K-1}\Delta\big]/(K-1) \\ \ge\, & n\Delta^K/(K(K-1)) \ge C\log n/(K(K-1)) \ge 3\log n \,, \end{aligned}$$

for $C \ge 3K(K-1)$, using (18), and $(\delta+\Delta)^K \ge \Delta^K + K\Delta\delta^{K-1} + \delta^K$.

For $0 \le i < m - m'$, the Poisson parameter is

$$n\big[1 - \delta^{K-1}\big]\Delta/(K-1) \ge n\Delta/(2(K-1)) \ge n\Delta^K/(K(K-1)) \ge 3\log n \,,$$

using $\delta \ge 1/2$ and $K \ge 2$.

For $(m - m') \le i < m$, the Poisson parameter is

$$n(\Delta - \Delta^K((1+i)^K - i^K)/K)/(K-1) \,.$$

A short calculation allows us to again bound this below by $(K+3)\log n$ (the bound is slack for $K > 2$): Note that

$$\begin{aligned} \Delta^K((1+i)^K - i^K) &\le \Delta^K(m^K - (m-1)^K) = 1 - (1-\Delta)^K \\ &\le K\Delta - K(K-1)\Delta^2/2 + K(K-1)(K-2)\Delta^3/6 \,, \end{aligned}$$

where we used that $(1+i)^K - i^K$ is monotone increasing in i for $i \ge 0$. Substituting back, we obtain that the Poisson parameter is bounded by

$$n(1-(K-2)\Delta/3)\Delta^2/2 \geq n\Delta^2/4$$

for $(K-2)\Delta/3 \leq 1/2$, which occurs for sufficiently large n. Finally, $\Delta^2 \geq \Delta^K$, hence $n\Delta^2/4 \geq n\Delta^K/4 \geq 3\log n$ for $C \geq 12$.

Choosing $C = 6K(K-1)$, in all cases the Poisson parameter is bounded below by $3\log n$. It follows that

$$\Pr([i\Delta,(i+1)\Delta] \cap \mathcal{V}^{k_1,k_2} = \emptyset) \leq \exp(-3\log n) = 1/n^3\,.$$

We deduce by union bound over i and De Morgan's law on (22) that

$$\Pr\Big(\overline{\mathcal{B}^{k_1,k_2}}\Big) \leq 2m/n^3 \leq n^{1/K}/n^3 \leq 1/n^2\,, \tag{23}$$

for large enough n. Using union bound over (k_1,k_2) we deduce that

$$\Pr\Big(\cup_{(k_1,k_2)}\overline{\mathcal{B}^{k_1,k_2}}\Big) \leq K(K-1)/n^2 \tag{24}$$

Combining (23) and (24) by union bound and using De Morgan's law, we deduce that

$$\Pr\Big[\Big(\cap_k \mathcal{B}^k\Big) \cap \Big(\cap_{(k_1,k_2)}\mathcal{B}^{k_1,k_2}\Big)\Big] \geq 1 - K^2/n^2$$

for large enough n. This implies that for large enough n, with probability at least $1-K^2/n^2$ we have

$$\max\left(\max_k V^k, \max_{k_1,k_2} V^{k_1,k_2}\right) \leq 2\Delta \leq 3(C\log n/n)^{1/K} = O^*(1/n^{1/K})\,,$$

implying the main result for large enough n (note that $K^2/n^2 < 1/n$ for large enough n). For small values of n, we can simply choose $f(n,k)$ large enough to ensure that the bound holds with sufficient probability. $\square$

.

Lemma B.2. *For $k \in \mathcal{K}$, let $\tilde{\mathcal{R}}^k(\delta)$, $\tilde{\mathcal{V}}^k(\delta)$ and $\tilde{V}^k(\delta)$ be as defined by Eqs.* (4), (5) *and* (6) *respectively. Fix $K \geq 1$. There exists $f(n) = O^*(1/n)$ such that for any $\delta \in (0, 1]$, the following occurs: Let*

$$\mathcal{B}_2 \equiv \left\{ \max_{k \in \mathcal{K}} \tilde{V}^k(\delta) \leq f(n)/\delta^{K-1} \right\} . \tag{25}$$

Then

$$\Pr(\mathcal{B}_2) \geq 1 - 1/n .$$

Proof. The values in the set $\tilde{\mathcal{V}}^k \subset [0, 1]$ follow a one-dimensional Poisson process with rate $n\delta^{K-1}$. Choose $f(n) = 6 \log n/n$. If $6 \log n/(n\delta^{K-1}) \geq 1$ there is nothing to prove, since $\max_{k \in \mathcal{K}} \tilde{V}^k(\delta) \leq 1$ by definition. Hence assume $6 \log n/(n\delta^{K-1}) < 1$. Divide $[0, 1]$ into intervals of length $\Delta = f(n)/(3\delta^{K-1}) = 3 \log n/(n\delta^{K-1})$ (to simplify notation, we assume $1/\Delta \geq 2$ is an integer. The argument can easily be adapted to handle $n\delta^{K-1}/(3 \log n)$ not an integer). The probability that any particular interval of length Δ does not contain a point is no more than $\exp(-3 \log n) = 1/n^3$. The number of intervals of length Δ is $1/\Delta = n\delta^{K-1}/(3 \log n) \leq n$ for large enough n. By union bound, with probability at least $1 - 1/n^2$, each Δ-interval contains at least one point, implying that $\tilde{V}^k(\delta) \leq 2\Delta = f(n)/\delta^{K-1}$ with probability at least $1 - 1/n^2$, as required. □

In this section so far we considered the rate n Poisson process in $[0, 1]^K$ for convenience. However, the results we proved can easily be transported to the closely related model of n points distributed i.i.d. uniformly in $[0, 1]^K$.

Lemma B.3. *Consider n points distributed i.i.d. uniformly in $[0, 1]^K$. Lemmas B.1 and B.2 hold for this model as well.*

Proof. We use a standard coupling argument along with monotonicity of the considered random variables with respect to additional points. Let $\mathcal{P}$ be a rate $n/2$ Poisson process in $[0, 1]^K$. The N points are distributed i.i.d. uniform $[0, 1]^K$ conditioned on the value of N. Let $\mathcal{B}$ be the event

$N \leq n$. Clearly, $\mathcal{B}$ occurs with probability at least $1 - 1/n^2$. Let $\mathcal{U}$ be the process consisting of n points distributed i.i.d. in $[0, 1]^K$. Conditioned on $\mathcal{B}$, we can couple the process $\mathcal{P}$ with the process $\mathcal{U}$ such that for every point in the Poisson process, there is an identically located point in $\mathcal{U}$.

We now show how to establish Lemma B.2 for process $\mathcal{U}$ using such a coupling. Note that $\max_{k\in\mathcal{K}} \tilde{V}^k(\delta)$ is monotone non-increasing as we add more points. As such, an upper bound on this quantity continues to hold if more points are added. For instance, consider $\max_{k\in\mathcal{K}} \tilde{V}^k(\delta)$. Let $\mathcal{B}'$ be the event that

$$\max_{k\in\mathcal{K}} \tilde{V}^k(\delta) \leq f(n/2)/\delta^{K-1}$$

under $\mathcal{P}$. The proof of Lemma B.2 shows that $\Pr(\mathcal{B}') \geq 1 - (2/n)^2$. By union bound on $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}'}$, we deduce that $\Pr(\mathcal{B} \cap \mathcal{B}') \geq 1 - 5/n^2 \geq 1 - 1/n$, for large enough n. We deduce, using a coupling as described above, that with probability at least $1 - 1/n$, for process $\mathcal{U}$ we have

$$\max_{k\in\mathcal{K}} \tilde{V}^k(\delta) \leq \tilde{f}(n)/\delta^{K-1}\,,$$

where $\tilde{f}(n) = f(n/2)$, for large enough n. (For small values of n, we can simply choose $\tilde{f}(n)$ large enough to ensure that the bound holds with sufficient probability.) Thus we have shown that Lemma B.2 holds for process $\mathcal{U}$.

Lemma B.1 can similarly be established for process $\mathcal{U}$ using that

$$\max\left(\max_{k\in\mathcal{K}} V^k, \max_{(k_1,k_2)\in\mathcal{K}^{(2)}} V^{k_1,k_2}(\delta)\right)$$

is monotone non-increasing as we add more points. $\square$

We now establish another result about n points $(\epsilon_j)_{j=1}^n$ distributed i.i.d. uniformly in $[0, 1]^K$. This result is key to the proof of the tightness of Theorem 1 (Proposition 3).

For $\delta \in [0,1]$ let

$$\widehat{\mathcal{R}}^{k_1,k_2}(\delta) = \{x \in [0,1]^K : x^{k_1} \geq x^{k_2} - \delta\,;\, x^{k_1} \geq x^k \,\forall k \notin \{k_1,k_2\}, k \in \mathcal{K}\} \tag{26}$$

Let $n^{k_1,k_2}(\delta)$ be the number of points in $\widehat{\mathcal{R}}^{k_1,k_2}(\delta)$.

Lemma B.4. *Let $\mathcal{B}_3$ be the event that there for all $k_1, k_2 \in \mathcal{K}$ we have $n^{k_1,k_2} \geq 1 + n/K$. For $\delta = \delta(n) \geq 1/n^{0.49}$, we have that $\mathcal{B}_3$ occurs with high probability.*

Proof. A short calculation shows that the volume of $\widehat{\mathcal{R}}^{k_1,k_2}(\delta)$ is

$$\begin{aligned} v &= \frac{1}{K-1}\left(1 - \frac{(1-\delta)^K}{K}\right) && (27)\\ &\geq \frac{1}{K} + \frac{\delta}{K-1} - \frac{\delta^2}{2} && (28)\\ &\geq \frac{1+\delta}{K} && (29) \end{aligned}$$

for $\delta \leq 2/(K(K-1))$. Now, the probability of $\epsilon_j \in \widehat{\mathcal{R}}^{k_1,k_2}(\delta)$ is exactly v. It follows that n^{k_1,k_2} is distributed as Binomial(n,v). Notice $\mathrm{E}[n^{k_1,k_2}] = nv \geq n(1+\delta)/K$. We obtain

$$\Pr(n^{k_1,k_2} < 1 + n/K) \leq \exp\{-\Omega(n\delta^2)\} = \exp\{-\Omega(n^{0.02})\} = o(1) \tag{30}$$

using a standard Chernoff bound (e.g., see Durrett (2010)). Using union bound over pairs k_1, k_2 we deduce that $\overline{\mathcal{B}_3}$ occurs with probability $o(1)$, i.e., event $\mathcal{B}_3$ occurs with high probability. $\square$

We now prove Theorem 4.

Proof of Theorem 4. By invoking Lemma B.1, Lemma B.2 and Lemma B.3, for each t we have that w.p. at least $1 - \frac{2}{n_t}$ the event $\big(\mathcal{B}_1(t,\delta) \cap \mathcal{B}_2(t,\delta)\big)$ occurs. As the total number of types is upper bounded by $K + Q$, we apply an union bound to conclude that w.p. at least $1 - \frac{2(K+Q)}{n^*}$, the event $\bigcap_{t \in \mathcal{T}_\mathcal{L} \cup \mathcal{T}_\mathcal{E}} \big(\mathcal{B}_1(t,\delta) \cap \mathcal{B}_2(t,\delta)\big)$ occurs. $\square$

C Proof of Theorem 1 upper bound

Proof of Proposition 1. The proposition follows immediately from stability. $\square$

Throughout the rest of the appendix, we shall use the following characterization of $\mathcal{C}$.

Lemma C.1 (Size of the core)**.** *Let M be the unique maximum weight type-matching. Assume Assumption 2, fixed type-type compatibilities $u(\cdot,\cdot)$, idiosyncratic productivities being i.i.d. $U(0,1)$ and that at least one $u(\cdot,\cdot)$ is greater than -2 (otherwise there will no matched agent pairs). For each pair of types $(k,q) \in \mathcal{T}$, let $N(k,q)$ denote the number of matches between agents of type k and agents of type q. Then, with probability $1-1/n$, the size of the core $\mathcal{C}$, cf. Definition 1, is bounded as*

$$\mathcal{C} = \Theta\left(\frac{\sum_k \sum_q N(k,q)|\alpha_{kq}^{\max} - \alpha_{kq}^{\min}|}{\sum_k \sum_q N(k,q)}\right).$$

Proof. To prove this lemma, we need to show that the total number of matches $\sum_k \sum_q N(k,q)$ is within a constant factor of the social welfare with probability $1-1/n$. The total social welfare is bounded above by $n(2+\max u(\cdot,\cdot)) = O(n)$ and total number of matches is bounded above by n. We will show that both these quantities are also $\Omega(n)$ with probability $1-1/n$, which will then imply the lemma. Suppose $u(k,q) > -2$. By Assumption 2, we know that there are at least Cn agents each of type k and type q. Let $\delta = (u(k,q)+2)/3$. Let

$$\mathcal{M}_k = \mathcal{M}_k(\delta) = \{i \in k : \eta_i^q \geq 1-\delta\}, \tag{31}$$

and similarly,

$$\mathcal{M}_q = \mathcal{M}_q(\delta) = \{j \in q : \epsilon_j^k \geq 1-\delta\}. \tag{32}$$

Clearly, one can find a matching (not necessarily a stable outcome) consisting $\min(|\mathcal{M}_k|, |\mathcal{M}_q|)$ pairs, each with one agent from $\mathcal{M}_k$ and the other agent from $\mathcal{M}_q$. The weight of every edge/pair in this matching is at least $u(k,q) + 2(1-\delta) = \delta$, hence the total weight of the matching is at least $\delta \min(|\mathcal{M}_k|, |\mathcal{M}_q|)$. A standard Chernoff bound yields that $|\mathcal{M}_k| \geq Cn \min(\delta/2, 1)$ with probability $1-1/n^2$, and similarly for $|\mathcal{M}_q|$. It follows that the total weight of the matching is at least $Cn\delta \min(\delta/2,1) = \Omega(n)$ with probability $1-1/n$ as needed. Since stable outcomes maximize social welfare it follows that the total social welfare is $\Omega(n)$ in stable outcomes.

It remains to bound the number of matches in stable outcomes. We claim that either all agents in $\mathcal{M}_k$ are matched or all agents in $\mathcal{M}_q$ are matched. If not, there is a blocking pair consisting of an unmatched agent from each set. It follows that there are at least $\min(|\mathcal{M}_k|, |\mathcal{M}_q|)$ matched agents in maximum weight matching (the matching in all stable outcomes), and in particular at least $Cn\delta \min(\delta/2, 1) = \Omega(n)$ matched pairs under the lower bound obtained above on $|\mathcal{M}_k|$ and $|\mathcal{M}_q|$. $\square$

The above characterization only differs from the original definition (Definition 1) in that the size of the core is scaled by the number of matched agents instead of being scaled by the social welfare. Lemma C.1 allows us to bound $\mathcal{C}$ by controlling $|\alpha_{kq}^{\max} - \alpha_{kq}^{\min}|$.

We now present the complete proof of Theorem 1.

Before moving on to the key lemmas, we introduce some definitions. For every type $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$, we define $\vartheta(t)$ as $\vartheta(t) = \{k \in \mathcal{T}_{\mathcal{L}} : \ N(k,t) > 0\}$ when $t \in \mathcal{T}_{\mathcal{E}}$ and $\vartheta(t) = \{q \in \mathcal{T}_{\mathcal{E}} : \ N(t,q) > 0\}$ when $t \in \mathcal{T}_{\mathcal{L}}$. That is, $\vartheta(t)$ is the set of neighbors of t in the graph $G(M)$. Recall that, given a type $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$ we denote by $D(t)$ the *dimension* of the idiosyncratic productivity vector associated to agents of type t. That is, $D(t) = K$ if $t \in \mathcal{T}_{\mathcal{E}}$ and $D(t) = K$ if $t \in \mathcal{T}_{\mathcal{L}}$.

Given a type $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$ we denote by $\nu(t)$ or simply ν, the points in t. That is, for each agent j of type t, we define ν_j as follows:

$$\nu_j = \begin{cases} \epsilon_j & \text{if } t \in \mathcal{T}_{\mathcal{E}} \\ \eta_j & \text{if } t \in \mathcal{T}_{\mathcal{L}} \end{cases}$$

For a fixed $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$ and $t' \in \vartheta(t)$, let $\beta_{tt'}$ be defined as:

$$\beta_{tt'} = \begin{cases} -\alpha_{tt'} & \text{if } t \in \mathcal{T}_{\mathcal{E}} \\ \alpha_{tt'} - u(t,t') & \text{if } t \in \mathcal{T}_{\mathcal{L}} \end{cases} \tag{33}$$

Note that $\beta_{tt'}$ can be interpreted as the price that agents of type t pay for matching with type t'.

Using the above notation, we can re-write the conditions in Proposition 2 associated to a fixed type $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$ as follows:

(ST) For every $k, k' \in \vartheta(t)$:

$$\min_{j \in t \cap M(k)} \nu_j^k - \nu_j^{k'} \geq \beta_{kt} - \beta_{k't} \geq \max_{j \in t \cap M(k')} \nu_j^k - \nu_j^{k'}.$$

(IM) For every $k \in \vartheta(t)$:

$$\min_{j \in t \cap M(k)} \nu_j^k \geq \beta_{kt} \geq \max_{j \in q \cap U} \nu_j^k.$$

As all the ν variables are in $[0, 1]$, then the above conditions can be interpreted as geometric conditions in the $[0, 1]^{D(t)}$-hypercube.

Lemma C.2. *Consider the unique maximum weight type-matching M and a type $t \in \mathcal{T}_{\mathcal{L}} \cap \mathcal{T}_{\mathcal{E}}$. Let $\mathcal{F}_1(t)$ be the event*

$$\mathcal{F}_1(t) = \{t \text{ is marked in } G(M) \text{ and at least one agent in } t \text{ is matched}\}, \tag{34}$$

that is, t has at least one unmatched and one matched agent. Let the events $\mathcal{B}_1(t, \delta)$ and $\mathcal{B}_2(t, \delta)$ be as defined by Eqs. (15) *and* (16) *respectively. Under $\mathcal{F}_1(t) \cap \mathcal{B}_1(t, \delta) \cap \mathcal{B}_2(t, \delta)$, we have*

$$\max_{t' \in \vartheta(t)} \left(\alpha_{t,t'}^{\max} - \alpha_{t,t'}^{\min}\right) \leq \max\left(2f_1(n_t, D(t)) + \delta, f_2(n_t)/\delta^{D(t)-1}\right),$$

where f_1 and f_2 agree with those in the definitions of events $\mathcal{B}_1(t, \delta)$ and $\mathcal{B}_2(t, \delta)$ respectively.

The proof of Lemma C.2 is partitioned into two lemmas. Given a core solution (M, α), let the event $\mathcal{D}(t, \delta)$ be defined as:

$$\mathcal{D}(t, \delta) = \{\beta_{tz} \geq \delta \quad \forall z \in \vartheta(t)\}. \tag{35}$$

We interpret $\mathcal{D}(t, \delta)$ as the event that *all* prices seen by type t agents (weakly) exceed δ. Lemma C.3 below deals with $\mathcal{D}(t, \delta)$ whereas Lemma C.4 deals with the complement $\overline{\mathcal{D}(t, \delta)}$. Together they imply Lemma C.2.

Lemma C.3. *Consider a core solution (M, α) and a type t. Let the events $\mathcal{F}_1(t)$, $\mathcal{D}(t, \delta)$ and $\mathcal{B}_2(t, \delta)$ be as defined by Eqs.* (34), (35) *and* (16) *respectively. Under $\mathcal{F}_1(t) \cap \mathcal{D}(t, \delta) \cap \mathcal{B}_2(t, \delta)$, we have $\max_{t' \in \vartheta(t)} \left(\alpha^{\max}_{t,t'} - \alpha^{\min}_{t,t'}\right) \le f_2(n_t)/\delta^{D(t)-1}$, where f_2 is as defined in the statement of Lemma B.2.*

Proof. The intuition is that when all prices seen by type t agents (weakly) exceed δ, we can individually bound the variation within the core of each price: For each type $k \in \vartheta(t)$ on the other side, we consider agents of type t who are not interested in types other than k and bound the gap between consecutive order statistics of ν_j^k for such agents.

Let $D = D(t)$. Fix $k \in \vartheta(t)$ and consider the orthotope $\tilde{\mathcal{R}}^k = \tilde{\mathcal{R}}^k(t, \delta)$ as defined by Eq. (4). As $\mathcal{D}(t, \delta)$ occurs, $\beta_{tz} \ge \delta$ for all $z \in \vartheta(t)$ and therefore $\tilde{\mathcal{R}}^k$ can only contain points corresponding to agents in $M(k) \cup U$. By using the notation introduced above, condition (IM) in Proposition 2 implies: $\alpha^{\max}_{kt} - \alpha^{\min}_{kt} \le \min_{j \in t \cap M(k)} \nu_j^k - \max_{j \in q \cap U} \nu_j^k$. However, $\min_{j \in t \cap M(k)} \nu_j^k - \max_{j \in q \cap U} \nu_j^k \le \min_{j \in \tilde{\mathcal{R}}^k \cap M(k)} \nu_j^k - \max_{j \in \tilde{\mathcal{R}}^k \cap U} \nu_j^k \le \tilde{V}^k(t, \delta)$, where $\tilde{V}^k(t, \delta)$ is as defined by Eq. (6). Therefore, for each $k \in \vartheta(t)$ we must have $\alpha^{\max}_{kt} - \alpha^{\min}_{kt} \le \tilde{V}^k(t, \delta)$. Finally, under $\mathcal{B}_2(t, \delta)$ we have $\max_{k \in \vartheta(t)} \tilde{V}^k(t, \delta) \le f_2(n_t)/\delta^{D-1}$, which completes the result. □

Lemma C.4. *Consider a core solution (M, α) and a type t. Let $\mathcal{F}_1(t)$ be the event defined in Eq.* (34). *Let the event $\mathcal{B}_1(t, \delta)$ be as defined by Eq.* (15), *and let the event $\overline{\mathcal{D}(t, \delta)}$ denote the complement of the event defined by Eq.* (35). *Under $\mathcal{F}_1(t) \cap \overline{\mathcal{D}(t, \delta)} \cap \mathcal{B}_1(t, \delta)$, we have $\max_{t' \in \vartheta(t)} \left(\alpha^{\max}_{t,t'} - \alpha^{\min}_{t,t'}\right) \le 2 f_1(n_t, D(t)) + \delta$, where f_1 is as defined in the statement of Lemma B.1.*

Proof. Here at least one price seen by type t is less than δ, since we consider $\overline{\mathcal{D}(t, \delta)}$. The first part of our proof controls the variation in the core of the price which is the smallest at α; we call the corresponding type k^*. The second part of our proof controls relative variation in prices faced by type t, and then uses the control on the price of type k^* from the first part to control prices of other types $k \neq k^*$.

Suppose $t \in \mathcal{T}_{\mathcal{E}}$. Consider the unit hypercube in $\mathbb{R}^K$. For each $j \in \mathcal{E}$ such that $\tau(j) = t$, let $\epsilon_j \in [0,1]^K$ denote the vector of realizations of ϵ_j^k for every $k \in \mathcal{T}_{\mathcal{L}}$. By condition (ST) in

Proposition 2, we can partition the $[0,1]^K$ hypercube into $|\vartheta(t)|$ regions such that all the points ϵ corresponding to agents matched to $k \in \vartheta(t)$ must be contained in the corresponding region. In particular, for each $k \in \vartheta(t)$, we define $Z(k) \subseteq [0,1]^K$ to be the region corresponding to type k, with $Z(k) = \cap_{k' \in \vartheta(t),\ k' \neq k} \{x \in [0,1]^K : x_k - x_{k'} \geq \alpha_{k't} - \alpha_{kt}\}$. Note that the region $Z(k)$ can only contain points corresponding to agents matched to k or unmatched.

Let $k^* = \text{argmax}_{k \in \mathcal{T_L}} \{\alpha_{tk^*} :\ k \in \vartheta(t)\}$, and let $\mathcal{R}^{k^*} = \mathcal{R}^{k^*}(t)$ be as defined by Eq. (9). By condition (ST) in Proposition 2, we have that for all $k \in \vartheta(t)$:

$$\min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} - \epsilon_j^k \geq \alpha_{kt} - \alpha_{k^*t} \geq \max_{j \in q \cap M(k)} \epsilon_j^{k^*} - \epsilon_j^k.$$

As $\alpha_{kt} - \alpha_{k^*t} \leq 0$ for all $k \in \vartheta(t)$, we must have $\mathcal{R}^{k^*} \subseteq Z(k^*)$. Let $V^{k^*} = V^{k^*}(t)$ be as defined in Eq. (12). We claim that $\alpha_{k^*,t}^{\max} - \alpha_{k^*,t}^{\min} \leq V^{k^*}$. To see why this holds, consider two separate cases. First, suppose there is at least one point corresponding to an unmatched agent in $\mathcal{R}^{k^*}$. By condition (IM) in Proposition 2, we must have $\min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} \geq -\alpha_{k^*t} \geq \max_{j \in t \cap U} \epsilon_j^{k^*}$. Hence, $\alpha_{k^*,t}^{\max} - \alpha_{k^*,t}^{\min} \leq \min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} - \max_{j \in t \cap U} \epsilon_j^{k^*} \leq V^{k^*}$ as desired. For the second case, suppose that all points in $\mathcal{R}^{k^*}$ correspond to matched agents. As $\max_{j \in t \cap U} \epsilon_j^{k^*} \geq 0$, we must have $\alpha_{k^*,t}^{\max} - \alpha_{k^*,t}^{\min} \leq \min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} \leq \min_{j \in \mathcal{R}^{k^*}} \epsilon_j^{k^*} \leq V^{k^*}$, as the difference between 0 and the $\min_{j \in \mathcal{R}^{k^*}} \epsilon_j^{k^*}$ is upper bounded by V^{k^*}. Therefore, we conclude $\alpha_{k^*,t}^{\max} - \alpha_{k^*,t}^{\min} \leq V^{k^*}$.

Next, we consider the bound for any arbitrary type $k \in \vartheta(t)$. By condition (ST) in Proposition 2, we have that for all $k \in \vartheta(t)$:

$$\alpha_{k^*t}^{\max} + \min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} - \epsilon_j^k \geq \alpha_{kt} \geq \alpha_{k^*t}^{\min} + \max_{j \in q \cap M(k)} \epsilon_j^{k^*} - \epsilon_j^k.$$

Therefore,

$$\alpha_{kt}^{\max} - \alpha_{kt}^{\min} \leq \alpha_{k^*t}^{\max} - \alpha_{k^*t}^{\min} + \min_{j \in t \cap M(k^*)} \left(\epsilon_j^{k^*} - \epsilon_j^k\right) - \max_{j \in q \cap M(k)} \left(\epsilon_j^{k^*} - \epsilon_j^k\right).$$

From our previous bound, we have that $\alpha_{k^*t}^{\max} - \alpha_{k^*t}^{\min} \leq V^{k^*}$. We now want an upper bound on $\min_{j \in t \cap M(k^*)}(\epsilon_j^{k^*} - \epsilon_j^k) - \max_{j \in q \cap M(k)}(\epsilon_j^{k^*} - \epsilon_j^k)$. Let $\mathcal{R}^{k^*,k} = \mathcal{R}^{k^*,k}(t,\delta)$ and $V^{k^*,k} = V^{k^*,k}(t)$ be as defined by Eqs. (10) and (14). We shall show that $\min_{j \in t \cap M(k^*)}(\epsilon_j^{k^*} - \epsilon_j^k) - \max_{j \in q \cap M(k)}(\epsilon_j^{k^*} -$

$\epsilon_j^k) \leq V^{k^*,k} + \delta$. Recall that, under $\overline{\mathcal{D}(t,\delta)}$, we have $\delta \geq -\alpha_{k^*t}$.

To that end, note that all points in $\mathcal{R}^{k^*,k}$ must correspond to agents matched to k^* or matched to k, as the region $\mathcal{R}^{k^*,k}$ cannot contain unmatched agents without violating condition (IM). Furthermore, as $\mathcal{R}^{k^*} \subseteq Z(k^*)$ and $\mathcal{R}^{k^*} \cap \mathcal{R}^{k^*,k} \neq \emptyset$, at least one point in $\mathcal{R}^{k^*,k}$ corresponds to an agent matched to k^*. We now consider two separate cases, depending on whether $\mathcal{R}^{k^*,k}$ contains at least one point matched to k. First, suppose $\mathcal{R}^{k^*,k}$ contains at least one point matched to k. Then, the bound trivially applies as

$$\min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} - \epsilon_j^k - \max_{j \in t \cap M(k)} \epsilon_j^{k^*} - \epsilon_j^k \leq \min_{j \in \mathcal{R}^{k^*,k} \cap M(k^*)} \epsilon_j^{k^*} - \epsilon_j^k - \max_{j \in \mathcal{R}^{k^*,k} \cap M(k)} \epsilon_j^{k^*} - \epsilon_j^k \leq V^{k^*,k}.$$

Otherwise, $\mathcal{R}^{k^*,k}$ contains only points matched to k^*. In that case,

$$\min_{j \in t \cap M(k^*)} \epsilon_j^{k^*} - \epsilon_j^k - \max_{j \in q \cap M(k)} \epsilon_j^{k^*} - \epsilon_j^k \leq \min_{j \in \mathcal{R}^{k^*,k}} \epsilon_j^{k^*} - \epsilon_j^k - (1 + \alpha_{k^*t}) \leq V^{k^*,k} + \delta,$$

as desired. Overall, we have shown that:

$$\max_{k \in \vartheta(t)} \left(\alpha_{tk}^{\max} - \alpha_{tk}^{\min} \right) \leq \max \left(V^{k^*}, \max_{k \in \vartheta(t)} \left(V^{k^*} + V^{k^*,k} + \delta \right) \right).$$

Under $\mathcal{B}_1(t,\delta)$ we have $\max\left(V^{k^*}, \max_k V^{k^*,k}\right) \leq f_1(n_t, K)$, implying

$$\max \left(V^{k^*}, \max_{k \in \vartheta(t)} \left(V^{k^*} + V^{k^*,k} + \delta \right) \right) \leq 2 f_1(n_t, K) + \delta,$$

as desired.

To conclude, we briefly discuss the changes when $t \in \mathcal{T}_{\mathcal{L}}$. Consider the unit hypercube in $\mathbb{R}^Q$. For each $j \in \mathcal{L}$ such that $\tau(j) = t$, let $\eta_j \in [0,1]^Q$ denote the vector of realizations of η_j^q for every $q \in \mathcal{T}_{\mathcal{E}}$. For each $q \in \vartheta(t)$, we define $Z(q) \subseteq [0,1]^Q$ to be the region corresponding to type q. The main difference with the case in which $t \in \mathcal{T}_{\mathcal{E}}$ is that we need to define the regions $Z(q)$ in terms of the $\tilde{\eta}$ instead of η. To that end, let $\beta_{kq} = \alpha_{kq} - u(k,q)$. By the (ST) condition in

Proposition 2, we must have:

$$\min_{i\in t\cap M(q')} \tilde{\eta}_i^{q'} - \tilde{\eta}_i^q \geq \alpha_{tq'} - \alpha_{tq} \geq \max_{i\in t\cap M(q)} \tilde{\eta}_i^{q'} - \tilde{\eta}_i^q,$$

or equivalently,

$$\min_{i\in t\cap M(q')} \eta_i^{q'} - \eta_i^q \geq \beta_{tq'} - \beta_{tq} \geq \max_{i\in t\cap M(q)} \eta_i^{q'} - \eta_i^q.$$

By using β instead of α, the same geometric intuition as before applies. Then, we define $Z(q) = \cap_{q'\in\vartheta(t),\ q'\neq q}\{x \in [0,1]^Q : x_q - x_{q'} \geq \beta_{qt} - \beta_{q't}\}$. To select q^*, we just select the one with smallest β_{qt}. The rest of the proof remains the same. □

Proof of Lemma C.2. Lemma C.2 immediately follows from Lemmas C.3 and C.4. □

Lemma C.5. *Consider the unique maximum weight type-matching M and a type $t \in \mathcal{T}_{\mathcal{L}} \cap \mathcal{T}_{\mathcal{E}}$. Let $\mathcal{F}_2(t)$ be the event*

$$\mathcal{F}_2(t) = \{\textit{all agents in } t \textit{ are matched}\}.$$

Let the event $\mathcal{B}_1(t,\delta)$ be as defined by Eq. (15)*. Under $\mathcal{F}_2(t) \cap \mathcal{B}_1(t,\delta)$, for every $t^* \in \vartheta(t)$ we have $\max_{t'\in\vartheta(t)} \left(\alpha_{t,t'}^{\max} - \alpha_{t,t'}^{\min}\right) \leq \left(\alpha_{t,t*}^{\max} - \alpha_{t,t*}^{\min}\right) + 2f_1(n_t, D(t)) + 2\delta$, where f_1 agrees with the one in the definition of $\mathcal{B}_1(t,\delta)$.*

Proof. This proof is similar to the proof of Lemma C.4. Consider a core solution (M,α). Let $D = D(t)$. Fix a type $t^* \in \vartheta(t)$, and let $k^* = \mathrm{argmax}_{k\in\vartheta(t)}\beta_{tk}$. We start by showing that, under $\mathcal{F}_2(t) \cap \mathcal{B}_1(t,\delta)$, we must have $\alpha_{tk^*}^{\max} - \alpha_{tk^*}^{\min} \leq \left(\alpha_{t,t*}^{\max} - \alpha_{t,t*}^{\min}\right) + f_1(n_t, D(t)) + 2\delta$. If $k^* = t^*$, the claim follows trivially. Otherwise, let $\mathcal{R}^{k^*,t^*} = \mathcal{R}^{k^*,t^*}(t,\delta)$ and $V^{k^*,t^*} = V^{k^*,t^*}(t,\delta)$ be as defined by Eqs. (10) and (14). We show that $\min_{j\in t\cap M(k^*)} \nu_j^{k^*} - \nu_j^{t^*} - \max_{j\in q\cap M(t^*)} \nu_j^{k^*} - \nu_j^{t^*} \leq V^{k^*,t^*} + \delta$.

To that end, note that all points in $\mathcal{R}^{k^*,t^*}$ must correspond to agents matched to k^* or matched to t^*, as under $\mathcal{F}_2(t)$ all agents in t are matched. Furthermore, by the definition of k^*, $\mathcal{R}^{k^*,t^*}$ must contain a point corresponding to an agent matched to k^*. We now consider two separate cases, depending on whether $\mathcal{R}^{k^*,t^*}$ contains at least one point corresponding to an agent matched to t^*. First, suppose $\mathcal{R}^{k^*,t^*}$ contains at least one point corresponding to an agent matched to t^*. Then,

$$\min_{j\in t\cap M(k^*)} \nu_j^{k^*} - \nu_j^{t^*} - \max_{j\in t\cap M(t^*)} \nu_j^{k^*} - \nu_j^{t^*} \leq \min_{j\in \mathcal{R}^{k^*,t^*}\cap M(k^*)} \nu_j^{k^*} - \nu_j^{t^*} - \max_{j\in \mathcal{R}^{k^*,t^*}\cap M(k)} \nu_j^{k^*} - \nu_j^{t^*} \leq V^{k^*,t^*}.$$

Otherwise, $\mathcal{R}^{k^*,t^*}$ contains only points matched to k^*. In that case,

$$\min_{j\in t\cap M(k^*)} (\nu_j^{k^*} - \nu_j^{t^*}) - \max_{j\in t\cap M(t^*)} (\nu_j^{k^*} - \nu_j^{t^*}) \leq \min_{j\in \mathcal{R}^{k^*,t^*}} (\nu_j^{k^*} - \nu_j^{t^*}) - 1 \leq V^{k^*,t^*} + \delta,$$

as desired. By condition (ST) in Proposition 2, we must have:

$$\alpha_{tk^*}^{\max} - \alpha_{tk^*}^{\min} \leq \alpha_{tt^*}^{\max} - \alpha_{tt^*}^{\min} + \min_{j\in t\cap M(k^*)} (\nu_j^{k^*} - \nu_j^{t^*}) - \max_{j\in t\cap M(t^*)} (\nu_j^{k^*} - \nu_j^{t^*}) \leq \alpha_{tt^*}^{\max} - \alpha_{tt^*}^{\min} + V^{k^*,t^*} + \delta$$

Next, consider an arbitrary $k \in \vartheta(t)$ with $k \neq t^*, k^*$. By condition (ST) in Proposition 2, we must have:

$$\alpha_{kt}^{\max} - \alpha_{kt}^{\min} \leq \alpha_{k^*t}^{\max} - \alpha_{k^*t}^{\min} + \min_{j\in t\cap M(k^*)} (\nu_j^{k^*} - \nu_j^{k}) - \max_{j\in q\cap M(k)} (\nu_j^{k^*} - \nu_j^{k}).$$

Let $\mathcal{R}^{k^*,k} = \mathcal{R}^{k^*,k}(t,\delta)$ and $V^{k^*,k} = V^{k^*,k}(t)$ be as defined by Eqs. (10) and (14). By repeating the same arguments as before, we can show that $\min_{j\in t\cap M(k^*)} (\nu_j^{k^*} - \nu_j^{k}) - \max_{j\in q\cap M(k)} (\nu_j^{k^*} - \nu_j^{k}) \leq V^{k^*,k} + 2\delta$. Hence,

$$\alpha_{kt}^{\max} - \alpha_{kt}^{\min} \leq \alpha_{k^*t}^{\max} - \alpha_{k^*t}^{\min} + V^{k^*,k} + \delta \leq \alpha_{tt^*}^{\max} - \alpha_{tt^*}^{\min} + V^{k^*,t^*} + V^{k^*,k} + 2\delta.$$

To conclude, note that

$$\max_{k\in\vartheta(t)} \left(\alpha_{kt}^{\max} - \alpha_{kt}^{\min}\right) \leq \left(\alpha_{tt^*}^{\max} - \alpha_{tt^*}^{\min}\right) + 2\left(\max_{k\in\vartheta(t)} V^{k^*,k}\right) + 2\delta \leq \left(\alpha_{tt^*}^{\max} - \alpha_{tt^*}^{\min}\right) + 2f_1(n_t, D) + 2\delta,$$

where the last inequality follows from the fact that $\mathcal{B}_1(t,\delta)$ occurs by hypothesis. □

We can now proceed to the proof of the main theorem.

Proof of Theorem 1 (upper bound). Let $n^* = \min_{t\in\mathcal{T}_\mathcal{L}\cup\mathcal{T}_\mathcal{E}} n_t$. Under Assumption 2, we have that $n^* = \Theta(n)$. Let $\delta = 1/(n^*)^{1/\max(K,Q)}$. For each $t \in \mathcal{T}_\mathcal{L} \cup \mathcal{T}_\mathcal{E}$, let the events $\mathcal{B}_1(t,\delta)$ and $\mathcal{B}_2(t,\delta)$ be as defined by Eqs. (15) and (16) respectively. We start by showing that, under $\bigcap_{t\in\mathcal{T}_\mathcal{L}\cup\mathcal{T}_\mathcal{E}} \left(\mathcal{B}_1(t,\delta)\cap\mathcal{B}_2(t,\delta)\right)$, we must have $\mathcal{C} \le O^*\left(\frac{1}{\sqrt[\max(K,Q)]{n}}\right)$.

To that end, construct the type-adjacency graph $G(M)$ as defined in Section 4. For each vertex v, we denote by $d(v)$ the minimum distance between v and any marked vertex (that is, $d(v) = 0$ if v is marked, $d(v) = 1$ if v is unmarked and has a marked neighbour, and so on). By Lemma 1, we know that w.p. 1, each connected component of $G(M)$ must contain at least one marked vertex, so $d(v)$ is well-defined for all v. Let $C_d = \{v \in C : d(v) = d\}$, that is C_d is the set of vertices that are at distance d from a marked vertex. We now show the result by induction in d. In particular, we show that, under $\bigcap_{t\in\mathcal{T}_\mathcal{L}\cup\mathcal{T}_\mathcal{E}} \left(\mathcal{B}_1(t,\delta)\cap\mathcal{B}_2(t,\delta)\right)$, for each $t \in C_d$ we have that $\max_{k\in\vartheta(t)}\left(\alpha_{tk}^{\max} - \alpha_{tk}^{\min}\right) \le g_d(n^*, \max(K,Q))$ for some $g_d(n^*, \max(K,Q)) = O^*(\frac{1}{n^{1/\max(K,Q)}})$.

We start by showing that the claim holds for the base case $d = 0$. For each $t \in C_0$, either all agents in t are unmatched or at least one agent is matched. In the former case, we can just ignore type t as it will not contribute to the size of the core. In the latter, we note that w.p.1 the event $\mathcal{F}_1(t)$ as defined in the statement of Lemma C.2 must hold. Therefore, we can apply Lemma C.2 to obtain $\max_{t'\in\vartheta(t)}\left(\alpha_{t,t'}^{\max} - \alpha_{t,t'}^{\min}\right) \le \max\left(2f_1(n_t, D(t)) + \delta, f_2(n_t)/\delta^{D(t)-1}\right)$, where f_1 and f_2 are as defined in the statement of the lemma. To conclude the proof of the base case, let

$$g_0(n^*, \max(K,Q)) = \max\left(2f_1(n^*, \max(K,Q)) + \delta, f_2(n^*)/\delta^{\max(K,Q)-1}\right).$$

By the definition of f_1, f_2, and δ, together with Assumption 2, we have $g_0(n^*, \max(K,Q)) = O^*(\frac{1}{n^{1/\max(K,Q)}})$. Therefore, we have shown that, for every $t \in C_0$, we have

$$\max_{k\in\vartheta(t)}\left(\alpha_{tk}^{\max} - \alpha_{tk}^{\min}\right) \le g_0(n^*, \max(K,Q)).$$

Now suppose the result holds for all $d' \le d$, we want to show it holds for $d+1$. Fix $t \in C_{d+1}$. By definition of C_{d+1}, we have that all agents in t must be matched and therefore w.p. 1, the event $\mathcal{F}_2(t)$ as defined in the statement of Lemma C.5 occurs. Moreover, there must exist a t^* such that the vertex corresponding to t^* is C_d and $t^* \in \vartheta(t)$. By induction, we have that $\left(\alpha_{tt^*}^{\max} - \alpha_{tt^*}^{\min}\right) \leq g_d(n^*, \max(K,Q))$ for $g_d(n^*, \max(K,Q)) = O^*(\frac{1}{n^{1/\max(K,Q)}})$. Further, by Lemma C.5, we know that under $\mathcal{F}_2(t) \cap \mathcal{B}_1(t,\delta)$, we have $\max_{t' \in \vartheta(t)} \left(\alpha_{t,t'}^{\max} - \alpha_{t,t'}^{\min}\right) \leq \left(\alpha_{t,t*}^{\max} - \alpha_{t,t*}^{\min}\right) + 2f_1(n_t, D(t)) + 2\delta$, where $\mathcal{B}_1(t,\delta)$ as defined by Eq. (17) and f is as defined in the statement of Lemma B.1. Therefore, by letting $g_{d+1}(n^*, \max(K,Q)) = g_d(n^*, \max(K,Q)) + 2f_1(n^*, \max(K,Q)) + 2\delta$, we have show that with probability at least $1 - \frac{d+1}{n^*}$, we have

$$\max_{k \in \vartheta(t)} \left(\alpha_{tk}^{\max} - \alpha_{tk}^{\min}\right) \leq g_{d+1}(n^*, \max(K,Q)),$$

with $g_{d+1}(n^*, \max(K,Q)) = O^*(\frac{1}{n^{1/\max(K,Q)}})$.

Next, we note that $\max_v d(v)$ is upper bounded by $K+Q$. Hence, for every $t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}$, we have $\max_{k \in \vartheta(t)} \left(\alpha_{tk}^{\max} - \alpha_{tk}^{\min}\right) \leq g_{K+Q}(n^*, \max(K,Q))$ for $g_{K+Q}(n^*, \max(K,Q)) = O^*(\frac{1}{n^{1/\max(K,Q)}})$ and therefore

$$\max_{t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}} \max_{k \in \vartheta(t)} \left(\alpha_{tk}^{\max} - \alpha_{tk}^{\min}\right) \leq g_{K+Q}(n^*, \max(K,Q)).$$

To conclude, by Theorem 4 we have that with probability at least $1 - \frac{2(K+Q)}{n^*}$, the event $\bigcap_{t \in \mathcal{T}_{\mathcal{L}} \cup \mathcal{T}_{\mathcal{E}}} \left(\mathcal{B}_1(t,\delta) \cap \mathcal{B}_2(t,\delta)\right)$ occurs. In all other cases, we just use the fact that the size of the core is upper-bounded by a constant $C < \infty$. Hence,

$$\begin{aligned}
\mathrm{E}[\mathcal{C}] &= \frac{\sum_{(k,q) \in \mathcal{T}_{\mathcal{L}} \times \mathcal{T}_{\mathcal{E}}} N(k,q) \left(\alpha_{kq}^{\max} - \alpha_{kq}^{\min}\right)}{\sum_{(k,q) \in \mathcal{T}_{\mathcal{L}} \times \mathcal{T}_{\mathcal{E}}} N(k,q)} \\
&\leq (K+Q) g_{K+Q}(n^*, \max(K,Q)) + C\frac{2(K+Q)}{n^*} \\
&= O^*\left(\frac{1}{\sqrt[\max(K,Q)]{n}}\right)
\end{aligned}$$

implying the main result for large enough n (note that $\frac{2(K+Q)}{n^*} = \Theta^*(1/n)$). $\square$

D Theorem 1 lower bound: Proof of Proposition 3

Again, we make use of the characterization of $\mathcal{C}$ in Lemma C.1.

Proof of Proposition 3. We consider the sequence of markets described in Section 4.2. We remark that the gaps in values of n in the described sequence can easily be filled in, leaving the core and its size essentially unaffected.[34]

Claim D.1. *For this market, all labor agents of types different from k_* will be matched in the core.*

Proof. We know that there is some employer j who is either unmatched or matched to a labor agent i' of type k_*. Consider any matching where a labor agent i of type $k \neq k_*$ is unmatched. Now $\Phi(i', j) = \epsilon_{i'} + \eta_j^{k_*} \leq 1 + 1 = 2$, whereas $\Phi(i, j) \geq u(k, 1) = 3$, hence the weight of such a matching can be increased by instead matching j to i. It follows that in any maximum weight matching, all labor agents with type different from k_* are matched. Finally, recall that every core outcome lives on a maximum weight matching, cf. Proposition 2 $\square$

Among agents $i \in k_*$, exactly one agent will be matched, specifically agent $i_* = \arg\max_{i \in k_*} \eta_i$. Let j_* be the agent matched to i_* (break ties arbitrarily). Recall that core solutions always live on a maximum weight matching, and in case of multiple maximum weight matchings, the set of vectors α such that (M, α) is a core solution is the same for any maximum weight matching M. This allows us to suppress the matching, and talk about a vector α being in the core, cf. Proposition 2. The (IM) condition in Proposition 2 for the pair of types $(k_*, 1)$ are

$$\eta_{i_*} \geq \alpha_{k_*} \geq \max_{i \in k_* \setminus i_*} \eta_i \,, \tag{36}$$

and the slack condition $\alpha_{k_*} \geq -\epsilon_{j_*}^{k_*}$. The (IM) conditions for types $(k, 1)$ for $k \neq k_*$ are

$$3 + \min_{i \in k} \eta_{i_*} \geq \alpha_k \geq - \min_{j \in M(k)} \epsilon_j^k \,. \tag{37}$$

The stability conditions are

$$\min_{j \in M(k)} \epsilon_j^k - \epsilon_j^{k'} \geq \alpha_{k'} - \alpha_k \geq \max_{j \in M(k')} \epsilon_j^k - \epsilon_j^{k'} \,, \tag{38}$$

[34]In the described construction, we have $n = (2K - 1)\tilde{n} + 1$ for $\tilde{n} = 1, 2, \ldots$ but intermediate values of n can be handled by having slightly fewer workers of type k_*, which leaves our analysis essentially unaffected.

for all $k \neq k'$. It is easy to see that Eq. (38) with $k' = k_*$ implies $\alpha_k \leq 2$ for all $k \neq k_*$. Hence, the upper bound in Eq. (37) is slack. Consider the left stability inequality with $k' = k_*$. As Eq.(36) implies $\alpha_{k_*} \geq 0$, we must have

$$\alpha_k \geq -\min_{j \in M(k)} \epsilon_j^k - \epsilon_j^{k_*} \geq -\min_{j \in M(k)} \epsilon_j^k$$

implying that the lower bound in (37) is also slack. Thus a vector α is in the core if and only if conditions (36) and (38) are satisfied.

For simplicity, we start with the special case $K = 2$, with the two types of labor being k and k_*. To obtain intuition, notice that from Eq. (36) we have $\alpha_{k_*} \xrightarrow{\tilde{n}\to\infty} 1$ in probability, and when we use this together with Eq. (38) we obtain $\alpha_k \xrightarrow{\tilde{n}\to\infty} 2$ in probability. (We do not use these limits in our formal analysis below.) Hence, we focus on Eq. (36) together with

$$\min_{j \neq j_*} \epsilon_j^k - \epsilon_j^{k_*} \geq \alpha_{k_*} - \alpha_k \geq \epsilon_{j_*}^k - \epsilon_{j_*}^{k_*} . \tag{39}$$

where $j_* = \arg\min_j \epsilon_j^k - \epsilon_j^{k_*}$. Now, $X_j = \epsilon_j^k - \epsilon_j^{k_*}$ are distributed i.i.d. with density $U[0,1] * U[-1,0]$ which is

$$f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

(Note that if we draw $\tilde{n}+1$ samples from this distribution, it is not hard to see that $\mathrm{E}[(\min_{j \neq j_*} X_j) - X_{j_*}] = \Theta(1/\sqrt{\tilde{n}})$.) We lower bound the expected core size as follows: Let $X_j = \epsilon_j^k - \epsilon_j^{k_*}$. Let $\mathcal{B}$ be the event that exactly one of the X_j's is in $[-1, -1 + 1/\sqrt{\tilde{n}}]$, and no X_j is in $[-1 + 1/\sqrt{\tilde{n}}, -1 + 2/\sqrt{\tilde{n}}]$. Under f the probability of being in $[-1, -1 + 1/\sqrt{\tilde{n}}]$ is $1/(2\tilde{n})$ and the probability of being in $[-1 + 1/\sqrt{\tilde{n}}, -1 + 2/\sqrt{\tilde{n}}]$ is $3/(2\tilde{n})$. It follows that

$$\Pr(\mathcal{B}) = \binom{\tilde{n}+1}{1, 0, \tilde{n}} \frac{1}{2n} \left(1 - 2/\tilde{n}\right)^{\tilde{n}} = \Omega(1) . \tag{41}$$

Claim D.2. *Consider the case $K = 2$. Under event $\mathcal{B}$, for any core vector (α_{k_*}, α_k), for any* *value* $\alpha'_k \in [\alpha_{k_*} + 1 - 2/\sqrt{\tilde{n}}, \alpha_{k_*} + 1 - 1/\sqrt{\tilde{n}}]$, *we have that vector* $(\alpha_{k_*}, \alpha'_k)$ *is in the core. In particular,* $\mathcal{C} = \Omega(1/\sqrt{\tilde{n}})$.

Proof. Eq. (39) is satisfied since event $\mathcal{B}$ holds. Since, $\alpha_{k'}$ can take any value in an interval of length $1/\sqrt{\tilde{n}}$, it follows that $\mathcal{C} = \Omega(1/\sqrt{\tilde{n}})$ under $\mathcal{B}$. □

Combining Claim D.2 with Eq. (41), we obtain that $\mathrm{E}[\mathcal{C}] = \Omega(1/\sqrt{\tilde{n}})$ as desired.

We now construct a similar argument for $K > 2$, with $\mathcal{K} = \mathcal{T}_{\mathcal{L}} \backslash \{k_*\}$ being the other labor types, all of whose agents are matched. It again turns out that $\alpha_{k_*} \xrightarrow{\tilde{n}\to\infty} 1$ in probability, and when we use this together with Eq. (38) we obtain $\alpha_k \xrightarrow{\tilde{n}\to\infty} 2$ $\forall k \in \mathcal{K}$ in probability (but we do not prove or use these limits).

Considering only the dimensions in $\mathcal{K}$ (recall $|\mathcal{K}| = K - 1$ here) of each ϵ_j, let $\mathcal{B}_3$ be the event as defined in Lemma B.4 with $\delta = 1/n^{0.51}$.

Claim D.3. *Let* $\underline{k} = \arg\min_{k\in\mathcal{K}} \alpha_k$ *and let* $\bar{k} = \arg\max_{k\in\mathcal{K}} \alpha_k$. *Under event* $\mathcal{B}_3$, *we claim that*

$$\alpha_{\bar{k}} - \alpha_{\underline{k}} \leq \delta \tag{42}$$

Proof. From Proposition 2, we know that the set of core α's is a linear polytope, hence it is immediate to see that the set of θ's is an interval. Let $\underline{k} = \arg\min_{k\in\mathcal{K}} \alpha_k$ and let $\bar{k} = \arg\max_{k\in\mathcal{K}} \alpha_k$. Under event $\mathcal{B}_3$, we claim that $\alpha_{\bar{k}} - \alpha_{\underline{k}} \leq \delta$. We can argue this by contradiction: Suppose $\alpha_{\bar{k}} - \alpha_{\underline{k}} > \delta$. One can see that all j's such that $\epsilon_j^{\mathcal{K}} \in \widehat{\mathcal{R}}^{\bar{k},\underline{k}}(\delta)$, cf. (26), will be matched to type $\bar{k}$, with the possible exception of j_*. Thus, under $\mathcal{B}_3$, the number of employers matched to type $\bar{k}$ is bounded below by

$$n^{\bar{k},\underline{k}} - 1 \geq ((K-1)\tilde{n} + 1)/(K-1) > \tilde{n}\,,$$

which is a contradiction, implying (42). □

The above claim bounds the maximum difference between α's corresponding to any pair of types in $\mathcal{K}$. Intuitively, note that all types in $\mathcal{K}$ have the same u and therefore the same distribution for the θ variables of the agents in such type. Moreover, all types in $\mathcal{K}$ have the same number of agents. Hence, one would expect the α's to be equal. While true in the limit, for each finite n we need to account for the stochastic fluctuations in given realization. Therefore, we can show that no pair of α's in $\mathcal{K}$ can differ by more than δ. The next claim follows immediately from Claim D.3.

Claim D.4. *Let $k \in \mathcal{K}$ be an arbitrary type. Under event $\mathcal{B}_3$, we claim that*

$$\max_{k' \in \mathcal{K}} \left(\epsilon_j^{k'} - \epsilon_j^k \right) \leq \delta \quad \forall j \in M(k) \tag{43}$$

Proof. By Claim D.3, we have that under $\mathcal{B}_3$, $|\alpha_k - \alpha_{k'}| \leq \delta$ for all $k' \in cK$. By the stability condition in Eq. (38), we have

$$\delta \geq \alpha_k - \alpha_{k'} \geq \epsilon_j^{k'} - \epsilon_j^k \quad \forall j \in M(k),\ \forall k' \in \mathcal{K}.$$

Therefore, for every $j \in M(k)$ we must have $\delta \geq \max_{k' \in \mathcal{K}} \epsilon_j^{k'} - \epsilon_j^k$ as desired. □

Next, we focus on the stability conditions involving type k_*. For each $k \in \mathcal{K}$, the stability condition is:

$$\epsilon_{j_*}^{k_*} - \epsilon_{j_*}^k \geq \alpha_k - \alpha_{k_*} \geq \max_{j \in M(k)} \epsilon_j^{k_*} - \epsilon_j^k, . \tag{44}$$

where j_* is the employer matched to i_*. For each $j \in \mathcal{E}$, let X_j be defined as $X_j = (\max_{k \in \mathcal{K}} \epsilon_j^k) - \epsilon_j^{k_*}$. The X_j are distributed i.i.d. with cumulative distribution $F(-1+\theta) = \theta^K/K$ for $\theta \in [0,1]$ (we will not be concerned with the cumulative for positive values). Let $\mathcal{B}$ be the event that exactly one of the X_j's is in $[-1, -1 + 1/\tilde{n}^{1/K}]$ (this will be X_{j_*}), and no X_j is in $[-1 + 1/\tilde{n}^{1/K}, -1 + 2/\tilde{n}^{1/K}]$. Under cumulative F, the probability of being in $[-1, -1 + 1/\tilde{n}^{1/K}]$ is $1/(K\tilde{n})$ and the probability of being in $[-1 + 1/\tilde{n}^{1/K}, -1 + 2/\tilde{n}^{1/K}]$ is $2^K/(K\tilde{n})$. It follows that

$$\Pr(\mathcal{B}) = \binom{\tilde{n}+1}{1, 0, \tilde{n}} \frac{1}{Kn} \left(1 - 2^K/(K\tilde{n})\right)^{\tilde{n}} = \Omega(1)\,. \tag{45}$$

Clearly, under $\mathcal{B}$, we must have $j_* = \arg\min_{j \in \mathcal{E}} X_j$. Keeping this in mind, we state and prove our last claim.

Claim D.5. *Suppose $\mathcal{B}_3 \cap \mathcal{B}$ occurs. Take any core vector $(\alpha_{k_*}, (\alpha_k)_{k\in\mathcal{K}})$. Then*

$$\{\theta \in \mathbb{R} : (\alpha_{k_*}, (\alpha_k + \theta)_{k\in\mathcal{K}}) \text{ is in the core}\} \tag{46}$$

is an interval of length at least $1/n^{1/K} - 2\delta = \Omega(1/n^{1/K})$. In particular, $\mathcal{C} \geq \Omega(1/n^{1/K})$.

Proof. Define

$$\underline{\theta} = 1 - 2/\tilde{n}^{1/K} + \delta - \alpha_{\underline{k}} + \alpha_{k_*}$$
$$\overline{\theta} = 1 - 1/\tilde{n}^{1/K} - \alpha_{\bar{k}} + \alpha_{k_*}$$

We claim that, under $\mathcal{B}_3 \cap \mathcal{B}$, we have that $\alpha(\theta) = (\alpha_{k_*}, (\alpha_k + \theta)_{k\in\mathcal{K}})$ is in the core for all $\theta \in [\underline{\theta}, \overline{\theta}]$. To establish this, we need to show that conditions (36) and (38) are satisfied. Since α belongs to the core, we immediately infer that (36) holds, and also (38) when $k_* \notin \{k, k'\}$ by definition of $\alpha(\theta)$. That leaves us with (44). Now, for any $k \in \mathcal{K}$ and $\theta \in [\underline{\theta}, \overline{\theta}]$ we have

$$\alpha_k(\theta) = \alpha_k + \theta \leq \alpha_{\bar{k}} + \theta \leq \alpha_{\bar{k}} + \overline{\theta} = 1 - 1/\tilde{n}^{1/K} + \alpha_{k_*} \leq \epsilon_{j_*}^{k_*} - \epsilon_{j_*}^{k} + \alpha_{k_*}\,,$$

where used the definitions of $\bar{k}$ and $\overline{\theta}$, and the fact that $\mathcal{B}$ occurs (so $1 - 1/\tilde{n}^{1/K} \leq \epsilon_{j_*}^{k_*} - \epsilon_{j_*}^{k}$). This establishes the left inequality in (44). Similarly, for any $k \in \mathcal{K}$ we have

$$\begin{aligned} \alpha_k(\theta) &= \alpha_k + \theta \geq \alpha_{\underline{k}} + \theta \geq \alpha_{\underline{k}} + \underline{\theta} = 1 - 2/\tilde{n}^{1/K} + \delta + \alpha_{k_*} \\ &\geq \epsilon_j^{k_*} - \max_{k'\in\mathcal{K}} \epsilon_j^{k'} + \delta + \alpha_{k_*} \geq \epsilon_j^{k_*} - \epsilon_j^{k} + \alpha_{k_*} \qquad \forall j \in M(k)\,, \end{aligned}$$

where used the definitions of $\underline{k}$ and $\underline{\theta}$ for the first two inqualities, and the fact that $\mathcal{B}$ occurs (so $1 - 2/\tilde{n}^{1/K} \geq \epsilon_j^{k_*} - \max_{k'\in\mathcal{K}} \epsilon_j^{k'}$, $\forall j \in M(k)$). Finally, the last inequality follows from $\mathcal{B}_3$ and Claim D.4 (which implies $-\max_{k'\in\mathcal{K}} \epsilon_j^{k'} + \delta \geq -\epsilon_j^{k}$ for $j \in M(k)$). This establishes the right inequality in (44). Thus, we have shown that $\alpha(\theta)$ is in the core for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The length of this interval is $1/\tilde{n}^{1/K} - (\alpha_{\bar{k}} - \alpha_{\underline{k}}) - \delta \geq 1/\tilde{n}^{1/K} - 2\delta = \Omega(1/\tilde{n}^{1/K})$, using (42). Therefore, that $\mathrm{E}[\mathcal{C}] = \Omega(1/\tilde{n}^{1/K})$ under $\mathcal{B}_3 \cap \mathcal{B}$. ☐

Using Lemma B.4 and Eq. (45) we have

$$\Pr(\mathcal{B}_3 \cap \mathcal{B}) = \Omega(1)\,.$$

Combining with the claim above we obtain that $\mathrm{E}[\mathcal{C}] = \Omega(1/n^{1/K})$. □

E Proof of Theorem 2

We start by restating Theorem 2 and discussing the structure of the proof. Recall that we use the characterization of the size of the core from Lemma C.1.

Theorem (Theorem 2)**.** *Let idiosyncratic productivities be drawn i.i.d. from any fixed distribution F that is supported on an interval of the form $[0, C_u]$ or $[0, \infty)$ where $C_u < \infty$, and whose density f is strictly positive and continuous everywhere on the support.*[35] *In addition, suppose that $u(k, 1) \geq 0$ for all $k \in \mathcal{T}_{\mathcal{L}}$. Consider the setting in which $K \geq 2$, $Q = 1$, $n_{\mathcal{E}} > n_{\mathcal{L}}$ and let $m = n_{\mathcal{E}} - n_{\mathcal{L}}$. Under Assumption 2, we have, with high probability, that $\mathcal{C} \leq O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right)$. Also, $E[\mathcal{C}] \leq O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right)$.*

Note that Assumption 1 is automatically satisfied under the hypotheses of the theorem.

As in the proof of Theorem 1, we first prove the result for F being Uniform$(0, 1)$, and later extend our proof to more general F. The idea of the proof is as follows. First, we show a bound on the expectation of $\min_{k \in \mathcal{T}_{\mathcal{L}}} \{\alpha_k^{\max} - \alpha_k^{\min}\}$. In particular, we show that $\mathrm{E}\left[\min_{k \in \mathcal{T}_{\mathcal{L}}} \{\alpha_k^{\max} - \alpha_k^{\min}\}\right] = O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right)$. To do so, we note that by condition (IM) in Proposition 2, we must have

$$\min_k \left(\alpha_k^{\max} - \alpha_k^{\min}\right) \leq \min_{k \in \mathcal{T}_{\mathcal{L}}} \left(\min_{j \in M(k)} \epsilon_j^k - \max_{j \in U} \epsilon_j^k\right).$$

Then, we consider two separate cases to prove the result, depending the size of the imbalance. When $m \leq \log(n)$, the result is shown in Lemma E.1, which we prove via an upper bound on $\min_{k \in \mathcal{T}_{\mathcal{L}}} \left(\min_{j \in M(k)} \epsilon_j^k\right)$. On the contrary, when $m \geq \log(n)$, the result is shown in Lemma E.4.

[35]If the lower limit of the support of F is positive, this can be absorbed into the u's, hence this case is covered.

The proof of Lemma E.4 relies mainly on the geometry of a core solution which (roughly) allows us to first control the largest of the α's (all α's must be negative i in the core since some employers are unmatched, and we control, roughly, the least negative α).

Next, we then show that, for every pair of types $k, q \in \mathcal{T}_{\mathcal{L}}$ we must have

$$\mathrm{E}\left[\min_{j\in M(k)}(\epsilon_j^k - \epsilon_j^q) - \max_{j\in M(q)}(\epsilon_j^k - \epsilon_j^q)\right] = O^*\left(\frac{1}{n}\right).$$

By Condition (ST) in Proposition 2, this implies that for fixed $k, q \in \mathcal{T}_{\mathcal{L}}$, the expected maximum variation in $\alpha_k - \alpha_q$ in the core is bounded by $O^*\left(\frac{1}{n}\right)$.

Finally, we use the bounds in the first two steps to argue that, for every type $k \in \mathcal{T}_{\mathcal{L}}$,

$$\mathrm{E}\left[\alpha_k^{\max} - \alpha_k^{\min}\}\right] = O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right),$$

which implies $\mathrm{E}[\mathcal{C}] = O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right)$. This is done in the proof of Theorem 2.

We now show our bound on $\mathrm{E}\left[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}\right]$. To that end, let $Z_k = \min_{j\in M(k)}\epsilon_j^k$ and $U_k = \max_{j\in U}\epsilon_j^k$. By Condition (IM) in Proposition 2, $\mathrm{E}\left[\min_k|\alpha_k^{\max} - \alpha_k^{\min}|\right] \leq \mathrm{E}[\min_k\{Z_k - U_k\}]$, and therefore we will focus on bounding $\mathrm{E}[\min_k\{Z_k - U_k\}]$. As a reminder, we have defined $m = n_{\mathcal{E}} - n_{\mathcal{L}}$ and $\delta_n = \frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}$. Also, in all lemmas we are working under the assumptions of the theorem, that is, $K \geq 2, Q = 1$, $n_{\mathcal{E}} > n_{\mathcal{L}}$ and Assumption 2.

Lemma E.1. *Suppose $m \leq 6K\log(n_{\mathcal{E}})$. Then, there exists a constant $C_3 = C_3(K) < \infty$ such that $E\left[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}\right] \leq 2C_3\frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}$.*

Proof. Let $Z_k = \min_{j\in M(k)}\epsilon_j^k$, $U_k = \max_{j\in U}\epsilon_j^k$ and $\delta_n = \frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}$. By Condition (IM) in Proposition 2, $\mathrm{E}[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}] \leq \mathrm{E}[\min_k\{Z_k - U_k\}]$. As U_k is a non-negative random variable, we have $\mathrm{E}[\min_k\{Z_k - U_k\}] \leq \mathrm{E}[\min_k\{Z_k\}]$. Therefore,

$$\mathrm{E}[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}] \leq \mathrm{E}\left[\min_k\{Z_k - U_k\}\right] \leq \mathrm{E}\left[\min_k\{Z_k\}\right] \leq C_3\delta_n + \Pr\left(\min_k Z_k \geq C_3\delta_n\right),$$

using $Z_k \leq 1$.

To finish the proof, it suffices to show that $\Pr(\min_k Z_k \geq C_3\delta_n) \leq C_3\delta_n$. Hence, our next step is to bound $\Pr(\min_k Z_k \geq C_3\delta_n)$. Now $\min_k Z_k \geq C_3\delta_n$ implies that all j such that $\epsilon_j \in [0, C_3\delta_n]^K$ are unmatched. But there are only m unmatched employers. It follows that

$$\Pr\left(\min_k Z_k \geq C_3\delta_n\right) \leq \Pr\left(\text{at most } m \text{ points in the hypercube } [0, C_3\delta_n]^K\right)$$

Let $X \sim \text{Bin}\left(n_{\mathcal{E}}, (C_3\delta_n)^K\right)$ be defined as the number of points, out of $n_{\mathcal{E}}$ in total, that fall in the hypercube $[0, C_3\delta_n]^K$. By assumption, $m \leq 6K\log(n) \Rightarrow (C_3\delta_n)^K \geq (C_3 \log n/m)^K/n \geq 2^K \log n^K/n \geq 4(\log n)^2/n$ defining $C_3 \geq 12K$ and using $K \geq 2$. Further using $n \leq 2n_{\mathcal{E}}$ we obtain $\mathrm{E}[X] = n_{\mathcal{E}}\,(C_3\delta_n)^K \geq (n/2)4(\log n)^2/n = 2(\log n)^2$. It follows that

$$\Pr\left(\min_k Z_k \geq C_3\delta_n\right) \leq \Pr\left(X \leq 6K\log(n)\right) \leq \exp(-\Omega((\log n)^2)) \leq \frac{1}{n} \leq C_3\delta_n$$

where the second inequality was obtained by applying the Chernoff bound. Hence, we have shown that

$$\mathrm{E}[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}] \leq \mathrm{E}\left[\min_k\{Z_k\}\right] \leq C_3\delta_n + \Pr\left(\min_k Z_k \geq C_3\delta_n\right) \leq 2C_3\delta_n,$$

which completes the proof. □

We now establish an upper bound for the case in which $m \geq 6K\log(n_{\mathcal{E}})$. For the following results up to Lemma E.4 we shall assume $m \geq 6K\log(n_{\mathcal{E}})$.

Before we move on, we briefly give some geometric intuition regarding the problem. For each agent $j \in \mathcal{E}$, let $\epsilon_j = (\epsilon_j^1, \ldots, \epsilon_j^K)$ denote the profile of values assigned by the K types of agents in $\mathcal{L}$ to agent j. Given our stochastic assumptions, all points ϵ_j will be distributed in the $[0,1]^K$ hypercube. Using Proposition 2, we can partition the $[0,1]^K$-hypercube into $K+1$ disjoint regions: K of them containing the n_k points corresponding to agents matched to type k $(1 \leq k \leq K)$, and one region containing all unmatched agents. Furthermore, the region containing the unmatched agents is an orthotope[36] that has the origin as a vertex. This follows for the (IM) constraints in Proposition 2.

To that end, let $\mathcal{O}$ be the set of K-orthotopes contained in $[0,1]^K$ that have the origin as a vertex. Suppose R is expanded by the same amount θ in each coordinate direction. Define $D(R)$ as the smallest value of θ such that an additional point ϵ_j is contained in the expanded orthotope. (If one of the side lengths becomes 1 before an additional point is reached, then define $D(R) = 0$. This will never occur for R that contains only the unmatched agents.) As usual, let $Z_k = \min_{j \in M(k)} \epsilon_j^k$ and $U_k = \max_{j \in U} \epsilon_j^k$. We want to show that $\mathrm{E}\left[\min_k\{Z_k - U_k\}\right] \leq C_5 \delta_n$, for some constant $C_5 = C_5(K) < \infty$. To that end, note that $\min_k\{Z_k - U_k\}$ is equal to $D(R)$ for some orthotope $R \in \mathcal{O}$. In particular, $\min_k\{Z_k - U_k\}$ is equal to $D(R)$ when R is the orthotope that "tightly" contains all the m points in U.

For $R \in \mathcal{O}$, let $V(R)$ be defined as the volume of R. In addition, we define $|R|$ to be the number of points contained in R. We start by showing that, given that $m \geq 6K\log(n)$, an orthotope in $\mathcal{O}$ of volume less than $\frac{m}{4n_{\mathcal{E}}}$ in extremely unlikely to contain m points.

Lemma E.2. *Suppose $m \geq 6K\log(n)$. For $R \in \mathcal{O}$ such that $V(R) < \frac{m}{4n_{\mathcal{E}}}$, we have $\Pr\left(|R| = m\right) \leq \frac{1}{n^{K+1}}$, where $V(R)$ denotes the volume and $|R|$ denotes the number of points in R.*

Proof. Let X denote number of points in an orthotope in $\mathcal{O}$ of volume $\frac{m}{4n_{\mathcal{E}}}$. Then, $X \sim \mathrm{Bin}\left(n_{\mathcal{E}}, \frac{m}{4n_{\mathcal{E}}}\right)$. We have $\mu = \mathrm{E}[X] = m/4$. Using a Chernoff bound we have,

$$\Pr(X \geq m) = \Pr(X \geq 4\mu) \leq (e^3/4^4)^{m/4} \leq \exp(-m/4)$$

Now $m/4 \geq 6K\log n/4 \geq (K+1)\log n$, using $K \geq 2$. Substituting back we obtain $\Pr(X \geq m) \leq \exp(-(K+1)\log n) = 1/n^{K+1}$. But $|X|$ stochastically dominates $|R|$ since $V(R) < \frac{m}{4n_{\mathcal{E}}}$. The result follows. □

Our next step will be to bound $\Pr\left(D(R) > C_4\delta \,\middle|\, E\right)$, for $R \in \mathcal{O}$ and some constant $C_4 = C_4(K) < \infty$ where E is the event defined as $E = \{|R| = m,\ V(R) \geq \frac{m}{4n_{\mathcal{E}}}\}$.

[36] An orthotope (also called a hyperrectangle or a box) is the generalization of a rectangle for higher dimensions

Lemma E.3. *There exists some constant $C_4 = C_4(K) < \infty$ such that, for all $R \in \mathcal{O}$ with $V(R) \geq \frac{m}{4n_{\mathcal{E}}}$, we have that $P\left(D(R) > C_4\delta_n \,\middle|\, |R| = m\right) \leq \frac{1}{n^{K+1}}$, where $\delta_n = \frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}$.*

Proof. Conditioned on $|R| = m$, the remaining $n_{\mathcal{L}} = n_{\mathcal{E}} - m$ points are distributed uniformly i.i.d. in the complementary region of volume $(1 - V(R))$.

Let $F_{C_4\delta_n}$ denote the region swept when R is expanded by $C_4\delta_n$ along each coordinate axis. Clearly, $D(R) > C_4\delta_n$ if and only if region $F_{C_4\delta_n}$ contains no points.

Let X denote the number of points in $F_{C_4\delta_n}$, and let p denote the volume of $F_{C_4\delta_n}$. Then, $X \sim \text{Bin}(n_{\mathcal{L}}, p/(1 - V(R)))$ and hence stochastically dominates $\text{Bin}(n_{\mathcal{L}}, p))$. Note that such a volume p is at least the volume obtained when expanding the hypercube of side $\ell = \sqrt[K]{\frac{m}{4n_{\mathcal{E}}}}$ by $C_4\delta_n$ along each direction and therefore, $p \geq K\ell^{(K-1)}C_4\delta_n$. Hence,

$$
\begin{aligned}
P(D(R) > C_4\delta_n) = \Pr(X = 0) \leq (1 - p)^{\mathcal{L}} \leq \exp\left\{-\Omega(np)\right\} \\
\leq \exp\{-\Omega(n(\frac{m}{n})^{(K-1)/K}C_4\delta_n)\} = \exp\{-\Omega(C_4 \log n)\} \leq \frac{1}{n^{K+1}}\,,
\end{aligned}
$$

for appropriate C_4, where we have used Assumption 2.

$\square$

Lemma E.4. *Suppose $m \geq 6K\log(n)$. Then, there exists a constant $C_5 = C_5(K) < \infty$, such that $E\left[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}\right] \leq C_5\frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}$.*

Proof. Let $Z_k = \min_{j\in M(k)} \epsilon_j^k$, $U_k = \max_{j\in U} \epsilon_j^k$ and $\delta_n = \frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}$. By Condition (IM) in Proposition 2, we know that $\alpha_k^{\max} - \alpha_k^{\min} \leq Z_k - U_k$. Then,

$$
\mathrm{E}\left[\min_k\{\alpha_k^{\max} - \alpha_k^{\min}\}\right] \leq \mathrm{E}\left[\min_k\{Z_k - U_k\}\right].
$$

In addition, $\min_k\{Z_k - U_k\}$ is equal to $D(R)$ for some orthotope $R \in \mathcal{O}$. In particular, $\min_k\{Z_k - U_k\}$ is equal to $D(R)$ when R is the orthotope that "tightly" contains all the m points in U.

Define $\mathcal{R} = \{R \in \mathcal{O} : \ |R| = m\}$. Then,

$$\mathrm{E}\left[\min_k\{Z_k - U_k\}\right] \ \leq \ \mathrm{E}\left[\max_{R\in\mathcal{R}}\{\mathrm{D(R)}\}\right].$$

To bound $\mathrm{E}\left[\max_{R\in\mathcal{R}}\{\mathrm{D(R)}\}\right]$, consider the grid that results from dividing each of the K coordinate axes in the hypercube into intervals of length $1/n$. Let Δ denote that grid. Suppose we just consider orthotopes in the grid, that is, the orthotopes whose sides are multiples of $\frac{1}{n}$. Let $\mathcal{R}_\Delta = \{R \in \mathcal{R} : \ R \in \Delta\}$. Then,

$$\max_{R\in\mathcal{R}}\{\mathrm{D(R)}\} \leq \max_{R\in\mathcal{R}_\Delta}\{\mathrm{D(R)}\} + \frac{1}{n},$$

and,

$$\mathrm{E}\left[\max_{R\in\mathcal{R}}\{\mathrm{D(R)}\}\right] \leq \mathrm{E}\left[\max_{R\in\mathcal{R}_\Delta}\{\mathrm{D(R)}\}\right] + \frac{1}{n}.$$

Hence, we just need a bound for $\mathrm{E}\left[\max_{R\in\mathcal{R}_\Delta}\{\mathrm{D(R)}\}\right]$. Let $V_* = \frac{m}{4n}$. Note that $\mathrm{D(R)} \leq 1$ for all $R \in \mathcal{O}$ and therefore,

$$\mathrm{E}\left[\max_{R\in\mathcal{R}_\Delta}\{\mathrm{D(R)}\}\right] \ \leq \ \mathrm{E}\left[\max_{R\in\mathcal{R}'_\Delta}\{\mathrm{D(R)}\}\right] + \Pr\left(\min_{R\in\mathcal{R}_\Delta} V(R) < V_*\right)$$

where $\mathcal{R}'_\Delta = \{R \in \mathcal{R}_\Delta : V(R) \geq V_*\}$. Now, by union bound

$$\Pr\left(\min_{R\in\mathcal{R}_\Delta} V(R) < V_*\right) \leq \sum_{R\in\Delta:V(R)<V_*} \Pr(|R| = m) \leq n^K \cdot 1/n^{K+1} = 1/n\,.$$

using $|\{R \in \Delta : V(R) < V_*\}| \leq |\{R \in \Delta\}| = n^K$ and Lemma E.2.

Further,

$$\mathrm{E}\left[\max_{R\in\mathcal{R}'_\Delta}\{\mathrm{D(R)}\}\right] \leq \mathrm{E}\left[\max_{R\in\Delta:V(R)\geq V_*}\{\mathrm{D(R)}\mathbb{I}(|R| = m)\}\right]$$

Now,

$$\begin{aligned}
&\Pr\left[\max_{R\in\Delta:V(R)\geq V_*}\left\{\mathrm{D(R)}\mathbb{I}(|R|=m)\right\}>C_4\delta_n\right]\\
\leq&\sum_{R\in\Delta:V(R)\geq V_*}\Pr(|R|=m)\Pr[\mathrm{D(R)}>C_4\delta_n||R|=m]\\
\leq&\sum_{R\in\Delta:V(R)\geq V_*}1\cdot 1/n^{K+1}\leq n^K/n^{K+1}=1/n
\end{aligned}$$

using a union bound and Lemma E.3 to bound the probability of $D(R)\geq C_4\delta_n$. It follows that

$$\mathrm{E}\left[\max_{R\in\mathcal{R}'_\Delta}\left\{\mathrm{D(R)}\right\}\right]\leq\ 1\cdot\Pr\left[\max_{R\in\Delta:V(R)\geq V_*}\left\{\mathrm{D(R)}\mathbb{I}(|R|=m)\right\}>C_4\delta_n\right]+C_4\delta_n=1/n+C_4\delta_n$$

Substituting the individual bounds back, we obtain

$$\mathrm{E}\left[\max_{R\in\mathcal{R}}\left\{\mathrm{D(R)}\right\}\right]=C_4\delta_n+2/n\leq C_5\delta_n\,.$$

defining $C_5=C_4+2$ and using $1/n\leq\delta_n$.

Overall,

$$\mathrm{E}\left[\min_k\{\alpha_k^{\max}-\alpha_k^{\min}\}\right]\ \leq\ \mathrm{E}\left[\min_k\{Z_k-U_k\}\right]\ \leq\ \mathrm{E}\left[\max_{R\in\mathcal{R}}\left\{\mathrm{D(R)}\right\}\right]\leq C_5\delta_n$$

as claimed. □

We now proceed to show that, for every pair of types $k,q\in\mathcal{T}_{\mathcal{L}}$ we have

$$\mathrm{E}\left[\min_{j\in M(k)}(\epsilon_j^k-\epsilon_j^q)-\max_{j\in M(q)}(\epsilon_j^k-\epsilon_j^q)\right]\leq C_2\frac{\log(n_{\mathcal{E}})}{n_{\mathcal{E}}}.$$

for appropriate $C_2=C_2(K)<\infty$. This result is shown in Lemma E.7. Along the way, we establish a couple of intermediate results.

Let $Z_k=\min_{j\in M(k)}\epsilon_j^k$ and $U_k=\max_{j\in U}\epsilon_j^k$. Note that Z_k is an upper bound for $-\alpha_k$. By the definition of Z_k, all the points corresponding agents in $M(k)$ must be contained in the orthotope $[1 - Z_k, 1] \times [0, 1]^{K-1}$. The following proposition establishes that Z_k cannot be arbitrarily close to 1.

Lemma E.5. *Given a constant $c \in \mathbb{R}$, let the event E_c be defined as $E_c = \{\max_k \min_{j \in M(k)} \epsilon_j^k \leq 1 - c\}$. Then, there exist constants $\theta = \theta(K) > 0$ and $C_6 = C_6(K) > 0$ such that, for large enough n, E_θ occurs with probability at least $1 - \exp(-C_6 n)$.*

Proof. Let $Z_k = \min_{j \in M(k)} \epsilon_j^k$. The proof follows from the previous observation that all the points corresponding to agents in $M(k)$ must be contained in the orthotope of volume $(1 - Z_k)$. Let $C < \infty$ be such that $\dfrac{n_{\mathcal{E}}}{n_{\mathcal{L}}} \leq C$. By Assumption 2, such a C must exist. Furthermore, by Assumption 2, there must exists $C_K \in \mathbb{R}$ such that $n_k \geq C_K n$ for all $k \in \mathcal{T}_{\mathcal{L}}$. Let $n_{\mathcal{E}}$ be the total number of points in the cube $[0, 1]^K$. Let X denote the number of points out of the $n_{\mathcal{E}}$ ones that fall in the rectangle defined by $[1 - \theta, 1][0, 1]^{K-1}$. Then, $X \sim \mathrm{Bin}(n_{\mathcal{E}}, \theta)$. Suppose we set $\theta < \frac{C_K}{2C}$.Then, for large enough n and appropriate $C_6 > 0$ we have

$$\Pr(Z_k > 1 - \theta) \leq \Pr(X \geq C_K n_{\mathcal{L}}) \leq \Pr\left(X \geq \frac{C_K n_{\mathcal{E}}}{C}\right) \leq \exp(-2C_6 n) \leq (1/K)\exp(-C_6 n)$$

where we have used a Chernoff bound, $2n_{\mathcal{E}} \geq n$, and $\exp(-C_6 n) \leq (1/K)$ for large enough n. The result follows from a union bound over possible k. $\square$

Remark E.1. *Let θ, E_θ and C_6 be as defined in the statement of Lemma E.5. Define $G_{k,q}$ as*

$$G_{k,q} = \left\{ x \in [0, 1]^K : \ \left(x_k \geq 1 - \frac{\theta}{2} \text{ or } x_q \geq 1 - \frac{\theta}{2} \right) \text{ and } x_r < \frac{\theta}{2} \text{ for all } 1 \leq r \leq K, \ r \neq k, q \right\}.$$

Under event E_θ, we must have $G_{k,q} \subseteq M(k) \cup M(q)$.

The above remark follows from Lemma E.5 and the definition of $G_{k,q}$. If $j : \epsilon_j \in G_{k,q}$ were matched to a type $k' \notin \{k, q\}$, that will contradict maximality of the matching as, by swapping the matches of $j' : j' \in M(k), \epsilon^k_{j'} = Z_k$ and j, the overall weight of the matching strictly increases. A similar argument rules out j being unmatched.

Lemma E.6. *Let $G_{k,q}$ be as in Remark E.1, and let θ be as defined in Lemma E.5. Define $G'_{k,q}$ as follows:*

$$G'_{k,q} = G_{k,q} \cap \{x \in [0,1]^K, |x_k - x_q| \leq 1 - \theta\}$$

Let $\mathcal{V}^{kq} = \{x : \; x = \epsilon^k_j - \epsilon^q_j, \; \epsilon_j \in G'_{k,q}\}$, and let

$$V^{kq} = \max(\textit{Difference between consecutive values in } \mathcal{V}^{kq} \cup \{-1+\theta, 1-\theta\}).$$

Then, there exists a function $f(n) = O^*(1/n)$ such that $\Pr\left(\overline{\mathcal{B}^{kq}}\right) \leq 1/n$ where $\mathcal{B}^{kq}$ is the event that $V^{kq} \leq f(n)$.

The proof of Lemma E.6 is omitted as the required analysis is similar to (and much simpler than) that leading to Lemma B.1. Essentially, V^{kq} consists of values taken by $\Theta(n)$ points distributed uniformly and independently in $[-1+\theta, 1-\theta]$, so, with high probability, no two consecutive values are separated by more than $f(n) = O(\log n/n)$.

In the next lemma we bound the difference between every pair of α's.

Lemma E.7. *Consider types $k, q \in \mathcal{T}_{\mathcal{L}}$ and let f be as defined in the statement of Lemma E.6. Under event $E_\theta \cap \mathcal{B}^{kq}$, in every stable solution we must have that $(\alpha^{\max}_q - \alpha^{\min}_q) \leq 2f(n) + (\alpha^{\max}_k - \alpha^{\min}_k)$.*

Proof. We claim that under E_θ, we must have $\alpha_q - \alpha_k$ varies within a range of no more than V^{kq} within the core, where V^{kq} is as defined in the statement of Lemma E.6. By Remark E.1, under event E_θ we must have $G'_{kq} \subset M(k) \cup M(q)$, where G'_{kq} is as defined in the statement of Lemma E.6. Suppose that G'_{kq} contains at least one vertex matched to type k and one to type

q. Then, by Condition (ST) in Proposition 2 we must have:

$$
\begin{aligned}
(\alpha_q - \alpha_k)^{\max} - (\alpha_q - \alpha_k)^{\min} &\leq \min_{j \in M(k)} \{\epsilon_j^k - \epsilon_j^q\} - \max_{j \in M(q)} \{\epsilon_j^k - \epsilon_j^q\} \\
&\leq \min_{j \in M(k) \cap G'_{k,q}} \{\epsilon_j^k - \epsilon_j^q\} - \max_{j \in M(q) \cap G'_{k,q}} \{\epsilon_j^k - \epsilon_j^q\} \\
&\leq V^{kq}
\end{aligned}
$$

Next, consider the case in which all vertices in G'_{kq} are matched to type k (the analogous argument follows if they are all matched to type q). Under event E_θ, by Condition (IM) in Proposition 2 we must have $0 \leq -\alpha_k \leq 1 - \theta$ and $0 \leq -\alpha_q \leq 1 - \theta$. Therefore, $\alpha_q - \alpha_k \in [-1 + \theta, 1 - \theta]$. In addition, by Condition (ST) in Proposition 2 we must have $\alpha_q - \alpha_k \leq \min_{j \in M(k)} \{\epsilon_j^k - \epsilon_j^q\}$. However,

$$
\begin{aligned}
(\alpha_q - \alpha_k)^{\max} - (\alpha_q - \alpha_k)^{\min} &\leq \min_{j \in M(k)} \{\epsilon_j^k - \epsilon_j^q\} - (-1 + \theta) \\
&\leq \min_{j \in M(k) \cap G'_{k,q}} \{\epsilon_j^k - \epsilon_j^q\} - (-1 + \theta) \\
&= \min_{j \in G'_{k,q}} \{\epsilon_j^k - \epsilon_j^q\} - (-1 + \theta) \\
&\leq V^{kq}
\end{aligned}
$$

It follows that $(\alpha_q^{\max} - \alpha_q^{\min}) \leq 2V^{kq} + (\alpha_k^{\max} - \alpha_k^{\min})$. By definition, under $\mathcal{B}^{kq}$ we have $V^{kq} \leq f(n)$, which completes the proof. $\square$

Finally, we complete the last step of the proof by showing the main theorem.

Proof of Theorem 2. By definition, $\mathcal{C} = \sum_{k=1}^{K} \dfrac{N(k)|\alpha_k^{\max} - \alpha_k^{\min}|}{n_{\mathcal{L}}}$, where $N(k)$ is defined to be the number of agents of type k that are matched. For a given instance, let $k^* = \operatorname{argmin}_k \{\alpha_k^{\max} - \alpha_k^{\min}\}$. Let $\mathcal{B} = E_\theta \cap (\cap_{k,q} \mathcal{B}^{k,q})$. Note that using Lemmas E.1 and E.6 and a union bound, we obtain that

$$
\Pr(\overline{\mathcal{B}}) \leq \Pr(\overline{E_\theta}) + \sum_{k,q \in \mathcal{K}: k \neq q} \Pr(\overline{\mathcal{B}^{kq}}) = O(1/n)\,.
$$

By Lemma E.7, under $\mathcal{B}$, for every $k \in \mathcal{T}_{\mathcal{L}}$ we have

$$\alpha_k^{\max} - \alpha_k^{\min} \leq 2f(n) + \alpha_{k^*}^{\max} - \alpha_{k^*}^{\min}.$$

Therefore,

$$\begin{aligned}
\mathrm{E}[\mathcal{C}] &\leq \mathrm{E}\left[\alpha_{k^*}^{\max} - \alpha_{k^*}^{\min}\right] + 2f(n) + \Pr(\overline{\mathcal{B}}) \cdot O(1) \\
&\leq O\left(\frac{\log(n)}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right) + O^*(1/n) + O(1/n) \\
&= O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right)
\end{aligned}$$

where the first inequality follows from the above together with using the upperbound of $O(1)$ for the core size; the second inequality is obtained by using the bound on $\mathrm{E}\left[\alpha_{k^*}^{\max} - \alpha_{k^*}^{\min}\right]$ from Lemma E.1 for $m \leq 6K\log(n)$ and Lemma E.4 for $m \geq 6K\log(n)$, as well as the definition of $f(n)$ and $\Pr(\overline{\mathcal{B}}) = O(1/n)$ shown above. Finally, we note that each of the steps yields a bound that holds with high probability (instead of a bound on expected value) if we multiply by an additional $\log n$ factor. This yields that, with high probability,

$$\mathcal{C} \leq O^*\left(\frac{1}{n^{\frac{1}{K}} m^{\frac{K-1}{K}}}\right).$$

This completes the proof for F being Uniform$(0,1)$.

Now consider general F satisfying the conditions stated in the theorem. We extend our proof just as we extended the proof of Theorem 1 to general F. Lemma 2 gives us a lower bound of $-U \leq \alpha_{kq}$ for any core α, whereas we already know that $-C_u \geq \alpha_{kq} \leq 0$, since some employers are unmatched and others are matched. Overall, we know $\alpha_{kq} \in [-\min(U, C_u), 0]$. We scale utilities down by a factor $\min(U, C_u)$. Now the density of firm productivities is uniformly lower bounded in $[0,1]$, and firm productivities values in $[0,1]$ are the only relevant ones. Moreover, $\Omega(n)$ firms have all productivities lying in $[0,1]$. Hence, our bound on the gap between consecutive order statistics and their linear combinations in the unit hypercube implies the same bound (up to constant factors) for general F, and hence we get the same bound on core size for general

F.

□

F Proof of Theorem 3

In this appendix, we state and prove a more detailed version of Theorem 3.

Theorem 5. *Fix K and Q and a distribution F. Also fix the fraction ρ_k of each agent type k, where k can be a type of worker or a type of employer. We draw a market with n agents by independently drawing the type of each agent, and then independently drawing the idiosyncratic productivities from the distribution F. We obtain the following bounds as a function of n.*

- **Limit characterization of α.** *There exists $\alpha^* = \alpha^*(K, Q, \rho, F)$ such that as $n \to \infty$, we have that both $\alpha^{\max}$ and $\alpha^{\min}$ converge (in probability and almost surely) to α^*. In fact, for any $h_n = \omega(1)$ we have that with high probability, for every core outcome (M, α),*

$$\|\alpha - \alpha^*\| \leq h_n/\sqrt{n}\,. \tag{47}$$

- **Size of the core.** *There exists $C = C(K, Q, \rho, F) < \infty$ such that with high probability, the size of the core is bounded as*

$$\mathcal{C} \leq C \log n/n\,. \tag{48}$$

We formally establish the claims stated as part of the proof at the end of this appendix.

Proof of Theorem 5. We define a discrete tatonnement operator $\mathsf{T} : \mathbb{R}^{K+Q} \to \mathbb{R}^{K+Q}$ that takes a price vector α to an updated price vector $\mathsf{T}\alpha$ in a limit market (we do not formally define the limit market, but it guides our intuition). The operator T will be monotone, as a result of a gross substitutes property on both sides of the market. Its fixed point α^* will be turn out to be the limiting core solution. We also define a corresponding operator T_n for the n-th market to relate core outcomes α in that market to α^*. Our approach draws on the order-theoretic/lattice-theoretic approach that has yielded rich dividends in matching theory (Gale and Shapley 1962, Kelso and Crawford 1982, Hatfield and Milgrom 2005) and in the study of equilibrium more broadly.

We start by defining the "limit" operator T. The idea is that when T acts on a price vector α to produce an updated price vector $\mathsf{T}\alpha$, it computes each price $(\mathsf{T}\alpha)_{kq}$ as follows: treating all other prices as given, it computes the limiting "demand curve" $D_{kq}(p)$ of the number of type k workers who want to match with type q employers as a function of the price between type-pair (k, q) (here p is the dummy variable for this price), and the limiting "supply curve" $S_{kq}(p)$ of the number of type q employers who want to match with type k workers. Then $(\mathsf{T}\alpha)_{kq}$ is set to the value of p at which the demand and supply curves intersect. (There will be a unique such point, since $D_{kq}(p)$ will be strictly decreasing in p and $S_{kq}(p)$ will be strictly increasing in p.) We define

$$D_{kq}(p; \alpha) = \rho_k \int_0^\infty f(z + p - u(k, q)) \prod_{q' \neq q} F(z + \alpha_{kq'} - u(k, q'))\, \mathrm{d}z \tag{49}$$

(where z serves as a dummy variable for the utility earned by an agent of type k, who finds type q at price p more attractive to match with than any other type q' at price $\alpha_{kq'}$) and

$$S_{kq}(p; \alpha) = \rho_q \int_0^\infty f(z - p) \prod_{k' \neq k} F(z - \alpha_{k'q})\, \mathrm{d}z\,. \tag{50}$$

We now suppress the α dependence, simply writing $D_{kq}(p)$ and $S_{kq}(p)$. It is easy to verify that $D_{kq}(\cdot)$ is strictly monotone decreasing and that $S_{kq}(\cdot)$ is strictly monotone increasing, given that $f(\cdot)$ is positive everywhere. Further, both $D_{kq}(\cdot)$ and $S_{kq}(\cdot)$ are positive everywhere, and $\lim_{p\to\infty} D_{kq}(\cdot) = 0$ and $\lim_{p\to-\infty} S_{kq}(\cdot) = 0$. Combining, we deduce that there is a unique price where the demand curve $D_{kq}(\cdot)$ and the supply curve $S_{kq}(\cdot)$ intersect, and we define $(\mathsf{T}\alpha)_{kq}$ to be equal to this price.

Strict gross substitutes property on both sides of the limit market: Write $\alpha \preceq \alpha'$ if $\alpha_{kq} \le \alpha'_{kq}$ for all $k \in \mathcal{T}_{\mathcal{L}}, q \in \mathcal{T}_{\mathcal{E}}$, and $\alpha \prec \alpha'$ if at least one of these inequalities is strict. Now viewing $D_{kq}(p; \cdot)$ as a function of α, it is easy to verify that $D_{kq}(p; \cdot)$ is monotone increasing with respect to the defined ordering on α's (the demand for type q increases if the price of other employer types increases); i.e., if $\alpha \preceq \alpha'$, then $D_{kq}(p;\alpha) \leq D_{kq}(p;\alpha')$ for all p. In other words, the demand from each type of worker satisfies strict gross substitutes. Similarly, $S_{kq}(p;\cdot)$ is monotone decreasing, by the strict gross substitutes property for each type of employer.

Thus, increasing α causes the demand curve to move up and the supply curve to move down, and hence it causes the price at which they intersect to move up. It follows that T is monotone increasing in α.

Fact F.8. *If $\alpha \preceq \alpha'$ then $\mathsf{T}\alpha \preceq \mathsf{T}\alpha'$.*

We now show existence of a fixed point of T, thereby overcoming the challenge that the set of possible α's is not compact. To do this, we show that there is a (large) price P such that with $\alpha_P := (P, P, \ldots, P)$ we have $\mathsf{T}\alpha_P \preceq \alpha_P$, and another (large negative) price P' such that with $\alpha_{P'} := (P', P', \ldots, P')$ we have $\mathsf{T}\alpha_{P'} \succeq \alpha_{P'}$. Then, starting with α_P and iteratively applying T, we get a fixed point α^* using monotonicity.

Set all prices to be equal to P. Now P is the only variable. We have

$$D_{kq}(P;\alpha_P) = \rho_k \int_P^\infty f(w - u(k,q)) \prod_{q' \neq q} F(w - u(k,q'))\, \mathrm{d}w\,. \tag{51}$$

Note that $D_{kq}(P;\alpha_P)$ is positive everywhere, decreasing in P, and $\lim_{P\to\infty} D_{kq}(P;\alpha_P) = 0$. We also have that $\min_{k,q} D_{kq}(P;\alpha_P)$ is positive everywhere, decreasing in P, and

$$\lim_{P\to\infty} \min_{k,q} D_{kq}(P;\alpha_P) = 0\,.$$

Similarly,

$$S_{kq}(P;\alpha_P) = \rho_q \int_{-P}^\infty f(w) \prod_{k' \neq k} F(w)\, \mathrm{d}w\,. \tag{52}$$

Note that $S_{kq}(P;\alpha_P)$ is positive everywhere, increasing in P and $\lim_{P\to-\infty} S_{kq}(P;\alpha_P) = 0$. We also have $\min_{k,q} S_{kq}(P;\alpha_P)$ is positive everywhere, increasing in P and

$$\lim_{P\to-\infty} \min_{k,q} S_{kq}(P;\alpha_P) = 0\,.$$

It follows that for large enough P we have $\mathsf{T}\alpha_P \preceq \alpha_P$ and for small enough P' we have $\mathsf{T}\alpha_{P'} \succeq \alpha_{P'}$. We deduce that T is a monotone self-mapping of $[P', P]^{K+Q}$, and it follows that it has a fixed point, and that the set of fixed points forms a complete lattice. (A fixed point can be computed by starting with α^P and iteratively applying T until convergence.) Combining this with the fact that any fixed point must balance supply and demand, we obtain uniqueness (proofs of all claims are at the end of this appendix).

Claim F.1. *The operator* T *has a unique fixed point* α^*.

Consider the fixed point α^* of T. Let $g(n) = h_n/\sqrt{n}$ for any $h_n = \omega(1)$. In the argument below we use $h_n = \sqrt{\log n} \Rightarrow g(n) = \sqrt{\log n/n}$, but this is only an example of a possible h_n that can be used. Define α^{lb} by $\alpha^{\text{lb}}_{kq} = \alpha^*_{kq} - g(n)$ for all k, q and α^{ub} by $\alpha^{\text{ub}}_{kq} = \alpha^*_{kq} + g(n)$. We will show that, for appropriate C, there is a core outcome in the n-th market satisfying $\alpha^{\text{lb}} \preceq \alpha \preceq \alpha^{\text{ub}}$, and that all core outcomes are within $C \log n/n$ of α. Define $\nu^*_{kq} = D_{kq}(\alpha^*_{kq}; \alpha^*)$, and show that $N(k,q)/n$ approaches ν^*_{kq}.

We now define an operator T_n such that its fixed point will capture a core solution in the n-th market. Analogous to Eq. (51) and Eq. (52), we define demand and supply for the n-the market to be[37]

$$\hat{D}_{kq}(p; \alpha) = \left|\{i : \tau(i) = k, u(k,q) - p + \eta_i^q \geq u(k,q') - \alpha_{kq'} + \eta_i^{q'} \text{ for all } q' \neq q\}\right| \tag{53}$$

and

$$\hat{S}_{kq}(p; \alpha) = \left|\{j : \tau(j) = q, p + \epsilon_j^k \geq \alpha_{k'q} + \epsilon_j^{k'} \text{ for all } k' \neq k\}\right| \tag{54}$$

for the n-th market. We define $(\mathsf{T}_n\alpha)_{kq} = \inf\{p : \hat{S}_{kq}(p;\alpha) > \hat{D}_{kq}(p;\alpha)\}$. As before, D_{kq} is (weakly) monotone decreasing in p and (weakly) monotone increasing in α, whereas S_{kq} is (weakly) monotone increasing in p and (weakly) monotone decreasing in α. (The monotonicities in α constitute a weak gross substitutes property for each type of agent on both sides of the

[37] To be perfectly precise, we should ensure that each worker counts toward exactly one unit of total demand by using a mix of strict and weak inequalities in the definition of D_{kq}, based on the direction in which ties between q and q' are broken in each case. To make all the details work out, we require agents to break ties in favor of the type they are matched to under the maximum weight matching M.

market.) We deduce monotonicity of T_n.

Fact F.9. *If $\alpha \preceq \alpha'$ then $\mathsf{T}_n\alpha \preceq \mathsf{T}_n\alpha'$.*

Our next claim implies that T_n has a fixed point that lies between α^{lb} and α^{ub}.

Claim F.2. *With high probability, we have $\mathsf{T}_n\alpha^{\mathrm{ub}} \preceq \alpha^{\mathrm{ub}}$ and $\mathsf{T}_n\alpha^{\mathrm{lb}} \succeq \alpha^{\mathrm{lb}}$. Hence, T_n has a fixed point α satisfying $\alpha^{\mathrm{lb}} \preceq \alpha \preceq \alpha^{\mathrm{ub}}$.*

The proof of this claim uses the convergence of the empirical distribution of (type, productivity vector) for agents to the limiting distribution. The main part of the proof is to show that w.h.p., at prices α^{ub}, the demand is less than the supply for each type pair (whereas the opposite is true at α^{lb}). This is accomplished by showing that the realized demand (supply) at α^{ub} is less (more) than the limiting demand (supply) at α^*. In turn, the definition of α^{ub} helps us show this, because the relative prices of different types are the same as under α^*, but each of them is now more expensive relative to remaining unmatched.

It is not hard to show that the fixed points of T correspond to core price vectors.

Claim F.3. *Let α be a fixed point of T_n. Then (M, α) is a core outcome, where M is the maximum weight matching.*

Though we don't need the converse, we remark that it holds: for any core outcome (M, α), the prices α are a fixed point of T_n.

Combining Claims F.2 and F.3 immediately gives a core solution α close to α^*, i.e., the bound (1) for *some* core solution. We are also in a position to prove (2), leveraging the following claim that controls the demand and supply in the n-th market for all price vectors between α^{lb} and α^{ub}. In particular, the bounds in the claim apply to the core solution α, where $\hat{D}_{kq}(\alpha_{kq}; \alpha) = \hat{S}_{kq}(\alpha_{kq}; \alpha) = N(k, q)$, thus yielding (2).

Claim F.4. *Let $\nu^*_{kq} = D_{kq}(\alpha^*_{kq}; \alpha^*)$. Then, there exists $f(n) = O^*(\sqrt{n})$ such that, with high probability, for all $\alpha^{\mathrm{lb}} \preceq \alpha \preceq \alpha^{\mathrm{ub}}$ we have, for all $(k, q) \in \mathcal{T}$, that*

$$\left|\hat{D}_{kq}(\alpha_{kq}; \alpha) - n\nu^*_{kq}\right| \leq f(n)\,,$$

$$\left|\hat{S}_{kq}(\alpha_{kq}; \alpha) - n\nu^*_{kq}\right| \leq f(n)\,.$$

For the second part of the theorem (the bound (3)), we prove the following claim, and use it together with the fact that weight$(M) = \Omega(n)$ with high probability, which follows from $\rho_k > 0$ for all k, and $f(x) > 0$ for all x. The claim also completes the proof of (1) for *all* core solutions.

Claim F.5. *There exists $C = C(K, Q, \rho, f) < \infty$ such that for all k, q we have*

$$|\alpha_{kq}^{\max} - \alpha_{kq}^{\min}| \leq C \frac{\log n}{n} \,. \tag{55}$$

The proof of this claim relies on the fact that all core α_{kq}'s must lie between some consecutive order statistics of ϵ_i^q's for workers i who are type k, and are either unmatched or matched to type q. Further, these order statistics are close together in the vicinity of α^*, and we already know that it is sufficient to consider this vicinity. $\square$

We now provide proofs of all the claims above that facilitated our proof of Theorem 5.

Proof of Claim F.1. The set of $\alpha \in [P', P]^{K+Q}$ is a complete sublattice. Since T is a monotone self-mapping of this set, it has a nonempty set of fixed points that themselves form a complete lattice. Take the two extreme fixed points $\alpha^{\max}(T)$ and $\alpha^{\min}(T)$. Suppose $\alpha^{\max}(T) \succ \alpha^{\min}(T)$. Fix a price vector α. A direct calculation yields the following intuitive expression for the total demand from worker type k:

$$\sum_q D_{kq}(\alpha_{kq}; \alpha) = \rho_k \Big(1 - \prod_q F(\alpha_{kq} - u(k, q))\Big) \,. \tag{56}$$

Similarly, the total supply from employer type q is

$$\sum_k S_{kq}(\alpha_{kq}; \alpha) = \rho_q \Big(1 - \prod_k F(-\alpha_{kq})\Big) \,. \tag{57}$$

For ease of notation, we write $D_{kq}^{\max} = \sum_q D_{kq}(\alpha_{kq}^{\max}(\mathsf{T}); \alpha^{\max}(\mathsf{T}))$, and so on. It follows from

$\alpha^{\max}(T) \succ \alpha^{\min}(T)$ and Eqs. (56) and (57) that

$$\sum_k \sum_q D_{kq}^{\max} < \sum_k \sum_q D_{kq}^{\min}$$
$$\sum_k \sum_q S_{kq}^{\max} > \sum_k \sum_q S_{kq}^{\min}.$$

At any fixed point of T, we know that $D_{kq} = S_{kq} \Rightarrow \sum_k \sum_q D_{kq} = \sum_k \sum_q S_{kq}$. Hence,

$$\begin{aligned} &\sum_k \sum_q D_{kq}^{\min} = \sum_k \sum_q S_{kq}^{\min} \\ \Rightarrow &\sum_k \sum_q D_{kq}^{\max} < \sum_k \sum_q S_{kq}^{\max}, \end{aligned}$$

a contradiction. $\square$

Proof of Claim F.2. We will use the fact that $\mathrm{E}[\hat{D}_{kq}(p;\alpha)] = nD_{kq}(p;\alpha)$ for any p and α, along with concentration bounds, to obtain the claim. We will show that w.h.p., we have $\hat{D}_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}}) < \hat{S}_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}})$ which will immediately imply $(\mathsf{T}_n\alpha^{\mathrm{ub}})_{kq} \le \alpha_{kq}^{\mathrm{ub}}$, i.e., $\mathsf{T}_n\alpha^{\mathrm{ub}} \preceq \alpha^{\mathrm{ub}}$. (The proof of $\mathsf{T}_n\alpha^{\mathrm{lb}} \succeq \alpha^{\mathrm{lb}}$ is analogous and we omit it.) The existence of the fixed point will immediately follow by starting with α^{ub} and iteratively using T_n until (monotone) convergence to α is obtained.

Note that $\hat{D}_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}})$ is distributed as $\mathrm{Binomial}(n, D_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}}))$. It follows that

$$\Pr[\hat{D}_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}}) < n(D_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}}) + c_n)] \ge 1 - \exp(-2nc_n^2)\,, \tag{58}$$

for any c using a standard Chernoff bound. Further, a direct calculation[38] gives

$$\begin{aligned} D_{kq}(\alpha_{kq}^*;\alpha^*) - D_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}}) &= \rho_k \int_0^{g(n)} f(z + \alpha_{kq}^* - u(k,q)) \prod_{q' \ne q} F(z + \alpha_{kq'}^* - u(k,q'))\,\mathrm{d}z \\ \Rightarrow \lim_{n\to\infty} \frac{D_{kq}(\alpha_{kq}^*;\alpha^*) - D_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{ub}})}{g(n)} &= \rho_k f(\alpha_{kq}^* - u(k,q)) \prod_{q' \ne q} F(\alpha_{kq'}^* - u(k,q')), \end{aligned}$$

[38] It is important here that α^{ub} and α^* differ by the same amount in each coordinate, meaning that the relative attractiveness of types on the other side is unchanged.

using the continuity of $f(\cdot)$ and that $\lim_{n\to\infty} g(n) = 0$. Define

$$c_n = \frac{g(n)}{2} \min_{k,q} \Big[\rho_k f(\alpha^*_{kq} - u(k,q)) \prod_{q'\neq q} F(\alpha^*_{kq'} - u(k,q'))\Big]$$

Note that $nc_n^2 = \Omega((\log n)^2)$. Then, for large enough n, we have

$$D_{kq}(\alpha^*_{kq}; \alpha^*) \le D_{kq}(\alpha^{\mathrm{ub}}_{kq}; \alpha^{\mathrm{ub}}) + c_n \qquad \text{for all } k, q\,.$$

Substituting in Eq. (58), we obtain

$$\Pr[\hat{D}_{kq}(\alpha^{\mathrm{ub}}_{kq}; \alpha^{\mathrm{ub}}) < nD_{kq}(\alpha^*_{kq}; \alpha^*)] \ge 1 - \exp\big(-\Omega(\log n)\big) = 1 - o(1) \qquad \text{for all } k, q\,; \tag{59}$$

i.e., with high probability,

$$\hat{D}_{kq}(\alpha^{\mathrm{ub}}_{kq}; \alpha^{\mathrm{ub}}) < nD_{kq}(\alpha^*_{kq}; \alpha^*) \qquad \text{for all } k, q\,.$$

Using an analogous argument, we establish that with high probability,

$$\hat{S}_{kq}(\alpha^{\mathrm{ub}}_{kq}; \alpha^{\mathrm{ub}}) > nS_{kq}(\alpha^*_{kq}; \alpha^*) \qquad \text{for all } k, q\,.$$

Now, by definition of α^* we have $D_{kq}(\alpha^*_{kq}; \alpha^*) = S_{kq}(\alpha^*_{kq}; \alpha^*)$ for all k, q. We deduce that, w.h.p.,

$$\hat{S}_{kq}(\alpha^{\mathrm{ub}}_{kq}; \alpha^{\mathrm{ub}}) > \hat{D}_{kq}(\alpha^{\mathrm{ub}}_{kq}; \alpha^{\mathrm{ub}}) \qquad \text{for all } k, q\,,$$

as needed. □

Proof of Claim F.3. Let α be a fixed point of T_n. Then α clears the market up to tie-breaking, and using the fact that this is a two-sided matching setting, the tie-breaking can be done so as to obtain a matching M' such that (M', α) is a core outcome. This also means that $M' = M$, the maximum weight matching. □

Proof of Claim F.4. Consider $\hat{S}_{kq}(\alpha_{kq};\alpha)$. Let $\hat{S}_{kq}^{\max}$ and $\hat{S}_{kq}^{\min}$ be the largest and smallest values, respectively, of $\hat{S}_{kq}$ for α in the specified range. Using the monotonicities of $\hat{S}_{kq}$ in its arguments, we have

$$\begin{aligned}
\hat{S}_{kq}^{\max} &= \hat{S}_{kq}(\alpha_{kq}^{\mathrm{ub}};\alpha^{\mathrm{lb}})\,,\\
\hat{S}_{kq}^{\min} &= \hat{S}_{kq}(\alpha_{kq}^{\mathrm{lb}};\alpha^{\mathrm{ub}})\,.
\end{aligned}$$

Using the definition (54) of $\hat{S}_{kq}$, we have that

$$\begin{aligned}
\hat{S}_{kq}^{\max} - \hat{S}_{kq}^{\min} \le & \left|\left\{j : \tau(j) = q, |\alpha_{kq}^* + \epsilon_j^k| \le g(n)\right\}\right| \\
& + \sum_{k' \neq k} \left|\left\{j : \tau(j) = q, |\epsilon_j^k - \epsilon_j^{k'} + \alpha_{kq}^* - \alpha_{k'q}^*| \le g(n)\right\}\right|\,. \qquad (60)
\end{aligned}$$

Now,

$$\begin{aligned}
\left|\left\{j : \tau(j) = q, |\alpha_{kq}^* + \epsilon_j^k| \le g(n)\right\}\right| &= \mathrm{Binomial}(n\rho_q, F(-\alpha_{kq}^* + g(n)) - F(-\alpha_{kq}^* - g(n)))\\
&\le O^*(\sqrt{n})\,,
\end{aligned}$$

w.h.p., using that $g(n) = \sqrt{\log n/n}$. Let F_1 be the distribution of the difference between two i.i.d. draws from F. The F_1 is also a distribution with positive and continuous density everywhere in $(-\infty, \infty)$. We can bound each of the other terms in Eq. (60) as

$$\begin{aligned}
& \left|\left\{j : \tau(j) = q, |\epsilon_j^k - \epsilon_j^{k'} + \alpha_{kq}^* - \alpha_{k'q}| \le g(n)\right\}\right| \\
&= \mathrm{Binomial}(n\rho_q, F_1(-\alpha_{kq}^* + \alpha_{k'q}^* + g(n)) - F_1(-\alpha_{kq}^* + \alpha_{k'q}^* - g(n)))\\
&\le O^*(\sqrt{n})\,.
\end{aligned}$$

Combining, we obtain that, with high probability,

$$\begin{aligned}\hat{S}_{kq}^{\max} - \hat{S}_{kq}^{\min} &\leq O^*(\sqrt{n}) \\ \Rightarrow |\hat{S}_{kq}(\alpha_{kq};\alpha) - \hat{S}_{kq}(\alpha_{kq}^*;\alpha^*)| &\leq O^*(\sqrt{n})\end{aligned} \tag{61}$$

for all $\alpha^{\text{lb}} \preceq \alpha \preceq \alpha^{\text{ub}}$.

Recall that $\nu_{kq}^* = D_{kq}(\alpha_{kq}^*;\alpha^*) = S_{kq}(\alpha_{kq}^*;\alpha^*)$ and $\mathrm{E}[\hat{S}_{kq}(\alpha_{kq}^*;\alpha^*)] = nS_{kq}(\alpha_{kq}^*;\alpha^*)$. A Chernoff bound gives that, with high probability,

$$|\hat{S}_{kq}(\alpha_{kq}^*;\alpha^*) - n\nu_{kq}^*| \leq O^*(\sqrt{n})\,. \tag{62}$$

Combining Eqs. (61) and (62) yields

$$|\hat{S}_{kq}(\alpha_{kq};\alpha) - n\nu_{kq}^*| \leq O^*(\sqrt{n})\,, \tag{63}$$

for all $\alpha^{\text{lb}} \preceq \alpha \preceq \alpha^{\text{ub}}$.

The proof of the corresponding bound on $\hat{D}_{kq}(\alpha_{kq};\alpha)$ is analogous.

□

Proof of Claim F.5. Having found a core solution α (the fixed point of T_n) that moreover satisfies $\alpha^{\text{lb}} \preceq \alpha \preceq \alpha^{\text{ub}}$, we now use these bounds on α to control the size of the core.

Recall that there is (with probability 1), a unique[39] maximum weight matching M. We focus on a subset of agents S that are of type k and either matched to type q or unmatched under M. We show that, w.h.p., the largest value of η_i^q among $i \in S\backslash M$ is within $C\log n/n$ of the smallest value of η_i^q among $i \in S \cap M$, yielding the desired bound on $|\alpha_{kq}^{\max} - \alpha_{kq}^{\min}|$.

Define

$$S = \{i : \tau(i) = k, \eta_i^{q'} < \alpha_{kq'}^* - g(n) - u(k,q')\ \forall\, q' \neq q\}. \tag{64}$$

If $\alpha \succeq \alpha^{\text{lb}}$ (which holds w.h.p.), we know that no agent in S is matched to $q' \neq q$. Now, the

[39]We do not need uniqueness of M to bound the core size, but we use it to simplify the argument.

likelihood of an agent being in S is

$$\rho_k \prod_{q' \neq q} F(\alpha^*_{kq'} - g(n) - u(k,q')) \geq 0.9\rho_k \prod_{q' \neq q} F(\alpha^*_{kq'} - u(k,q'))$$

for large enough n, using continuity of F and $\lim_{n\to\infty} g(n) = 0$. (Note that we do not need to reveal η_i^q to compute whether $i \in S$.) Now divide $[\alpha^*_{kq} - u(k,q) - 2g(n), \alpha^*_{kq} - u(k,q) + 2g(n)]$ into intervals of length $\Delta = (C/3)\log n/n$, where

$$C = 4 \Big/ \min_{k,q} \left[\rho_k f(\alpha^*_{kq} - u(k,q)) \prod_{q' \neq q} F(\alpha^*_{kq'} - u(k,q'))\right].$$

Using continuity of $f(\cdot)$, we have that

$$\inf_{x \in [\alpha^*_{kq} - u(k,q) - 2g(n), \alpha^*_{kq} - u(k,q) + 2g(n)]} f(x) \geq 0.9 f(\alpha^*_{kq} - u(k,q)).$$

Using the previous three equations, and independence between membership in a subinterval and membership in S, it follows that the number of agents in S who belong to any individual subinterval is Binomial(n, p_1) for

$$p_1 \geq \Delta \cdot 0.9 f(\alpha^*_{kq} - u(k,q)) \cdot 0.9\rho_k \prod_{q' \neq q} F(\alpha^*_{kq'} - u(k,q')) \geq \log n/n,$$

and hence each subinterval has at least one agent in S with probability at least $1 - 1/n$. We deduce using a union bound that w.h.p., all subintervals have at least one member of S. Now, in any core solution, each i such that $\eta_i < \alpha_{kq} - u(k,q)$ must remain unmatched, and $\eta_i > \alpha_{kq} - u(k,q)$ must be matched to type q. Further, we have that w.h.p., $\alpha_{kq} \in (\alpha^*_{kq} - g(n), \alpha^*_{kq} + g(n))$. Considering the subintervals on each side of the one in which α_{kq} occurs, since each of them has at least one agent (who also belongs to S), and considering that M is unique, we deduce that

$$|\alpha^{\max}_{kq} - \alpha^{\min}_{kq}| \leq 3\Delta = C\frac{\log n}{n}.$$

Using a union bound, we deduce that w.h.p., this holds simultaneously for all k, q. $\square$

G Proof of Lemma 2

Proof of Lemma 2. Let (M, α) be a core solution. Suppose $\alpha_{k^*q^*} = \max_{k,q} \alpha_{kq}$. Consider worker type q^*. Since $\alpha_{k^*q^*} \geq \alpha_{kq^*}$ for all $k \neq k^*$, we obtain that each agent i of type q^* has likelihood at least $1/K$ of counting toward $\hat{S}_{k^*q^*} = \hat{S}_{k^*q^*}(\alpha_{k^*q^*}; \alpha)$; see Eq. (54). It follows that w.h.p., $\hat{S}_{k^*q^*} \geq (\#\text{ agents of type } q^*)/K - O^*(\sqrt{n}) > Cn/2$, using Assumption 2. At prices α, an agent of type k^* prefers being unmatched to matching with type q^* with likelihood $F(\alpha_{k^*q^*} - u(k^*, q^*))$. Hence, w.h.p., we have $\hat{D}_{k^*q^*} = \hat{D}_{k^*q^*}(\alpha_{k^*q^*}; \alpha) <$ (number of agents of type k^*)$(1 - F(\alpha_{k^*q^*} - u(k^*, q^*))) + O^*(\sqrt{n}) \leq n(1 - F(\alpha_{k^*q^*} - u(k^*, q^*))) + O^*(\sqrt{n}) < Cn/2$ if $\alpha_{k^*q^*} > U$ for some $U < \infty$. (Here we used that $\lim_{P \to \infty} 1 - F(P - u(k^*, q^*)) = 0$.) But $\hat{D}_{k^*q^*} = \hat{S}_{k^*q^*}$, a contradiction. It follows that $\alpha_{k^*q^*} \leq U$, and hence $\alpha_{kq} \leq U$ for all k, q. The lower bound of $-U \leq \alpha_{kq}$ can be similarly established. $\square$