Unbalanced Random Matching Markets: The Stark Effect of Competition

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Abstract

We study competition in matching markets with random heterogeneous preferences and an unequal number of agents on either side. First, we show that even the slightest imbalance yields an essentially unique stable matching. Second, we give a tight description of stable outcomes, showing that matching markets are extremely competitive. Each agent on the short side of the market is matched with one of his top choices, and each agent on the long side is either unmatched or does almost no better than being matched with a random partner. Our results suggest that any matching market is likely to have a small core, explaining why small cores are empirically ubiquitous.

Keywords: Matching markets; Random markets; Competition; Stability; Core.

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1 Introduction

Stable matching theory has been instrumental in the study and design of numerous two-sided markets. Two-sided markets are described by two disjoint sets of men and women, where each agent has preferences over potential partners. Examples include entry-level labor markets, dating, and college admissions. The core of a two-sided market is the set of stable matchings, where a matching is stable if there is no man and woman who both prefer each other over their assigned partners. Stability is a useful equilibrium concept for these markets since it predicts observed outcomes in decentralized markets,[1] and since stability is a critical requirement for the success of centralized clearing-houses.[2]

This paper analyzes competition in matching markets to address two fundamental issues. First, we address the longstanding issue of multiplicity of stable matchings.[3] Previous studies show that the core is small only under restrictive assumptions on market structure, suggesting that the core is generally large. In contrast, matching markets have an essentially unique stable matching in practice.[4] Second, relatively little is known about how the structure of stable outcomes is determined by market characteristics. For example, it is known that increasing the number of agents on one side makes agents on the other side weakly better off (Crawford, 1991), but little is known about the magnitude of this effect.

We address these issues by looking at randomly drawn matching markets, allowing for competition arising from an unequal number of agents on either side. Influential works by Pittel (1989b) and Roth and Peranson (1999) study the same model with an equal number

[1]Hitsch et al. (2010) and Banerjee et al. (2013) use stable matchings to predict matching patterns in online dating and the Indian marriage market, respectively.

[2]Stable matching models have been successfully adopted in market design contexts such as school choice (Abdulkadiroglu et al., 2006; Abdulkadiroğlu et al., 2005) and resident matching programs (Roth and Peranson, 1999). Roth and Xing (1994) and Roth (2002) show that stability is important for the success of centralized clearing houses.

[3]The potential multiplicity of stable matchings is a central issue in the literature and has led to many studies about the structure of the core (Knuth, 1976), which stable matching to implement (Schwarz and Yenmez, 2011) and strategic behavior (Dubins and Freedman, 1981; Roth, 1982).

[4]A unique stable matching was reported in the National Resident Matching Program (NRMP) (Roth and Peranson, 1999), Boston school choice (Pathak and Sönmez, 2008), online dating (Hitsch et al., 2010) and the Indian marriage market (Banerjee et al., 2013). We are not aware of any evidence of a large core in a matching market.

of agents on both sides and find that the core is typically large. Our first contribution is showing that this is a knife-edge case: the competition resulting from even the slightest imbalance yields an essentially unique stable matching. Our results, which hold for both small and large markets, suggest that any matching market is likely to have a small core, thereby providing an explanation as to why small cores are empirically ubiquitous.

Our second contribution is that the essentially unique stable outcome can be almost fully characterized using only the distribution of preferences and the number of agents on each side. Roughly speaking, under any stable matching, each agent on the short side of the market is matched with one of his top choices. Each agent on the long side is either unmatched or does almost no better than being matched with a random partner. Thus, we find that matching markets are extremely competitive, with even the slightest imbalance greatly benefiting the short side. We present simulation results showing that the short side's advantage is robust to small changes in the model.

Formally, we consider a matching market with n men and $n+1$ women. For each agent we independently draw a complete preference list uniformly at random. We show that with high probability: (i) the core is small in that almost all agents have a unique stable partner (i.e., they are matched with the same partner in all stable matchings), and (ii) under any stable matching men are, on average, matched with their $\log n$-th most preferred woman, while matched women are on average matched with their $n/(\log n)$-th most preferred man. Thus agents on the short side rank their partners, on average, as they would if they were to choose partners in sequence.[5] Matched agents on the long side, on average, rank their partners approximately the same as if they each chose their match only from a limited, randomly drawn set of $\log n$ potential partners. For example, in a market with 1,000 men and 1,001 women, men are matched on average with their 7th ($\approx \log 1000$) most preferred woman, while women are matched on average with their 145th ($\approx 1000/\log 1000$) most preferred man. We further show that the benefit to the short side is amplified when the imbalance is greater.

Given that the smallest imbalance leads to a small core, even when preferences are heterogeneous and uncorrelated, we expect that matching markets will generally have a small

[5]The corresponding mechanism would be Random Serial Dictatorship (RSD), under which men are randomly ordered and each man is assigned in turn to his most preferred woman who was not previously assigned (see, e.g., Abdulkadiroğlu and Sönmez, 1998).

core. In real settings preferences are likely to be correlated, but it is generally thought that correlation reduces the size of the core,[6] and this notion is supported by simulation results in Section 4. We simulated markets with varying correlation structures, market sizes, and list lengths, and for all them found that imbalance leads to a small core.

A small core implies that there is limited scope for strategic behavior. Suppose agents report preferences to a central mechanism, which implements a matching that is stable with respect to reported preferences. Demange et al. (1987) consider the induced full information game, and show that agents who have a unique stable partner are unable to gain from misreporting their preferences. It follows that in unbalanced markets with incomplete information truthful reporting is an ε-Bayes–Nash equilibrium.[7] This suggests that agents should report their preferences truthfully to stable matching mechanisms, and may help explain the practical success of stable matching mechanisms.

To gain some intuition for the effect of imbalance in matching markets, it is useful to compare our setting with a competitive, homogeneous buyer-seller market. In a market with 100 homogeneous sellers who have unit supply and a reservation value of zero, and 100 identical buyers who have unit demand and value the good at one, every price between zero and one gives a core allocation. However, when there are 101 sellers, competition among sellers implies a unique clearing price of zero, since any buyer has an outside option of buying from the unmatched seller who will sell for any positive price.

Our results show a sharp phenomenon in random matching markets similar to the one in the homogeneous buyer-seller market, despite heterogeneous preferences and the lack of transfers. In particular, the direct outside option argument (used above in the buyer-seller market) does not explain the strong effect of a single additional woman: while the unmatched woman is willing to be matched with any man, she creates a useful outside option only for a few men who rank her favorably. However, this outside option makes these men better off, making their spouses worse off. The men who like these spouses must, in turn, be made better off. This effect ripples through the entire market, making most men better off.

Our proof requires developing some technical tools that may be of independent interest. We build on the work of McVitie and Wilson (1971) and Immorlica and Mahdian (2005)

[6] See, for example, Roth and Peranson (1999).

[7] Furthermore, agents with a unique stable partner will be unaffected by profitable manipulations by other agents.

to construct an algorithm that calculates the women-optimal stable matching through a series of rejection chains. Previous analysis of rejection chains by Immorlica and Mahdian (2005) and Kojima and Pathak (2009) analyzed each rejection chain independently. Our new algorithm accounts for the interdependence between different rejection chains, allowing us to analyze the ripple effect generated by a small imbalance. The progress of the modified algorithm on random preferences can be captured by a tractable stochastic process, whose analysis reveals that different chains are likely to be highly connected.

1.1 Related literature

Most relevant to our work are Pittel (1989a) and Knuth et al. (1990), who extensively analyze balanced random matching markets. They characterize the set of stable matchings for a random matching market with n men and n women, showing that the men's average rank of wives ranges from $\log n$ to $n/\log n$ in different stable matchings and that the fraction of agents with multiple stable partners approaches 1 as n grows large. Roughly, our results show that the addition of a single woman makes the core collapse, leaving only the stable matching which is most favorable for men.

Several papers study the size of the core to understand incentives to misreport preferences, analyzing matching markets where one side has short, randomly drawn preference lists.[8] Immorlica and Mahdian (2005) show that women cannot manipulate in a one-to-one marriage market, and Kojima and Pathak (2009) show that schools cannot manipulate in many-to-one matching markets. Our results differ in two ways. First, these papers are limited to studying manipulation and the size of the core, while we characterize outcomes and show that competition benefits the short side of the market. Second, their analysis relies on a specific market structure, which generates a large number of unmatched agents.[9] This

[8]The random, short-list assumption is motivated by the limited number of interviews as well as the NRMP restriction that allows medical students to submit a rank-ordered list of up to only 30 programs (http://www.nrmp.org/match-process/create-and-certify-rol-applicants/) and by the limited number of interviews.

[9]Their proof requires that randomly drawn preference lists are short enough for agents to have a significant probability of remaining unmatched. Thus, though they assume an equal number of seats and students, their market behaves like a highly unbalanced market. Unless being unmatched is an attractive option, this implies that agents are not submitting long enough lists.

market structure was necessary for the analysis in these papers, leading them to conclude that the core is likely to be small only under restrictive assumptions.

Lee (2011) and Coles and Shorrer (2014) study manipulation in asymptotically large balanced matching markets, making different assumptions about the utility functions of agents. Coles and Shorrer (2014) define agents' utilities to be equal to the rank of their spouse (varying between 1 and n), which grows linearly with the number of agents in the market. They show that women can profitably manipulate the men-proposing deferred acceptance mechanism. Lee (2011) allows for correlation in preferences but assumes that utilities are bounded and the market grows large. He shows that in large markets most agents cannot profitably manipulate the men-proposing deferred acceptance mechanism. The different results stem in part from the different utility parameterizations. In a balanced market an agent's rank of their spouse can range from $\log n$ to $n/\log n$, and this difference in rank can be small or large in terms of utilities.[10]

The literature on matching markets with transferable utility provides theoretical predictions on who is matched with whom based on market characteristics. Rao (1993) and Abramitzky et al. (2011) exploit random variation in female-male balance to show that outcomes in the marriage market favor the short side. Also related to this paper is the well-known phenomenon that increasing the relative number of agents of one type ("buyers") benefits agents of the other type ("sellers") (Shapley and Shubik, 1971; Becker, 1973). Our paper provides the first quantification of this effect in matching markets without transfers.

Several papers studied markets with strong correlation in preferences. It is well known that if all men have the same preferences over women, there is a unique stable matching. Holzman and Samet (2014) generalize this observation, showing that if the distance between any two men's preference lists is small, the set of stable matchings is small. Azevedo and Leshno (Forthcoming) look at large many-to-one markets with a constant number of schools and an increasing number of students; they find that the set of stable matchings generically converges to the unique stable matching of a continuum model. The core of these markets can equivalently be described as the core of a one-to-one matching market between students and seats in schools, where all seats in a school have identical preferences over students.

[10]Both $\log n/n$ and $1/\log n$ converge to 0, meaning that as the market gets large both men and women are matched with a partner in the top percentile of their preferences, although the convergence for women is very slow. For example, $\log n/n = 1\%$ for $n \approx 600$, but $1/\log n = 1\%$ for $n \approx 2.68 \times 10^{43}$.

1.2 Organization of the paper

Section 2 presents our model and results. Section 3 provides intuition for the results, outlines our proof, and presents the new matching algorithm that is the basis for our proof. Section 4 presents simulation results, which show that the same features occur in small markets, and verifies the robustness of our results. Section 5 gives some final remarks and discusses the limitations of our model.

Appendix A proves the correctness of our new matching algorithm. The proof of our main results is in Appendix B. In Appendix C, we discuss how our results may be extended to many-to-one random matching markets.

2 Model and results

2.1 Random matching markets

A two-sided matching market is composed of a set of men $\mathcal{M} = \{1, \ldots, n\}$ and a set of women $\mathcal{W} = \{1, \ldots, n + k\}$. Each man m has a complete[11] strict preference list $\succ_m$ over the set of women, and each woman w has a complete strict preference list $\succ_w$ over the set of men. A *matching* is a mapping μ from $\mathcal{M} \cup \mathcal{W}$ to itself such that for every $m \in \mathcal{M}$, $\mu(m) \in \mathcal{W} \cup \{m\}$, and for every $w \in \mathcal{W}$, $\mu(w) \in \mathcal{M} \cup \{w\}$, and for every $m, w \in \mathcal{M} \cup \mathcal{W}$, $\mu(m) = w$ implies $\mu(w) = m$. We use $\mu(w) = w$ to denote that woman w is unmatched under μ.

A matching μ is *unstable* if there are a man m and a woman w such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A matching is *stable* if it is not unstable. It is well known that the core of a matching market is the set of stable matchings. We say that m is a stable partner for w (and vice versa) if there is a stable matching in which m is matched with w.

A *random matching market* is generated by drawing a complete preference list for each man and each woman independently and uniformly at random. Thus, for each man m, we draw a complete ranking $\succ_m$ from a uniform distribution over the $|\mathcal{W}|!$ possible rankings.

A stable matching always exists, and can be found using the Deferred Acceptance (DA) algorithm by Gale and Shapley (1962).[12] They show that the men-proposing DA finds

[11] That is, each man prefers any woman over being unmatched.

[12] We describe the DA algorithm in Section 3.

the *men-optimal stable matching* (MOSM), in which every man is matched with his most preferred stable woman. The MOSM matches every woman with her least preferred stable man. Likewise, the women-proposing DA produces the women-optimal stable matching (WOSM) with symmetric properties.

We are interested in the size of the core, as well as how matched agents rank their assigned partners. Denote the rank of woman w in the preference list $\succ_m$ of man m by $\text{Rank}_m(w) \equiv |\{w' : w' \succeq_m w\}|$. A smaller rank is better, and m's most preferred woman has a rank of 1. Symmetrically, denote the rank of m in the preference list of w by $\text{Rank}_w(m)$.

Definition 2.1. *Given a matching μ, the* men's average rank of wives *is given by*

$$R_{\text{MEN}}(\mu) = \frac{1}{|\mathcal{M} \backslash \bar{\mathcal{M}}|} \sum_{m \in \mathcal{M} \backslash \bar{\mathcal{M}}} \text{Rank}_m(\mu(m)) \,,$$

where $\bar{\mathcal{M}}$ is the set of men who are unmatched under μ.

Similarly, the women's average rank of husbands *is given by*

$$R_{\text{WOMEN}}(\mu) = \frac{1}{|\mathcal{W} \backslash \bar{\mathcal{W}}|} \sum_{w \in \mathcal{W} \backslash \bar{\mathcal{W}}} \text{Rank}_w(\mu(w)) \,,$$

where $\bar{\mathcal{W}}$ is the set of women who are unmatched under μ.

We use two metrics for the size of the core. First, we consider the fraction of agents who have multiple stable partners. Second, we consider the difference between the men's average rank of wives under the MOSM and under the WOSM.

2.2 Previous results

Previous literature has analyzed balanced random matching markets, which have an equal number of men and women. We start by citing a key result on the structure of stable matchings in balanced markets:

Theorem [Pittel (1989a)]. *In a random matching market with n men and n women, the fraction of agents who have multiple stable partners converges to 1 as $n \to \infty$. Furthermore,*

$$\frac{R_{\text{MEN}}(\text{MOSM})}{\log n} \xrightarrow{p} 1 \,,$$

$$\frac{R_{\text{MEN}}(\text{WOSM})}{n / \log n} \xrightarrow{p} 1 \,,$$

where $\xrightarrow{p}$ *denotes convergence in probability.*

This result shows that in a balanced market, the core is large under both measures: most agents have multiple stable partners, and the men's average rank of wives under the MOSM and the WOSM are significantly different. We find that this does not extend to unbalanced markets.

2.3 The size of the core in unbalanced markets

In our main result, we show that in a typical realization of an unbalanced market, almost all agents have a unique stable partner, and the men's average rank of wives as well as the women's average rank of husbands are almost the same under all stable matchings. We omit quantifications for the sake of readability and give a stronger version of the theorem in Appendix B.

Theorem 1. *Consider a sequence of random matching markets, indexed by* n*, with* n *men and* $n + k$ *women, for arbitrary* $k = k(n) \geq 1$*. Fix any* $\varepsilon > 0$*. With high probability,*[13] *we have that*

(i) the fraction of men and the fraction of women who have multiple stable partners are each no more than ε*, and*

(ii) the men's average rank of wives is almost the same under all stable matchings,[14] *as is the women's average rank of husbands:*

$$R_{\text{MEN}}(\text{WOSM}) \leq (1+\varepsilon)R_{\text{MEN}}(\text{MOSM})\,,$$
$$R_{\text{WOMEN}}(\text{WOSM}) \geq (1-\varepsilon)R_{\text{WOMEN}}(\text{MOSM})\,.$$

For centralized unbalanced markets, this result implies that the choice of the proposing side in DA makes little difference. For decentralized markets, this implies that stability gives an almost unique prediction.

[13] Given a sequence of events $\{\mathcal{E}_n\}$, we say that this sequence occurs *with high probability* (whp) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$.

[14] For any stable matching μ it follows from the properties of the MOSM and WOSM that $R_{\text{MEN}}(\text{MOSM}) \leq R_{\text{MEN}}(\mu) \leq R_{\text{MEN}}(\text{WOSM})$.

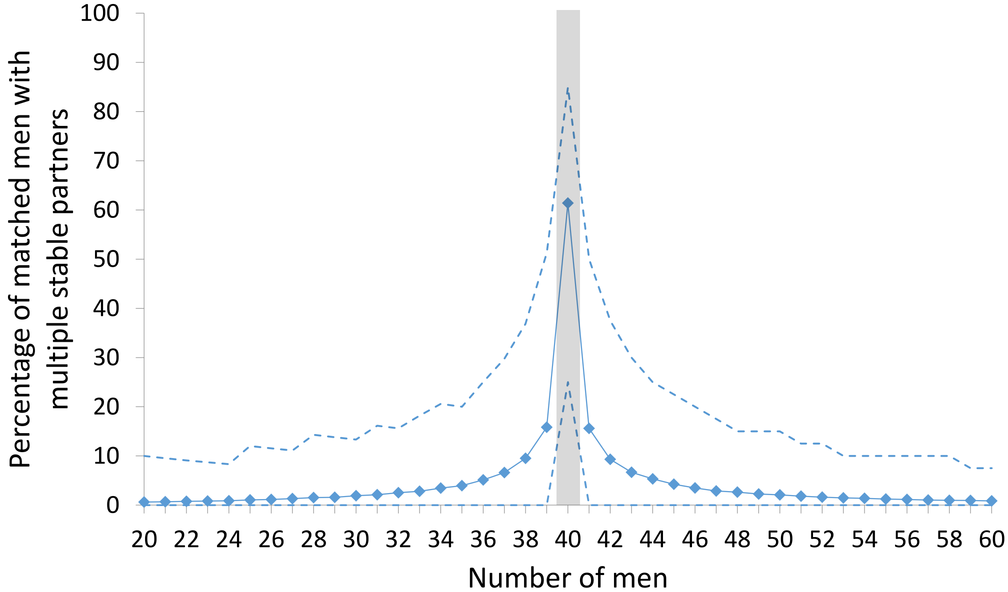


Figure 1: Percentage of men with multiple stable partners, in random markets with 40 women and a varying number of men. The main line indicates the average over 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentiles.

Figures 1 and 2 illustrate the results and show that the same features hold in small unbalanced markets. Figure 1 reports the fraction of men who have multiple stable partners in random markets with 40 women and 20 to 60 men. Figure 2 plots the men's average rank of wives under MOSM and WOSM. Observe that, even in such small markets, the large core of the balanced market (40 men and 40 women) is a knife-edge case.

2.4 Characterization of stable outcomes

The next theorem shows the advantage of the short side. When there are more women than men the men's average rank of wives is small, meaning that most men are matched with one of their top choices. On the other hand, the women's average rank of husbands is not much better than that resulting from random assignment. The theorem states the result for a general imbalanced market, and we give simplified expressions for special cases of interest in the next subsection. We give a stronger version of the theorem in Appendix B.

Theorem 2. *Consider a sequence of random matching markets, indexed by n, with n men*

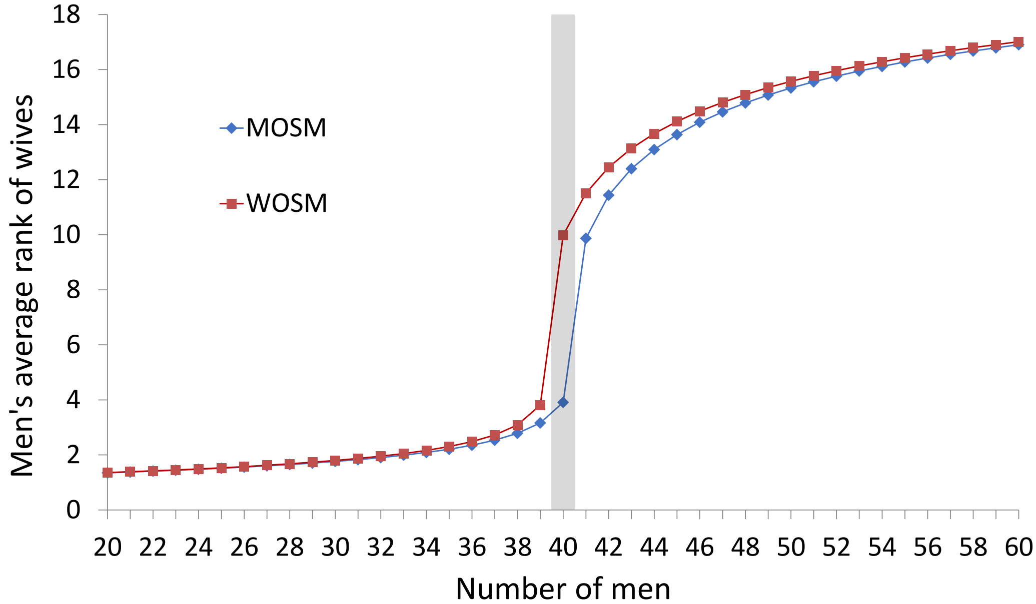


Figure 2: Men's average rank of wives under MOSM and WOSM in random markets with 40 women and a varying number of men. The lines indicate the average over 10,000 realizations.

and $n+k$ women, for arbitrary $k = k(n) \geq 1$. With high probability, the following hold for every stable matching μ:

$$R_{\text{MEN}}(\mu) \leq (1+\varepsilon)\left(\tfrac{n+k}{n}\right)\log\left(\tfrac{n+k}{k}\right),$$

$$R_{\text{WOMEN}}(\mu) \geq n\big/\left[1+(1+\varepsilon)\left(\tfrac{n+k}{n}\right)\log\left(\tfrac{n+k}{k}\right)\right].$$

For comparison, consider the assignments generated by the men's random serial dictatorship (RSD) mechanism. In RSD, men are ordered at random, and each man chooses his favorite woman who has yet to be chosen (ignoring women's preferences). The men's average rank of wives under RSD is approximately $\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$.[15] Thus, under any stable matching, the men's average rank of wives would be almost the same as under RSD. The women's average rank of husbands under any stable matching is better than getting a random husband by only a small factor of at most $\log n$. Thus, roughly speaking, in any stable matching, the short side "chooses" while the long side is "chosen."

[15] Following an analysis very similar to the proof of Lemma B.4(i), we can show that the men's average rank of wives under RSD is, with high probability, at least $(1-\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$ and at most $(1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$.

Figure 2 illustrates the advantage of the short side. When men are on the short side (there are fewer than 40 men), they are matched, on average, with one of their top choices. When men are on the long side, they are either unmatched or rank their partner only slightly better than a random match. See Figure 3 for a comparison with the men's average rank of wives under RSD. Section 4 provides simulation results indicating that Theorem 2 gives a good approximation for small markets, and that the advantage of the short side persists under correlated preferences.

2.5 Special cases: Small and large imbalances

To highlight two particular cases of interest, we present the following two immediate corollaries. We first focus on markets with minimal imbalance, where there is only one extra woman.

Corollary 2.2. *Consider a sequence of random matching markets with n men and $n+1$ women. Fix any $\varepsilon > 0$. With high probability, in every stable matching, the men's average rank of wives is no more than $(1+\varepsilon)\log n$, the women's average rank of husbands is at least $\frac{n}{(1+\varepsilon)\log n}$, and the fractions of men and women who have multiple stable partners are each no more than ε.*

The next case of interest is a random matching market with a large imbalance, taking $k = \lambda n$ for fixed λ.

Corollary 2.3. *For $\lambda > 0$, consider a sequence of random matching markets with $|\mathcal{M}| = n$, $|\mathcal{W}| = (1+\lambda)n$. Fix any $\varepsilon > 0$. Define the constant $\kappa = (1+\varepsilon)(1+\lambda)\log(1+1/\lambda)$. We have that with high probability, in every stable matching, the men's average rank of wives is at most κ, the women's average rank of husbands is at least $n/(1+\kappa)$, and the fractions of men and women who have multiple stable partners are each no more than ε.*

When there is a substantial imbalance in the market, the allocation is largely driven by men's preferences. For example, in a market with 5% extra women, men will be matched, on average, with roughly their 3rd most preferred woman. The women's average rank of husbands under WOSM is only a factor of $(1+\kappa)/2 = 2.1$ better than being matched with a random man. Thus, the benefit of being on the short side becomes more extreme when the imbalance is bigger.[16]

We can examine welfare through fractiles, in addition to rank. In markets with n men and $n+1$ women, on average, men receive a wife who is at the $(\log n/n)$-th fractile of their preference list, and women receive a husband who is at the $(1/\log n)$-th fractile of their preference list. Note that both sides of the market asymptotically receive their top fractile as n grows large, the rates of convergence are very different.[17] In markets with n men and $(1+\lambda)n$ women, even as n grows women do not receive their top fractile. For examination of welfare in matching markets see subsequent work by Yariv and Lee (2014) and Che and Tercieux (2014).

2.6 Implications for strategic behavior

In this section, we consider the implications of our results for strategic behavior in matching markets. A *matching mechanism* is a function that takes reported preferences of men and women and produces a matching. A *stable matching mechanism* is a matching mechanism that produces a matching that is stable with respect to reported preferences. A matching mechanism induces a direct revelation game, in which each agent reports a preference ranking and receives utility from his/her assigned partner.

Men-proposing DA (MPDA) is strategy-proof for men, but some women may benefit from misreporting their preferences.[18] Demange et al. (1987) show that a woman cannot profitably manipulate a stable matching mechanism if she has a unique stable partner. In a random unbalanced matching market, most women will have a unique stable partner and therefore cannot gain from misreporting their preferences.

Formally, we consider the following direct revelation game induced by a stable matching mechanism. Each man m independently draws utilities $u_m(w) \sim F$ for matching with each woman w. Symmetrically, each woman w draws utilities $u_w(m) \sim G$ for matching with each man m. We assume that F, G are non-atomic probability distributions with finite support

[16] As we would expect, $\kappa = \kappa(\lambda)$ is monotone decreasing in λ and $\lim_{\lambda \to \infty} \kappa(\lambda) = 1 + \varepsilon$.

[17] See footnote 10.

[18] A mechanism is said to be strategy-proof for men (women) if it is a dominant strategy for every man (woman) to report preferences truthfully. Roth (1982) shows that no stable matching mechanism is strategy-proof for both sides of the market.

$[0, \bar{u}]$. Each agent privately learns his/her own preferences, submits a ranking to the matching mechanism, and receives the utility of being matched with his/her assigned partner. We say that an agent reports truthfully if he/she submits a ranked list of spouses in the order of his/her utility of being matched with them.

Theorem 2.4. *Consider any stable matching mechanism. Let there be a sequence of random matching markets, indexed by n, with n men and $n + k$ women, for arbitrary $k = k(n) \geq 1$. For any $\varepsilon > 0$, there exists n_0 such that for any $n > n_0$ it is an ε-Bayes–Nash equilibrium for all agents to report truthfully.*

Note that Theorem 2.4 applies to *any* stable matching mechanism, regardless of the stable matching it selects. While the theorem is stated for large n, the simulations in Section 4 show that there is little scope for manipulation even in small markets.

Proof. For any $\delta > 0$, Theorem 1 tells us that the expected fraction of women with multiple stable partners is no more than δ with probability at least $1 - \delta$, for large enough n. Thus, the expected fraction of women with multiple stable partners is bounded by 2δ for large enough n. All women are ex ante symmetric; and since preferences are drawn independently and uniformly at random, the women are still symmetric after we reveal the preference list of one woman. Therefore, the interim probability that a woman has multiple stable partners, conditional on her realized preferences, is equal to the expected fraction of women with multiple stable partners, and thus bounded by 2δ. Choosing $\delta \leq \varepsilon/(2\bar{u})$ guarantees that any woman can gain at most ε by misreporting her true preferences, for large enough n. The argument for men is identical. □

Using the stronger version of Theorem 1 (Theorem 3 in Appendix B) one can produce tighter bounds on the gains from manipulation and extend the above result to utilities with unbounded support.

3 Proof idea and algorithm

This section presents the intuition for Theorems 1 and 2. We first provide intuition for the advantage of the short side in Section 3.1. Our proof follows a different and more constructive approach, which uses a new matching algorithm to trace the structure of the set of stable matchings. We present the main ideas of the proof in Section 3.2 and the new matching algorithm in Section 3.3.

3.1 Intuition for the small core and advantage of the short side

Suppose there are $n + 1$ women. By the Rural Hospital Theorem (Roth, 1986), the same woman $\bar{w}$ is unmatched in all stable matchings. Thus, every stable matching must also induce a stable matching for the balanced market that results from dropping $\bar{w}$. A stable matching of the balanced market remains stable after we add $\bar{w}$ only if all men prefer their assigned match over $\bar{w}$. Intuitively, a matching is more likely to satisfy this constraint when the men are better off. Typically only a few stable matchings (that are all close to the MOSM of the balanced market) satisfy this constraint, and the core of the original market is small.

3.2 Sketch of the proof

We prove our results by calculating both the MOSM and WOSM through a sequence of proposals by men, as specified in Algorithm 2. Since men receive a low average rank, the run of the algorithm on a randomly drawn market is a short and tractable stochastic process. By analyzing this stochastic process we uncover the desired properties of the core.

Algorithm 2 calculates the WOSM through the use of rejection chains. The algorithm maintains a stable matching, initiated to be the MOSM, and in each phase attempts to move to a stable matching that is more preferred by women. A phase starts by selecting a woman $\hat{w}$ and divorcing her husband m. This triggers a rejection chain in which man m continues to propose, possibly displacing other men who propose in turn. We name the phase according to how the chain ends: (a) an *improvement phase* ends with $\hat{w}$ accepting a proposal from a man she prefers over m, and (b) a *terminal phase* ends with a proposal to an unmatched woman. An improvement phase finds a new stable matching that matches $\hat{w}$ with a partner she prefers over m. A terminal phase implies that m is $\hat{w}$'s most preferred stable husband (and hence we roll it back before proceeding). Once we have found a terminal phase for every woman, the algorithm is at the WOSM and terminates.

Analyzing the run of the algorithm for markets of n men and $(1 + \lambda)n$ women is simpler. In this case, a phase beginning with arbitrary $\hat{w}$ is very likely to be terminal: the probability that a man in the chain will propose to an unmatched woman is roughly $\lambda/(1+\lambda)$, while the probability that he will propose to $\hat{w}$ is of order $1/n$. Thus, improvement phases are rare, and most women will be matched with the same man under the MOSM and WOSM.

The analysis for markets of n men and $n+1$ women is more involved and requires us to use links between different rejection chains.[19] Denote by S the set of women for whom the algorithm has already found their most preferred stable husbands. We initialize S to contain all unmatched women (by the Rural Hospital Theorem, they are unmatched in all stable matchings). When a terminal phase visits each woman at most once,[20] each woman in the chain can be added to S, as divorcing their husbands will result in a sub-chain that is also terminal. When a new chain reaches a woman $w' \in S$, that new chain must be terminal, since reaching $w' \in S$ implies that the new chain merges with a previous terminal chain. Thus, the set S allows us to track rejection chains, and how they merge together.

We first track the progress of the algorithm until the end of the first terminal phase, which occurs with the first proposal to the unmatched woman. Each proposal by a man has about $1/n$ chance to go to the unmatched woman, and therefore the first proposal to the unmatched woman occurs after order n proposals. That terminal phase will involve a chain of order $\sqrt{n}$ distinct[21] women, who are all added to S. After this, the likelihood that a proposal goes to a woman in S is substantially increased, meaning that most proposals are part of terminal phases, and the set S rapidly grows larger. Once S is large, almost every phase is terminal, and almost all women are already matched with their most preferred stable husband. The algorithm terminates when $S = \mathcal{W}$ is reached.

We calculate the men's average rank of wives under the MOSM using methods similar to Pittel (1989a), showing that the number of proposals by men is roughly equal to the solution of the coupon collector's problem. The men's average rank under the WOSM is calculated by adding the proposals during improvement phases of Algorithm 2. We deduce bounds on the women's average rank of husbands from the number of proposals each woman receives.

[19]In our proof, we simultaneously consider all possible values of k. However, we discuss the special case of $k = 1$ here for simplicity of exposition.

[20]If a chain includes a sub-chain that begins and ends with the same woman w', we call that sub-chain an Internal Improvement Cycle (IIC). An IIC is equivalent to a separate improvement phase for w', which can be cut out of the original chain. After removing all IICs, the chain of every terminal phase includes each woman at most once.

[21]These are the women that remain after removal of IICs, cf. footnote 20.

3.3 Matching algorithm

This section presents Algorithm 2, which calculates the WOSM from the MOSM through a sequence of proposals by men. This algorithm can be used to quickly calculate the WOSM when women are on the long side. All algorithms in this section assume that there are strictly more women than men (that is, $|\mathcal{W}| > |\mathcal{M}|$). For proofs and further details, refer to Appendix A.

For completeness, we first present the MPDA algorithm that outputs the MOSM.

Algorithm 1. Men-proposing deferred acceptance (MPDA)

1. *Every unmatched man proposes to his most preferred woman who has not already rejected him. If no new proposal is made, output the current matching.*

2. *If a women has multiple proposals, she tentatively keeps her most preferred man and rejects the rest. Go to step 1.*

We now present Algorithm 2, which calculates the WOSM. It maintains the most recent stable matching $\tilde{\mu}$ and a set S of women whose most preferred stable match has already been found. In each phase a rejection chain is initiated to check whether there another stable matching preferred by women. The variable m is used to hold the proposing man and μ is used to hold the temporary assignment. Each rejection chain is tracked using an ordered set $V = (v_1, v_2, .., v_J)$ of women. If a proposal is accepted by a woman in V, the algorithm updates $\tilde{\mu}$ to a new stable matching that makes women better off. When a proposal is accepted by a woman in S, the phase is terminal, and all the women in V are added to S. We use the notation $x \leftarrow y$ for the operation of copying the value of variable y to variable x.

Algorithm 2. MOSM to WOSM

- Input: *A matching market with n men and $n + k$ women.*

- Initialization: *Run the men-proposing deferred acceptance algorithm to get the men-optimal stable matching μ. Initialize S to be the set of women unmatched under μ. Select any $\hat{w} \in \mathcal{W} \backslash S$.*

- New phase:

1. *Set* $\tilde{\mu} \leftarrow \mu$*. Set* $v_1 \leftarrow \hat{w}$ *and* $V \leftarrow (\hat{w})$*.*

2. Divorce: *Set* $m \leftarrow \mu(\hat{w})$ *and have* $\hat{w}$ *reject* m*.*

3. Proposal: *Man* m *proposes to his most preferred woman* w *who has not already rejected him.*[22]

4. w's Decision:

   (a) *If* $w \notin V$ *and* w *prefers* $\mu(w)$ *over* m*, or if* $w \in V$ *and* w *prefers* $\tilde{\mu}(w)$ *over* m*, then* w *rejects* m*. Go to step 3.*

   (b) *If* $w \notin S \cup V$ *and* w *prefers* m *over* $\mu(w)$*, then* w *rejects her current partner. Set* $m' \leftarrow \mu(w)$*,* $\mu(w) \leftarrow m$*. Append* w *to the end of* V*. Set* $m \leftarrow m'$ *and go to step 3.*

   (c) New stable matching: *If* $w \in V$ *and* w *prefers* m *over* $\tilde{\mu}(w)$*, then we have found a stable matching. If* $w = \hat{w} = v_1$*, set* $\mu(\hat{w}) \leftarrow m$*. Select* $\hat{w} \in \mathcal{W} \backslash S$ *and start a new phase from step 1.*

   If $w = v_\ell$ *for* $\ell > 1$*, record her current husband as* $m' \leftarrow \mu(w)$*. Call the set of all proposals made after the proposal of* m' *to* w *an* internal improvement cycle *(IIC). Set* $\mu(w) \leftarrow m$ *and update* $\tilde{\mu}$ *for the women in the loop by setting* $\tilde{\mu}(v_j) \leftarrow \mu(v_j)$ *for* $j = \ell, \ell+1, ..., J$*. Remove* $v_\ell, .., v_J$ *from* V*, set the proposer* $m \leftarrow m'$*, and return to step 3, in which* m *(earlier* m'*) will again propose to* w*.*

   (d) End of terminal phase: *If* $w \in S$ *and* w *prefers* m *over* $\mu(w)$*, then restore* $\mu \leftarrow \tilde{\mu}$ *and add all the women in* V *to* S*. If* $S = \mathcal{W}$*, terminate and output* $\tilde{\mu}$*. Otherwise, select* $\hat{w} \in \mathcal{W} \backslash S$ *and begin a new phase from step 1.*

4 Computational experiments

This section presents simulation results that complement our theoretical results. We first present simulation results for small markets and test the accuracy of estimates based on our theoretical findings. We then investigate the effects of correlation in preferences and test the

[22] In markets with complete preferences and strictly more women than men, such a woman always exists. For general markets, if m prefers to be unmatched over any woman who has not already rejected him, the algorithm continues directly to step 4(d).

robustness of our results. Last, we present simulation results for unbalanced many-to-one matching markets.

4.1 Numerical results for our model

The first computational experiment illustrates the sharp effect of imbalance in a small market. We consider markets with 40 women and a varying number of men, from 20 to 60 men. For each market size we simulate 10,000 realizations by drawing uniformly random complete preferences independently for each agent. For each realization we compute the MOSM and WOSM. Figure 1 (Section 2) reports the fraction of men who have multiple stable partners (averaged across realizations, as well as the top and bottom 2.5th percentiles); this fraction is small in all unbalanced markets.

Figure 3 reports an average across realizations of the men's average rank of wives[23] under the MOSM and WOSM as well as under Random Serial Dictatorship (RSD).[24] The results for the balanced market (40 men and 40 women) replicate the previous analysis by Pittel (1989a) and Roth and Peranson (1999). But in any unbalanced market, the men's average rank of wives is almost the same under the MOSM and WOSM. When there are fewer men than women (that is, less than 40 men) the men's average rank of wives under any stable matching is almost the same as under RSD, with most men receiving one of their top choices. When there are more men than women in the market, the men's average rank of wives is not much better than 20.5, which would be the result of a random assignment.

Tables 1 and 2 report simulation results for markets with varying size and imbalance. Table 1 reports the men's average rank of wives under the MOSM and WOSM, showing the sharp effect of imbalance across different market sizes. Under both the MOSM and WOSM the men's average rank of wives is large when there are strictly fewer women (columns $-10, -5, -1$) and small when there strictly fewer men (columns $1, 2, 3, 5, 10$). In addition, Table 1 reports theoretical estimates of the average rank based on the asymptotic results in Section 2. The estimate from our theorem is asymptotically accurate for the men's average rank of wives when men are on the short side, while it is only an asymptotic lower bound

[23]The women's average rank of husbands is symmetrically given by switching the number of men and women.

[24]In the RSD mechanism men are ordered at random, and each man chooses his favorite woman who has yet to be chosen (thus RSD ignores women's preferences).

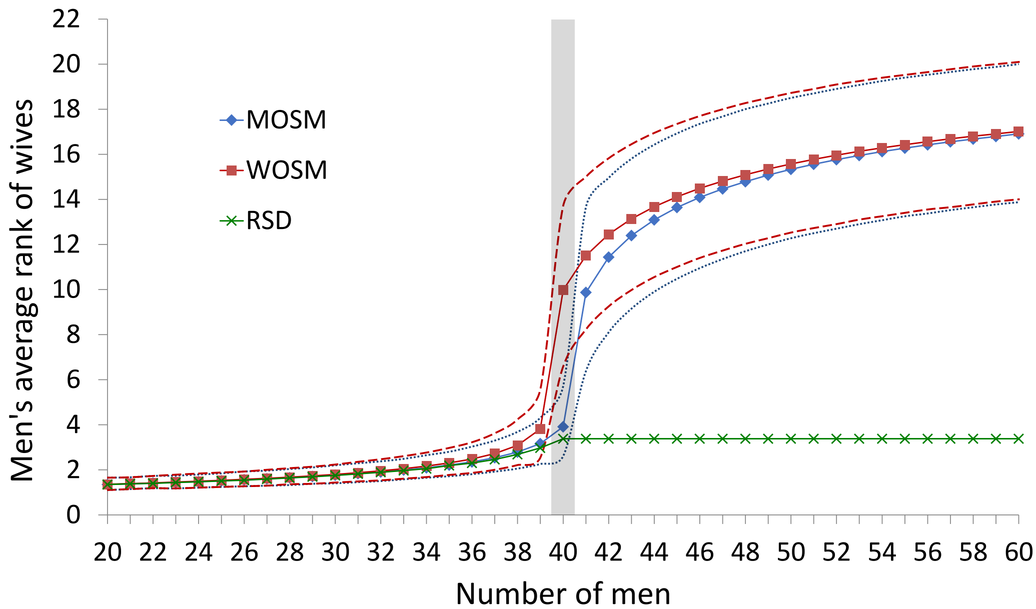


Figure 3: Men's average rank of wives under MOSM and WOSM in random markets with 40 women and a varying number of men. The main lines indicate the average over matched mens' average rank of wives in all 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentiles of the 10,000 realizations. The line labeled RSD gives the men's average rank under the Random Serial Dictatorship mechanism.

when men are the long side (see Appendix D for details). Observe that when men are on the short side of the market (the right side of the table) our estimate gives a surprisingly good approximation. When men are the long side (the left side of the table) our estimate is only about $10 - 25\%$ below the true values. Table 2 presents the percentage of men who have multiple stable partners. This percentage is large in balanced markets but is small in all unbalanced markets.

Using the matching algorithm from Section 3 we were able to run simulations of rather large unbalanced matching markets. Table 3 provides numerical results of 1,000 draws of matching markets of different sizes. The results show that in markets of any size a slight imbalance leads to a small core and agents on the short side are typically matched with one of their top choices.

$\lvert\mathcal{M}\rvert$ \ $\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert$		-10	-5	-1	0	1	2	3	5	10
100	MOSM	29.5	27.2	20.3	5.0	4.1	3.7	3.4	3.0	2.6
	WOSM	30.1	28.2	23.6	20.3	4.9	4.1	3.6	3.2	2.6
	EST	25.3	22.9	17.5		4.7	4.0	3.6	3.2	2.6
200	MOSM	53.6	48.1	35.3	5.7	4.8	4.3	4.1	3.7	3.1
	WOSM	54.7	49.9	41.0	35.5	5.7	4.7	4.4	3.8	3.2
	EST	45.7	40.8	31.5		5.3	4.7	4.3	3.8	3.2
500	MOSM	115.8	102.6	75.9	6.7	5.7	5.3	5.0	4.5	3.9
	WOSM	118.0	106.3	86.6	76.2	6.7	5.7	5.3	4.7	4.0
	EST	98.2	87.6	69.0		6.2	5.5	5.2	4.7	4.0
1000	MOSM	203.8	181.4	136.2	7.4	6.4	6.0	5.7	5.2	4.6
	WOSM	207.5	187.6	155.1	137.3	7.4	6.5	6.0	5.4	4.7
	EST	175.2	157.3	126.2		6.9	6.2	5.8	5.3	4.7
2000	MOSM	364.5	324.2	249.6	8.1	7.1	6.7	6.3	5.9	5.3
	WOSM	370.8	334.4	280.7	249.1	8.1	7.1	6.6	6.1	5.4
	EST	314.6	284.7	232.3		7.6	6.9	6.5	6.0	5.3
5000	MOSM	793.1	713.5	560.0	9.1	8.1	7.6	7.3	6.8	6.2
	WOSM	804.7	732.8	622.5	560.2	9.1	8.1	7.6	7.0	6.3
	EST	690.5	631.1	525.2		8.5	7.8	7.4	6.9	6.2

Table 1: Men's average rank of wives under the MOSM and WOSM for different market sizes. The numbers for each market size are averages over 1,000 realizations. A man's most preferred wife has rank 1, and a larger rank indicates a less preferred wife. EST is the theoretical estimate of the men's average rank based on Theorem 2 (see Appendix D for details).

4.2 Size of the core under correlated preferences

This section presents simulation results to examine the effects of correlation in preferences on the size of the core. We simulated a large class of distributions and found a large core only under balanced markets with little correlation in preferences.[25] Our simulations suggest that the core generally becomes smaller as preferences become more correlated, although there

[25] We did find somewhat unnatural correlation structures that also lead to a large core in balanced markets. For example, when generating preferences by drawing ε_{mw} independently for each pair (m, w) and setting $u_w(m) = \varepsilon_{mw} = -u_m(w)$. However, for all such structures we examined the core is small in unbalanced markets.

$\lvert\mathcal{M}\rvert$ \ $\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert$	-10	-5	-1	0	1	2	3	5	10
100	2.1	4.1	15.1	75.3	15.4	9.5	6.5	4.5	2.3
200	2.2	3.8	14.6	83.6	14.6	8.0	6.2	4.1	2.1
500	2.0	3.8	12.6	91.0	13.1	7.1	5.5	3.6	2.0
1000	1.9	3.5	12.3	94.5	12.2	7.2	5.1	3.4	2.0
2000	1.8	3.1	11.1	96.7	11.1	6.1	4.8	2.9	1.7
5000	1.5	2.7	10.1	98.4	10.2	6.0	4.3	2.8	1.5

Table 2: Percentage of men who have multiple stable partners for different market sizes. The numbers for each market size are averages over 1,000 realizations.

	$\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert = +1$			$\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert = +10$		
$\lvert\mathcal{M}\rvert$	Men's avg rank under MOSM	Men's avg rank under WOSM	% Men w. mul. stable partners	Men's avg rank under MOSM	Men's avg rank under WOSM	% Men w. mul. stable partners
10	2.0 (0.4)	2.3 (0.6)	13.8 (18.8)	1.3 (0.2)	1.3 (0.2)	1.2 (5.1)
100	4.1 (0.7)	4.9 (1.1)	15.2 (13.0)	2.5 (0.3)	2.6 (0.3)	2.3 (3.1)
1,000	6.5 (0.8)	7.4 (1.3)	11.9 (10.2)	4.6 (0.3)	4.7 (0.3)	1.9 (2.0)
10,000	8.8 (0.8)	9.8 (1.3)	9.4 (8.3)	6.9 (0.3)	7.0 (0.3)	1.5 (1.5)
100,000	11.1 (0.8)	12.1 (1.3)	7.7 (6.6)	9.1 (0.3)	9.3 (0.3)	1.1 (1.0)
1,000,000	13.4 (0.8)	14.4 (1.3)	6.6 (6.0)	11.5 (0.3)	11.6 (0.3)	0.8 (0.8)

Table 3: Men's average rank in different market sizes with a small imbalance. The numbers for each market size are averages over 1,000 realizations. Standard deviations are given in parentheses.

are examples of the opposite.[26]

We present results on preferences generated from the following random utility model, adapted from Hitsch et al. (2010). Each agent ℓ has two characteristics, x^A_ℓ and x^D_ℓ. The

[26]It is possible to construct specific examples such that an increase in correlation increases the size of the core. For example, if there are 41 men and 40 women with i.i.d. preferences we can create a balanced market (and hence a large core) by making all women rank the same man at the bottom of their list.

utility of agent i for being matched with agent j is given by

$$u_i(j) = \beta x_j^A - \gamma(x_i^D - x_j^D)^2 + \varepsilon_{ij}, \tag{1}$$

where ε_{ij} is an idiosyncratic term for the pair (i, j) independently drawn from the standard logistic distribution. We use x_i^A as a vertical quality that is desirable for all agents; with $x_i^A \sim U[0, 1]$ drawn independently for each agent. We use x_i^D as a location; with $x_i^D \sim U[0, 1]$ drawn independently for each agent. The values of the coefficient γ determines agents' preferences to be close to/far from their partner. The same values of the coefficients β, γ are used for both men and women.

When $\beta = \gamma = 0$ preferences are drawn independently and uniformly at random. As β increases (keeping γ fixed) preferences become more correlated, and if $\beta \to \infty$ all men will have the same preferences over women (and symmetrically). Taking $\gamma \neq 0$ generates alignment between $u_m(w)$ and $u_w(m)$.[27]

Figure 4 shows the size of the core for different levels of imbalance and a range of coefficient values. For each market we simulate 2,000 realizations and report the average percentage of men with multiple stable partners.[28] We consider markets with 40 women and either 40, 41, 45, or 60 men, $\beta \in [0, 20]$ and $\gamma \in [-20, 20]$. Observe that correlation tends to reduce the size of the core.[29] The only markets that have a large core are balanced markets with low levels of correlation in preferences. We interpret these results as complementary to our theoretical results, giving additional suggestive evidence that general preference distributions are likely to generate a small core.

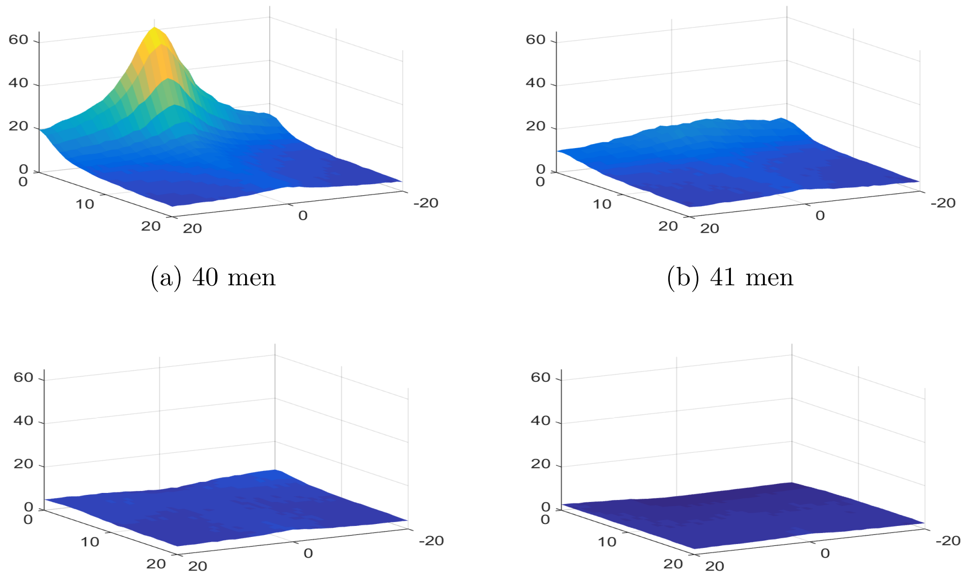
4.3 Robustness of the short side advantage

Theorem 2 shows that if there are strictly fewer men than women, and preferences are complete and uncorrelated, then men will have a large advantage: men will be matched with one of their top choices, while women will be matched with a husband who is not much better than a random man. But this result may not hold when preferences are correlated, for example, if all men have the same preferences, the top-ranked woman must be matched

[27] See Yariv and Lee (2014) for analysis of fully aligned random preferences.

[28] We also measured the size of the core using the difference/ratio between the men's rank of wives under MOSM and under WOSM. These measures produce very similar results and are therefore omitted.

[29] See footnotes 25 and 26.



(a) 40 men

(b) 41 men

(c) 45 men

(d) 60 men

Figure 4: The percentage of men with multiple stable partners for correlated preferences, generated as per Eq. (1). In each plot the z-axis (vertical and color) gives the percentage of agents with multiple stable partners, the x-axis is the value of the coefficient γ (ranging from -20 to 20), and the y-axis is the value of coefficient β (ranging from 0 to 20). The number of men is 40, 41, 45 or 60, as labeled for each plot. In all markets there are 40 women.

with her top choice (regardless of the number of men and women). We therefore conduct numerical experiments to investigate the extent of the men's advantage when there are fewer men and preferences are correlated.

Figure 5 reports the results for different markets generated using the utility model from Section 4.2. Each panel reports the men's average rank of wives under different values of β, γ for markets with 40 women and from 20 to 60 men. Each panel contains as a reference point the graph for $\beta = \gamma = 0$ (which replicates Figure 3). We plot the men's average rank only under the MOSM, as the men's average rank under the WOSM is almost identical (it differs only for balanced markets).

Panel 5a shows results with $\gamma = 0$ and different values of β as indicated. A higher value of β makes preferences of agents on the same side more correlated. The short side of the market retains an advantage for small values of β. When β is very large all men have almost the same preferences over women, and therefore the men's average rank is $(\min(|\mathcal{M}|, |\mathcal{W}|)+1)/2$.

Panel 5b shows results for varying values of γ, holding $\beta = 0$. If $\gamma > 0$ both agents prefer a close partner, generating alignment between the preferences of men and women, as well as correlation in the preferences of men. As γ increases the men's average rank of wives increases in markets with fewer than 40 men, but it decreases in markets with more than 40 men since matched men are likely to be favorably ranked by their wives, and therefore likely to rank their wives favorably.

Panel 5c shows the results when taking $\gamma = \beta$, and each line is labeled by the common value for both coefficients. The advantage of the short side gradually decreases as correlation increases, but is still evident even when there is a sizeable correlation.

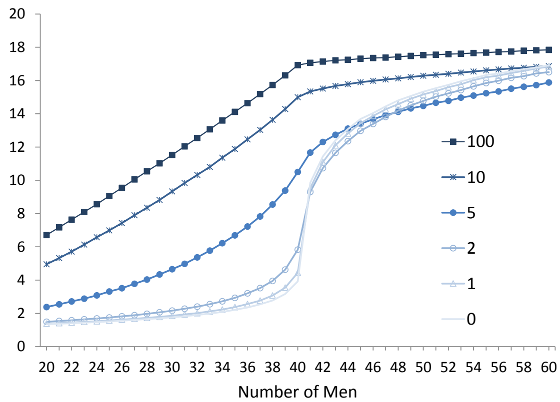
Put together, we find that for low levels of correlation, men tend to receive a lower average rank when there are fewer men. This advantage is continuously attenuated as agents' preferences become more correlated.

We further simulate markets in which agents find some potential partners to be unacceptable. For each man m and woman w we independently draw $u_m(w) \sim U[0, 1]$, where m's utility of remaining single is $\bar{u}^M$. Thus w is unacceptable to m if $u_m(w) < \bar{u}^M$, and if $\bar{u}^M = 0.33$ there is a 33% chance that a man will find a potential wife to be unacceptable. Preferences for women were drawn analogously, with $\bar{u}^W$ being the utility of remaining single. Figure 6 reports the men's average rank of wives under the MOSM [30] for markets with 40 women and from 20 to 60 men and $(\bar{u}^M, \bar{u}^W) \in \{0, .33, .66\}^2$. The shape of the marker indicates the value of $\bar{u}^M$, and the shade indicates the value of $\bar{u}^W$. The number of matched agents is at least 85% of $\min(|\mathcal{M}|, |\mathcal{W}|)$ in all reported markets. We find that only the selectivity of the long side matters: the lines merge according to shade when there are more women, and according to shape when there are more men. The benefit of the short side is still apparent, but it is continuously attenuated as the long side becomes more selective.

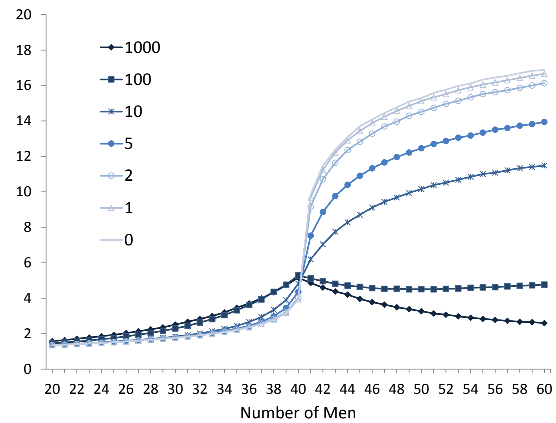
4.4 Many-to-one markets

We run computational experiments to investigate the effect of imbalance in many-to-one matching markets, in which colleges have responsive preferences (Roth, 1985) and each college is small relative to the size of the market (see Appendix C for further discussion). For

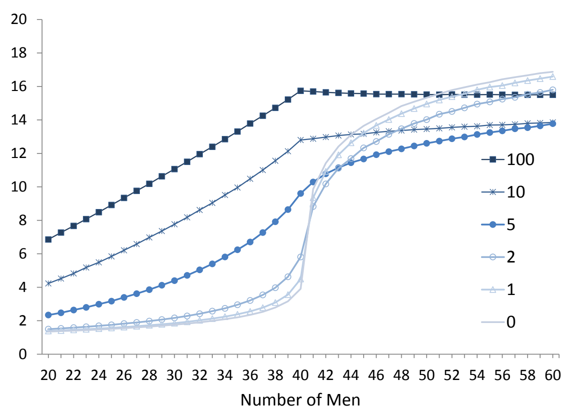
[30] The men's average rank under WOSM is omitted as it is almost identical in all unbalanced markets.



(a) Correlated Preferences



(b) Aligned Preferences



(c) Equal Coefficients, $\beta = \gamma$

Figure 5: Men's average rank of wives under MOSM for correlated preferences, generated as per Eq. (1), with 40 women and a varying number of men. Panel 5a plots the average rank when $\gamma = 0$ and β ranges from 0 to 100, with different lines corresponding to different values of β. Similarly, Panel 5b plots the average rank when $\beta = 0$ and γ ranges from 0 to 1000. Panel 5c plots the average rank for $\beta = \gamma$.

each student we independently draw a complete preference list over colleges uniformly at random. For each college we independently draw a complete preference list over individual students uniformly at random.

Denote by $\mathcal{S}$ the set of students, by $\mathcal{C}$ the set of colleges, and by q the number of seats in each college. We say that a market is unbalanced if the number of students differs from the

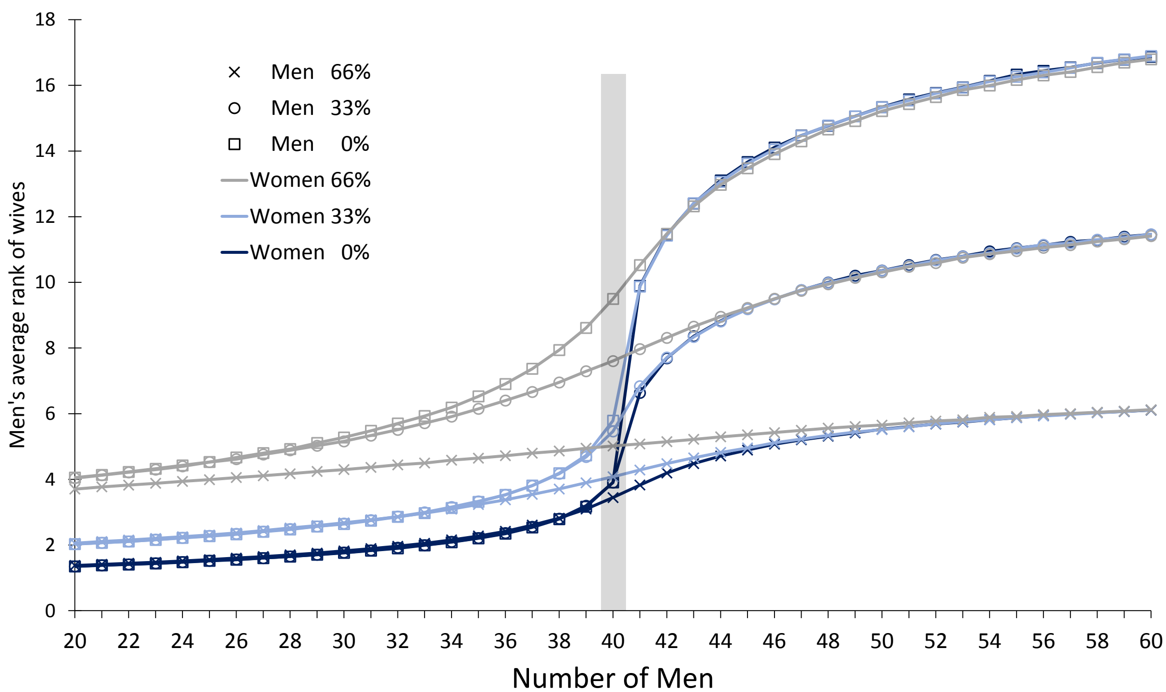


Figure 6: Men's average rank of wives for various levels of imbalance and selectivity in random markets with 40 women and a varying number of men. Each of the 3×3 lines indicates a different level of selectivity for men and women, marked by shade and shape. The shape indicates the selectivity of men, giving the chance a man will find a potential wife to be unacceptable. The shade indicates the selectivity of women, giving the chance that a woman will find a potential husband to be unacceptable.

total number of seats, i.e., $|\mathcal{S}| \neq |\mathcal{C}| \times q$. For each market size we draw 1,000 realizations and compute the extreme stable matchings: the student-optimal stable matching (SOSM) and the college-optimal stable matching (COSM). The students' average rank of their colleges (under both SOSM and COSM) is defined as before. The colleges' average rank of students is computed by averaging the rank of students in all filled seats.

Tables 4, 5, and 6 report simulation results for markets with an overall number of seats of $q \times |\mathcal{C}| = 100, 200, 500, 1000$, with a varying number of seats per college ($q = 2, 5$) and a varying number of students $|\mathcal{S}|$ (varying from $q|\mathcal{C}| - 10$ to $q|\mathcal{C}| + 10$). As in the one-to-one markets, we find that there is a big difference between the SOSM and COSM when the market is balanced, but in unbalanced markets the SOSM and COSM give almost the same average rank to both colleges and students. The percentage of students with multiple stable matches is large when the market is balanced, but it is small under imbalance.

If we increase q while holding $q \times |\mathcal{C}|$ and $|\mathcal{S}|$ fixed, we find that the core shrinks, and the college's average rank increases. Note that an increase in q effectively makes preferences over seats more correlated, and our findings here resemble the effects of increased correlations reported in Sections 4.2 and 4.3. The average rank of students is not directly comparable across different values of q, since the length of the students' preference list changes.

$|\mathcal{C}| \times q$ \ $|\mathcal{S}| - |\mathcal{C}| \times q$		-10	-5	-2	-1	0	1	2	5	10
100 (50×2)	SOSM	1.8	2.0	2.4	2.7	3.1	10.4	12.0	14.4	16.3
	COSM	1.8	2.1	2.6	3.1	10.5	12.0	13.0	14.9	16.6
200 (100×2)	SOSM	2.1	2.4	2.8	3.1	3.6	17.8	20.4	24.7	28.1
	COSM	2.2	2.5	3.1	3.5	17.9	20.6	22.3	25.6	28.7
500 (250×2)	SOSM	2.6	2.9	3.4	3.6	4.1	38.0	43.3	51.6	59.1
	COSM	2.6	3.0	3.6	4.1	38.2	43.5	46.9	53.4	60.2
1000 (500×2)	SOSM	3.0	3.3	3.8	4.0	4.5	69.1	78.0	91.7	103.7
	COSM	3.0	3.4	4.0	4.5	68.5	78.1	83.8	94.6	105.2
100 (20×5)	SOSM	1.3	1.5	1.6	1.7	1.9	4.5	5.1	6.0	6.8
	COSM	1.3	1.5	1.7	1.9	4.5	5.1	5.5	6.2	6.9
200 (40×5)	SOSM	1.5	1.7	1.8	2.0	2.1	7.5	8.5	10.2	11.6
	COSM	1.5	1.7	1.9	2.1	7.6	8.5	9.3	10.6	11.8
500 (100×5)	SOSM	1.8	1.9	1.1	2.3	2.4	15.6	17.7	21.0	23.9
	COSM	1.8	2.0	2.2	2.4	15.6	17.8	19.1	21.7	24.3
1000 (200×5)	SOSM	1.9	2.1	2.3	2.5	2.7	27.7	31.4	36.8	41.6
	COSM	2.0	2.2	2.5	2.6	28.0	31.4	33.8	38.0	42.3

Table 4: Students' average rank of colleges in the SOSM and COSM, for different markets. A student's most preferred college has rank 1, and a larger rank indicates a less preferred college.

5 Discussion

Competition has been at the heart of economic theory since Edgeworth (1881), and may have a stark effect as in the famous left-glove right-glove example by Shapley and Shubik

$\lvert\mathcal{C}\rvert \times q$ \ $\lvert\mathcal{S}\rvert - \lvert\mathcal{C}\rvert \times q$		-10	-5	-2	-1	0	1	2	5	10
100 (50×2)	SOSM	33.4	31.5	28.9	27.1	24.5	7.1	5.9	4.7	3.9
	COSM	32.9	30.6	26.8	23.9	7.2	6.0	5.4	4.5	3.8
200 (100×2)	SOSM	61.3	56.8	50.9	48.3	42.8	8.5	7.2	5.7	4.8
	COSM	60.4	55.1	47.4	42.9	8.5	7.2	6.5	5.4	4.6
500 (250×2)	SOSM	134.7	123.1	110.3	103.3	92.4	10.0	8.6	7.0	6.0
	COSM	132.6	119.1	102.9	92.7	10.1	8.6	7.9	6.8	5.8
1000 (500×2)	SOSM	242.3	220.4	199.1	188.3	168.1	11.1	9.6	8.0	7.0
	COSM	238.7	213.9	187	169.4	11.2	9.6	8.9	7.8	6.9
100 (20×5)	SOSM	38.1	36.9	34.8	33.4	31.5	13.4	11.3	9.0	7.5
	COSM	37.8	36.3	33.2	31.2	13.4	11.3	10.3	8.6	7.3
200 (40×5)	SOSM	72.2	68.1	63.3	60.5	56.4	16.1	13.8	11.0	9.3
	COSM	71.4	66.5	60.2	56.4	16.1	13.8	12.5	10.6	9.1
500 (100×5)	SOSM	162.6	152.1	140.8	132.7	123.7	19.4	16.8	13.8	11.9
	COSM	160.7	148.4	133.9	123.7	19.5	16.7	15.5	13.3	11.6
1000 (200×5)	SOSM	300.0	278.6	256.8	244.5	229.3	22.1	19.1	16.0	13.9
	COSM	296.7	272.3	244.7	228.4	21.9	19.2	17.7	15.5	13.7

Table 5: Colleges' average rank of students in the SOSM and COSM, for different markets. A college's most preferred student has rank 1, and a larger rank indicates a less preferred student.

(1969). We show that there is a similar stark effect of competition in random matching markets, despite the heterogeneity of preferences and the lack of monetary transfers. This allows us to characterize stable matchings and to provide an explanation as to why small cores are empirically ubiquitous.

Under more general preference distributions the structure of stable matchings may be more involved. As an example, consider a tiered one-to-one market with 30 workers and 40 firms in two tiers: 20 firms are "top" and 20 firms are "middle." Preferences are drawn uniformly at random, except that every worker prefers any top firm over any middle firm. Stable matchings can be decomposed to a stable matching between the top 20 firms and the 30 workers, and a stable matching between the remaining 10 unmatched workers and the 20 middle firms. Applying our results to each part implies that the core is small, and

$\lvert\mathcal{S}\rvert - \lvert\mathcal{C}\rvert \times q$ / $\lvert\mathcal{C}\rvert \times q$	-10	-5	-2	-1	0	1	2	5	10
100 (50×2)	1.8	3.4	7.8	12.4	70.7	14.1	8.4	3.9	2.1
200 (100×2)	1.8	3.3	7.3	11.4	79.8	14.1	8.6	4.0	2.2
500 (250×2)	1.7	3.4	6.9	10.4	89.0	12.7	7.9	3.7	2.0
1000 (500×2)	1.6	3.0	6.2	10.2	93.3	11.7	7.0	3.2	2.0
100 (20×5)	1.2	1.9	5.0	6.8	57.5	13.8	7.9	3.9	1.7
200 (40×5)	1.3	2.6	5.2	7.1	71.4	13.1	8.5	3.9	2.1
500 (100×5)	1.2	2.6	5.0	6.8	84.1	12.5	7.2	3.7	2.1
1000 (200×5)	1.1	2.3	4.8	6.7	90.3	12.0	7.1	3.4	1.8

Table 6: Percentage of students who have multiple stable matches.

stable matchings can roughly be described by the 20 top firms choosing their worker, and the remaining 10 workers choosing their firm from among the middle firms. A similar argument can be used to characterize stable matchings in any tiered market.

In Appendix C, we discuss how to extend our analysis to many-to-one markets between colleges and students, assuming that colleges have responsive preferences and that each college is small relative to the size of the market. The sharp effect of competition is also present in these markets: whenever there is even a slight difference between the number of seats and the number of students in the market, the core is small, and the short side will “choose”.

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A Matching algorithm

The goal of this section is to present Algorithm 2, which is the basis of our analysis. This algorithm allows us to calculate the WOSM by a process of successive proposals by men.

It first finds the men-optimal stable matching using the men-proposing DA algorithm (Algorithm 1 or MPDA), and then progresses to the women-optimal stable matching through a series of divorces of matched women followed by proposals by men. At the end of this section we show how the run of the algorithm on a random matching market is equivalent to a randomized algorithm. In Appendix B we analyze the randomized algorithm to prove Theorem 1. Throughout our analysis and this section we assume that there are strictly more women than men, that is, $|\mathcal{W}| > |\mathcal{M}|$.

Before presenting Algorithm 2 we first give a simplified version. The following algorithm, adapted from McVitie and Wilson (1971) and Immorlica and Mahdian (2005), produces the WOSM from the MOSM by finding each woman's most preferred stable match. It uses $\tilde{\mu}$ to maintain the most recent stable matching. Each phase instigates a rejection chain to check whether there is a more women-preferred stable matching, using m to hold the proposing man and μ to hold the temporary assignment. It also maintains a set S of women whose most preferred stable match has been found. Denote the set of women who are unmatched under the MOSM by[31] $\bar{\mathcal{W}}$. We initialize S to be $S = \bar{\mathcal{W}}$, as by the rural hospital theorem these women are unmatched under any stable matching. We use the notation $x \leftarrow y$ for the operation of copying the value of variable y to variable x.

Algorithm 3. MOSM to WOSM (simplified)

- Input: *A matching market with n men and $n + k$ women.*

- Initialization: *Run the men-proposing deferred acceptance to get the men-optimal stable matching μ. Set $S = \bar{\mathcal{W}}$ to be the set of women unmatched under μ. Select any $\hat{w} \in \mathcal{W} \backslash S$.*

- New phase:

  1. *Set $\tilde{\mu} \leftarrow \mu$.*

  2. Divorce: *Set $m \leftarrow \mu(\hat{w})$ and have $\hat{w}$ reject m.*

  3. Proposal: *Man m proposes to his most preferred woman w to whom he has not yet proposed.*[32]

[31] Since there are more women than men $\bar{\mathcal{W}} \neq \phi$.

[32] See footnote 22.

4. w's Decision:

(a) *If* $w \neq \hat{w}$ *prefers her current match* $\mu(w)$ *over* m*, or if* $w = \hat{w}$ *and she prefers* $\tilde{\mu}(\hat{w})$ *over* m*, she rejects* m*. Go to step 3.*

(b) *If* $w \notin \{\hat{w}\} \cup \bar{\mathcal{W}}$*, and* w *prefers* m *over* $\mu(w)$*, then* w *rejects her current partner and accepts*[33] m*. Go to step 3.*

(c) New stable matching: *If* $w = \hat{w}$ *and* $\hat{w}$ *prefers* m *over all her previous proposals,* w *accepts* m *and a new stable matching is found. Select* $\hat{w} \in \mathcal{W} \backslash S$ *and start a new phase from step 1.*

(d) End of terminal phase: *If* $w \in \bar{\mathcal{W}}$*, restore* $\mu \leftarrow \tilde{\mu}$*, erase rejections accordingly, and add* $\hat{w}$ *to* S*. If* $S = \mathcal{W}$*, terminate and output* $\tilde{\mu}$*. Otherwise, select* $\hat{w} \in \mathcal{W} \backslash S$ *and begin a new phase from step 1.*

It will be convenient to use the following terminology. A **phase** is a sequence of proposals made by the algorithm between visits to step 1. An **improvement phase** is a phase that terminates at step 4(c) (a new stable matching is found). A **terminal phase** is a phase that terminates at step 4(d) (there is no better stable husband for $\hat{w}$). We refer to the sequence of women who reject their husbands in a phase as the **rejection chain**.

In each phase the algorithm tries to find a more preferred husband for $\hat{w}$, which requires divorcing $m = \tilde{\mu}(\hat{w})$ and assigning him to another woman. In improvement phases the algorithm finds a more preferred stable husband for $\hat{w}$, and updates $\tilde{\mu}$ to the new stable matching. In terminal phases the algorithm finds that $\hat{w}$ cannot be assigned a man she prefers over m without creating a blocking pair, and therefore m is $\hat{w}$'s most preferred stable parter.

Proposition A.1. *Algorithm 3 outputs the women-optimal stable matching.*

Proof. The algorithm terminates as each man can make only a finite number of proposals, and there is at most one terminal phase (that is rolled back) per woman. Consider a phase that begins with a stable $\tilde{\mu}$. Immorlica and Mahdian (2005) show that if the phase ends at step 4(c), the matching $\tilde{\mu}$ at the end of the phase is stable as well. By induction, every $\tilde{\mu}$ is a stable matching. Immorlica and Mahdian (2005) also show that if a phase ends at step 4(d), then $\tilde{\mu}(\hat{w})$ is $\hat{w}$'s most preferred stable man. Any subsequent matching $\tilde{\mu}$ is a

[33] More precisely, we use a temporary variable m' as follows: $m' \leftarrow \mu(w)$, $\mu(w) \leftarrow m$ and $m \leftarrow m'$.

stable matching in which $\hat{w}$ is weakly better off, and therefore $\tilde{\mu}$ also matches $\hat{w}$ with her most preferred stable man. Finally, any woman in $\bar{\mathcal{W}}$ is unmatched under the WOSM by the rural hospital theorem (Roth, 1986). Thus the algorithm terminates with $\tilde{\mu}$ being the WOSM that matches all women with their most preferred stable husband. □

We now refine the algorithm to prune repetitions. Since proposals made during terminal phases are rolled back, we can end a phase (and roll back to $\tilde{\mu}$) as soon as we learn that the phase is a terminal phase. During the run of Algorithm 3 each woman in S is matched with her most preferred stable husband. Therefore we can terminate the phase (and roll back) if a woman in S accepts a proposal, as this can happen only in terminal phases. Furthermore, suppose that in a terminal phase the rejection chain includes each woman at most once. We show that every woman in the rejection chain is matched under $\tilde{\mu}$ with her most preferred stable husband, and can therefore be added to S. When the rejection chain includes a woman more than once there are improvement cycles in the chain. We can identify these improvement cycles and implement them as an **Internal Improvement Cycle** (IIC). Specifically, whenever a woman in the chain receives a new proposal we check whether she prefers the proposing man over the best stable partner she has found so far. If she prefers the proposing man, the part of the rejection chain between this proposal and her best stable partner so far forms an improvement cycle. We implement the IIC by recording the stable matching we found in $\tilde{\mu}$ and removing the cycle from the rejection chain. By removing these cycles from the rejection chain we are left with a rejection chain that includes each woman at most once, and when the phase is terminal the entire chain can be added to S. See Algorithm 2, step 4(c) below for a precise definition of IICs.

Applying these modifications to Algorithm 3 gives us Algorithm 2. It keeps track of the women in the current rejection chain as an ordered set $V = (v_1, v_2, .., v_J)$ of women, and adds all of them to S if the phase is terminal. If a woman in S accepts a proposal the phase ends as a terminal phase. As in Algorithm 3, the variable $\tilde{\mu}$ saves the most recent stable matching and μ stores a candidate matching which evolves through the phase. This version of the algorithm also keeps track of $\nu(w)$, the current number of proposals received by woman w, and $R(m)$, the set of women who rejected m so far. These counters were omitted in the version given in Section 3, and are added here to allow us to refer to them later in the proof (The two versions of Algorithm 2 are otherwise identical).

Algorithm 2. MOSM to WOSM

- Input: *A matching market with* n *men and* $n+k$ *women.*
- Initialization: *Run the men-proposing deferred acceptance algorithm to get the men-optimal stable matching* μ*, and set* $R(m)$ *and* $\nu(w)$ *accordingly. Set* $t = 0$ *since no proposals have occurred yet. Initialize* S *to be the set of women unmatched under* μ*. Select any* $\hat{w} \in \mathcal{W} \backslash S$*.*
- New phase:

  1. *Set* $\tilde{\mu} \leftarrow \mu$*. Set* $v_1 \leftarrow \hat{w}$ *and* $V \leftarrow (\hat{w})$*.*
  2. Divorce: *Set* $m \leftarrow \mu(\hat{w})$ *and have* $\hat{w}$ *reject* m *(add* $\hat{w}$ *to* $R(m)$*).*
  3. Proposal: *Man* m *proposes to his most preferred woman*[34] *in* $\mathcal{W} \backslash R(m)$*. Increment* $\nu(w)$ *and proposal number* t *by one each.*
  4. w's Decision:
     (a) *If* $w \notin V$ *and* w *prefers* $\mu(w)$ *over* m*, or if* $w \in V$ *and* w *prefers* $\tilde{\mu}(w)$ *over* m*, then* w *rejects* m *(add* w *to* $R(m)$*). Go to step 3.*
     (b) *If* $w \notin S \cup V$ *and* w *prefers* m *over* $\mu(w)$*, then* w *rejects her current partner. Set* $m' \leftarrow \mu(w)$*,* $\mu(w) \leftarrow m$*. Add* w *to* $R(m')$ *and append* w *to the end of* V*. Set* $m \leftarrow m'$ *and go to step 3.*
     (c) New stable matching: *If* $w \in V$ *and* w *prefers* m *over* $\tilde{\mu}(w)$*, then we have found a stable matching. If* $w = \hat{w} = v_1$*, set* $\mu(\hat{w}) \leftarrow m$*. Select* $\hat{w} \in \mathcal{W} \backslash S$ *and start a new phase from step 1.*
     If $w = v_\ell$ *for* $\ell > 1$*, record her current husband as* $m' \leftarrow \mu(w)$*. Call the set of all proposals made after the proposal of* m' *to* w *an* internal improvement cycle *(IIC). Set* $\mu(w) \leftarrow m$ *and update* $\tilde{\mu}$ *for the women in the loop by setting* $\tilde{\mu}(v_j) \leftarrow \mu(v_j)$ *for* $j = \ell, \ell+1, ..., J$*. Remove* $v_\ell, .., v_J$ *from* V*, set the proposer* $m \leftarrow m'$*, decrement* $\nu(w)$*, decrement* t*, and return to step 3, in which* m *(earlier* m'*) will again propose to* w*.*

[34] See footnote 22.

(d) End of terminal phase: *If* $w \in S$ *and* w *prefers* m *over* $\mu(w)$*, then restore* $\mu \leftarrow \tilde{\mu}$ *and add all the women in* V *to* S*. If* $S = \mathcal{W}$*, terminate and output* $\tilde{\mu}$*. Otherwise, select* $\hat{w} \in \mathcal{W} \backslash S$ *and begin a new phase from step 1.*

In step 4(c) we found a new stable matching. If the rejection chain cycles back to the original woman, we have an improvement phase. If the rejection chain cycles back to a woman v_ℓ in the middle of the chain we implement the IIC (implementing the IIC is equivalent to an improvement phase that begins with v_ℓ). Update the best stable matching $\tilde{\mu}$ for all women in the cycle, and make it the current assignment. Then take m' and make him propose again to v_ℓ, as we changed $\mu(v_\ell)$. Decrement t and $\nu(v_\ell)$ in order not to count this proposal twice.

Proposition A.2. *Algorithm 2 outputs the women-optimal stable matching.*

Proof. Consider the run of Algorithm 2 for a given sequence of selections of $\hat{w}$. We will find a sequence of selections for Algorithm 3 such that Algorithm 2 is equivalent to Algorithm 3. Since Algorithm 3 outputs the WOSM for every sequence of selections, this will prove that Algorithm 2 finds the WOSM as well.

We construct the sequence of selections of $\hat{w}$ for Algorithm 3 by following the run of Algorithm 2 and making the state of Algorithm 2 at the time of proposal t identical to the state of Algorithm 3 in some proposal (note that the two algorithms may have a different proposal count). After the initialization step both algorithms are in an identical state, and we start them both with the same selection of $\hat{w}$. Assume that under our sequence of selections the algorithms are identical up to proposal $t-1$, and consider the t-th proposal made by Algorithm 2. Both algorithms perform the same actions when Algorithm 3 performs steps 4(a), 4(b), and 4(c) when $w = \hat{w}$. When a phase ends we start a new phase for Algorithm 3 by selecting the same $\hat{w}$ as in Algorithm 2. Therefore it remains to consider step 4(c) with $w \neq \hat{w}$ and step 4(d).

Consider the case in which proposal t in Algorithm 2 performs step 4(c) with $w \neq \hat{w}$; that is, the algorithm found an IIC with $w = v_\ell$. We change the sequence of selections of $\hat{w}$ in Algorithm 3 so that v_ℓ is chosen to be $\hat{w}$ just before the current phase. That makes the previous phase of Algorithm 3 an improvement phase for $\hat{w} = v_\ell$ in which the cycle of the IIC is implemented. Continuing to run Algorithm 3 into the current phase will reveal the chain $(v_1, ..., v_{\ell-1})$ and reach the proposal to woman v_ℓ. Algorithm 2 will have the same chain and proposal following the implementation of the IIC, and since now $v_\ell \notin V \cup S$ the two algorithms will have an identical state at the end of the step.

Consider next the case in which proposal t in Algorithm 2 performs step 4(d); that is, the phase is declared to be a terminal phase because a woman $w \in S$ accepted the proposal. First, we show that if we continue to run Algorithm 3 it will uncover a rejection chain that ends in $\bar{\mathcal{W}}$ and declare the phase to be a terminal phase. If $w \in \bar{\mathcal{W}}$ this is immediate. Otherwise, we infer the remainder of the rejection chain as follows. Woman w was added to S in some previous terminal phase. Recursively build the rejection chain $C = \{w_1, w_2, ..., w_q\} \subset S$, where $w_1 = w$, $w_q \in \bar{\mathcal{W}}$, and each w_j was added to S in a terminal phase where w_j rejected her husband m_j, triggering a series of proposals by m_j that resulted in m_j making a proposal that was accepted by w_{j+1}. We next show that following proposal t Algorithm 3 will continue to uncover the rejection chain C.

From the construction of C, at the time of proposal t we have that $\mu(w_j) = m_j$ under Algorithm 2. To see that, recall that each w_j was added to S in a terminal phase which reverted her back to the husband m_j she rejected. Following that terminal phase the assignment of w_j did not change, as the algorithm never changes assignments of women who are already in S. Using the induction assumption, at the step of Algorithm 3 that corresponds to proposal t of Algorithm 2 we have that $\mu(w_j) = m_j$.

Now, consider the continuation of the phase under Algorithm 3. When $w = w_1$ rejects her husband m_1, he will make proposals in the same order as he did in the terminal phase in which w_1 was added to S under Algorithm 2. All women that m_1 prefers over w_2 rejected him back then, and since throughout the algorithm a woman's assignment can only be changed to a more preferred husband, these women will reject him again in the current phase. Therefore, m_1 will end up proposing to w_2. Since at that point $\mu(w_2) = m_2$ the proposal will be accepted by w_2, making m_2 the new proposer. By induction, m_j will make an accepted proposal to w_{j+1}, until a proposal is made to $w_q \in \bar{\mathcal{W}}$. At this step Algorithm 3 declares a terminal phase and rolls back to $\tilde{\mu}$, at which point the phase ends with the same μ and $\tilde{\mu}$ as in Algorithm 2.

The remaining difference between the algorithms is that under Algorithm 2 at step 4(d) all women in V are added to S. Continue to run Algorithm 3 by starting a phase with a selection of a new $\hat{w}$ from V, until we have that $V \subset S$ under Algorithm 3. We have shown above that all these phases will be terminal phases, and will therefore be rolled back. Following these selections, Algorithm 3 will have the same $\tilde{\mu}$ and S as in Algorithm 2 following proposal t.

Therefore, by the end of the run we find a sequence of selections of $\hat{w}$ for Algorithm 3 such that the two algorithms hold identical μ and $\tilde{\mu}$ at the end of Algorithm 2. Thus when Algorithm 2 terminates, Algorithm 3 terminates as well and outputs the same matching. By Proposition A.1 the output of Algorithm 3 is the WOSM. □

The following lemma shows how Algorithm 2 allows us to compare the WOSM and MOSM.

Lemma A.3. *The difference between the sum of mens' rank of wives under WOSM and the sum of mens' rank of wives under MOSM is equal to the number of proposals in improvement phases and IICs during Algorithm 2.*

Proof. Note that at the end of each terminal phase (Step 4(d)) we roll back all proposals made in that phase and return to $\tilde{\mu}$, the matching from the previous phase. Therefore, we can consider only improvement phases and IIC, in which each proposal increases the rank of the proposing man by one. □

A.1 Randomized algorithm

As we are interested in the behavior of Algorithm 2 on a random matching market, we transform the deterministic algorithm on random input into a randomized algorithm, which will be easier to analyze. The randomized, or coin flipping, version of the algorithm does not receive preferences as input, but draws them through the process of the algorithm.[35] This is often called the *principle of deferred decisions*.

The algorithm reads the next woman in the preference of a man in step 3 and whether a woman prefers a man over her current proposal in step 4. Since the algorithm ends a phase immediately when a woman $w \in S$ accepts a proposal, no man applies twice to the same woman during the algorithm, and therefore the algorithm never reads previously revealed preferences.[36] In step 3 the randomized algorithm selects the woman w uniformly at random from $\mathcal{W} \setminus R(m)$. In step 4 the probability that w prefers m over her current match can be given

[35] The initialization step of the randomized version of Algorithm 2 uses the randomized version of Algorithm 1.

[36] There is an apparent exception to this in the case of an IIC, where we chose to describe the algorithm with a rollback of the proposal by man m' to v_ℓ that occurred just before the IIC, followed by remaking that

directly from $\nu(w)$ for $w \notin S$ or bounded for $w \in S$.[37] Table 7 describes the probabilities of the possible decisions of w. Note that the event in Step 4(a) is the complement of the union of the events in Table 7.

Step	**Event**	**Probability**
4(b)	$w \notin S \cup \{\hat{w}\}$ prefers m over $\mu(w)$	$\frac{1}{\nu(w)+1}$
4(c)	$\hat{w}$ prefers m over $\tilde{\mu}(\hat{w})$	$\frac{1}{\nu(\hat{w})+1}$
4(d)	$w \in S \backslash \bar{\mathcal{W}}$ prefers m over $\mu(w)$	at least $\frac{1}{\nu(w)+1}$

Table 7: Probabilities in a run of Algorithm 2 on a random matching market.

B Proof

We will prove the following quantitative version of the main theorem:

Theorem 3. *Fix any $\epsilon > 0$. Consider a sequence of random matching markets, indexed by n, with n men and $n+k$ women, for arbitrary $1 \le k = k(n)$. There exists $n_0 < \infty$ such that for all $n > n_0$, with probability at least $1 - \exp\{-(\log n)^{0.4}\}$, we have*

(i) In every stable matching μ:

$$R_{\text{MEN}}(\mu) \le (1+\epsilon)\big((n+k)/n\big)\log\big((n+k)/k\big)$$

$$R_{\text{WOMEN}}(\mu) \ge n/\big[1+(1+\epsilon)\big((n+k)/n\big)\log\big((n+k)/k\big)\big]\,.$$

(ii) Less than $n/(\log n)^{0.5}$ men, and less than $n/(\log n)^{0.5}$ women have multiple stable partners.

proposal (which we know will be rejected by v_ℓ); cf. Step 4(c) of Algorithm 2. This was merely to simplify the exposition; in our stochastic analysis we proceed by directly considering the proposal by m' to his next most preferred woman after v_ℓ.

[37] The probability that a woman $w \in S$ accepts a man m can be calculated from the number of proposals she received during improvement phases or MPDA and the number of proposals she received during terminal phases. Since the bound on the acceptance probability we calculate from $\nu(w)$ is sufficient for our analysis we omit the additional counters from the algorithm.

(iii) The men are almost as well off under the WOSM as under the MOSM:

$$\frac{R_{\text{MEN}}(\text{WOSM})}{R_{\text{MEN}}(\text{MOSM})} \leq 1 + (\log n)^{-0.4}.$$

(iv) The women are almost as badly off under the WOSM as under the MOSM:

$$\frac{R_{\text{WOMEN}}(\text{WOSM})}{R_{\text{WOMEN}}(\text{MOSM})} \geq 1 - (\log n)^{-0.4}.$$

Remark 1. *In our proof of Theorem 3 (ii), we actually bound the number of different stable partners. We show that the sum over all men (or women) of the number of different stable partners of each man is no more than $n + n/\sqrt{\log n}$. Thus, in addition to the bound stated in Theorem 3 (ii), we rule out the possibility that there are a few agents who have a large number of different stable partners.*

Definition B.1. *Given a sequence of events $\{\mathcal{E}_n\}$, we say that this sequence occurs* with very high probability (wvhp) *if*

$$\lim_{n\to\infty} \frac{1 - \mathbb{P}(\mathcal{E}_n)}{\exp\left\{-(\log n)^{0.4}\right\}} = 0.$$

Clearly, it suffices to show that (i)-(iv) in Theorem 3 hold wvhp.

To prove Theorem 3, we analyze the number of proposals in Algorithm 1 followed by Algorithm 2, which will provide us the average rank of wives in the women-optimal stable match. We partition the run of Algorithm 2 leading to the WOSM into three parts (Parts II through IV below).

1. **Part I is the run of DA (Algorithm 1)**, which by an analysis similar to that in (Pittel (1989a)), takes no more than $3n\log(n/k)$ proposals wvhp.

2. **Part II are the proposals in Algorithm 2 that take place before the end of first terminal phase.** We show that wvhp,

   - Part II takes no more than $(n/k)(\log n)^{0.45} \leq n(\log(n/k))^{0.45}$ proposals.
   - When part II ends the set S contains at least $n^{(1-\varepsilon)/2}$ elements.

3. **Part III are the proposals in Algorithm 2 after Part II that take place until $|S| \geq n^{0.7}$. Thus, this part ends at the end of a terminal phase when $|S|$ exceeds $n^{0.7}$ for the first time.** We show that, wvhp, part III requires $O(n^{0.47})$ phases, and $o(n)$ proposals.[38]

4. **Finally, Part IV includes the remaining proposals from the end of part III until Algorithm 2 terminates or $50n \log n$ total proposals have occurred (including proposals made in Parts I and II), whichever occurs earlier.** Because the set S is large, most phases are terminal phases containing no IICs, and most acceptances lead to eventual inclusion in S. We show that, wvhp, part IV ends with termination of the algorithm, and that the number of proposals in improvement phases and IICs is $o(n)$. But the increase in sum of men's rank of wives from the MOSM to the WOSM is exactly the number of proposals in improvement phases and IICs, yielding the result.

The definition of Part III based on when $|S|$ exceeds $n^{0.7}$ is for technical reasons, with the exponent of 0.7 being a choice for which our analysis goes through. Throughout this section, we consider the preferences on both sides of the market as being revealed sequentially as the algorithm proceeds, as discussed in Appendix A.1.

Lemma B.2. *Consider a man m, who is proposing at step 3 of Algorithm 2. Consider a subset of women $\mathcal{A} \subseteq \mathcal{W} \backslash R(m)$. Let $\nu(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{\tilde{w} \in \mathcal{A}} \nu(\tilde{w})$ be the average number of proposals received by women in $\mathcal{A}$. The man m proposes to some woman w in the current step. Conditional on $w \in \mathcal{A}$ and all preferences revealed so far, the probability that m is the most preferred man who proposed w so far, is at least $\frac{1}{\nu(\mathcal{A})+1}$.*

Proof. For any woman $\tilde{w} \notin R(m)$ the probability that m is the most preferred man who applied to w so far is $\frac{1}{\nu(\tilde{w})+1}$. Conditional on $w \in \mathcal{A}$, the probability that m is the most preferred man who applied to w so far is at least

$$\frac{1}{|\mathcal{A}|} \sum_{\tilde{w} \in \mathcal{A}} \frac{1}{\nu(\tilde{w}) + 1} \geq \frac{1}{\nu(\mathcal{A}) + 1}$$

by Jensen's inequality. □

[38] For any two functions $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}_{>0}$ we write $f = o(g)$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ and $f = O(g)$ if there exist a constant a such that $f(n) \leq ag(n)$ for sufficiently large n. We write $f(n) = \Theta(g(n))$ if there exist constants $a \leq b, n_0$ such that $ag(n) \leq f(n) \leq bg(n)$ for all $n > n_0$.

The following lemma will be convenient and its proof is trivial:

Lemma B.3. *If all men have lists of length $|\mathcal{W}|$ and $|\mathcal{W}| > |\mathcal{M}|$, then no man ever reaches the end of his list in Algorithm 1 or Algorithm 2.*

B.1 Part I

For the analysis in this section we consider the following equivalent version of men proposing deferred acceptance:

Algorithm 4. *Index the men $\mathcal{M}$. Initialize $S_{\mathcal{M}} = \phi$, $\bar{\mathcal{W}} = \mathcal{W}$, $\nu(w) = 0\ \forall w \in \mathcal{W}$ and $R(m) = \phi\ \forall m \in \mathcal{M}$.*

1. *If $\mathcal{M} \backslash S_{\mathcal{M}}$ is empty then terminate. Else, let m be the man with the smallest index in $\mathcal{M} \backslash S_{\mathcal{M}}$. Add m to $S_{\mathcal{M}}$.*

2. *Man m proposes to his most preferred woman w whom he has not yet applied to (increment $\nu(w)$). If he is at the end of his list, go to Step 1.*

3. *Decision of w:*

   - *If $w \in \bar{\mathcal{W}}$, i.e, w is unmatched then she accepts m, remove w from $\bar{\mathcal{W}}$. Go to Step 1.*

   - *If w is currently matched, she accepts the better of her current match and m and rejects the other. Set m to be the rejected man, add w to $R(m)$ and continue at Step 2.*

Note that the output of Algorithm 4 is the same as the output of Algorithm 1, i.e., it is the man optimal stable match (we have just reordered the proposals). The output of Algorithm 4 is given as an input to Algorithm 2. Again, we think of preferences as being revealed as the algorithm proceeds, with the women only revealing preferences among the set of men who have proposed them so far.

The next lemma establishes upper bounds on the average and maximum men's rank of wives and a lower bound on the women's average rank of husbands. The upper bound for the worst possible men's rank of wifes is due to Pittel (1989a) who obtained this bound in a balanced market (by adding more women to the market men are only becoming better off).

Lemma B.4. *Fix any $\epsilon > 0$. Let μ be the men-optimal stable matching. The following hold wvhp:*

(i) the men's average rank of wives in μ is at most $(1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$ and is at least $(1-\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$,

(ii) $\max_{m\in\mathcal{M}} \mathrm{Rank}_m(\mu(m)) \leq 3(\log n)^2$,

(iii) the women's average rank of husbands in μ is at least $n/\left[1+(1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)\right]$.

Proof. We first prove the upper bound in (i), then (ii), then the lower bound in (i), and finally (iii).

Tracking Algorithm 4 like in Pittel (1989a), we claim that, wvhp, the sum of the men's rank of wives is at most $(1+\epsilon)(n+k)\log((n+k)/k)$ for small enough $\epsilon > 0$. This claim immediately implies the stated bound (i) on the men's average rank of wives. To prove the claim, we use the fact that the number of proposals is stochastically dominated by the number of draws in the coupon collector's problem, when n distinct coupons must be drawn from $n+k$ coupons. This latter quantity is a sum of Geometric$((n+k-i+1)/(n+k))$ random variables for $i = 1, 2, \ldots, n$. The mean is

$$\begin{aligned}\sum_{i=1}^{n} \frac{n+k}{n+k-i+1} &= (n+k)\Big(\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{n+k}\Big) \\ &= (n+k)\Big(\log((n+k)/k) + O(1/k)\Big) \\ &= (n+k)\log((n+k)/k)\Big(1 + O(1/(k\log((n+k)/k)))\Big)\,.\end{aligned}$$

A short analytical exercise[39] shows that $1/(k\log((n+k)/k))$ is monotone decreasing in k, and is thus maximized at $k = 1$. It follows that $1/(k\log((n+k)/k)) \leq 1/\log(n+1) \leq 1/(\log n)$, which establishes that the error term $O(1/(k\log((n+k)/k))) = O(1/\log n)$ vanishes in the limit. Now routine arguments (e.g., Durrett (2010)) can be used to show that, in fact, this sum exceeds $(1+\epsilon)(n+k)\log((n+k)/k)$ with probability $\exp(-\Theta(n))$. This establishes the upper bound in (i).

[39] Define $f(k) = k\log((n+k)/k)$, where we think of n as fixed, and $k \in (0,\infty)$ as varying. We obtain $f'(k) = \log(1+n/k) - 1 + 1/(1+n/k)$. It suffices to show that $f'(k) > 0$ for all $k > 0$. To show this, we define $g : (0,\infty) \to \mathbb{R}$ as $g(x) = \log(1+x) - 1 + 1/(1+x)$. Now $\lim_{x\to 0} g(x) = 0$ and $g'(x) = x/(1+x)^2 > 0$ for all $x > 0$, leading to $g(x) > 0$ for all $x > 0$. Hence, $f'(k) > 0$ for all $k > 0$, as required.

Intuitively, the upper bound in (ii) should hold since it holds for a balanced market (see Pittel (1989a)), and adding more women should presumably only make the bound tighter. We show that this is indeed the case. Our proof works as follows: we first show that for any man m, the probability that $\text{Rank}_m(\mu(m)) > 3(\log n)^2$ is bounded above by $1/n^{1.2}$. We then use a union bound over the men to establish (ii).

Fix a man m and consider Algorithm 4, where one additional man is processed at a time. From McVitie and Wilson (1971), we know that the final outcome is the MOSM, and this does not depend on the order in which the men are processed. Therefore, we can assume that m is processed last.

Let $t = 1, 2, \ldots$ be the index of the proposals. From the upper bound in (i) proved above, we know that with probability $1 - \exp(-\Theta(n))$, the MOSM is found before

$$
\begin{aligned}
t = T_* &= (1+\epsilon)(n+k)\log((n+k)/k) \\
&\le (1+\epsilon)(n\log(1+n/k) + n\lim_{k\to\infty}(k/n)\log((n+k)/k)) \\
&\le n(1+\epsilon)(\log(1+n/k)+1) \\
&\le 1.1n\log n\,,
\end{aligned}
$$

for small enough ϵ and large enough n. Let $\hat{\mathcal{E}}$ be the event that this bound holds. Then we know that

$$
\mathbb{P}(\hat{\mathcal{E}}^c) \le \exp(-\Theta(n))\,.
$$

We track[40] Algorithm 4 until it terminates or index t exceeds T_*. If the index t exceeds T_*, i.e., we observe $\hat{\mathcal{E}}^c$, we declare failure and stop. Hence, we can use $t \le T_*$ in what follows. Consider the i-th proposal by man m, and suppose $i \le 3(\log n)^2$. Then there are at least $n + k - 3(\log n)^2$ women that m has not yet proposed to, and these women have together received no more than T_* proposals in total so far. Using Lemma B.2, the probability of the proposal being accepted is at least

$$
1/(T_*/(n+k-3(\log n)^2)+1) \ge 1/(1.1n\log n/(n-3(\log n)^2)+1) \ge 1/(1.2\log n)
$$

[40]As usual, we reveal information about each preference list as it is needed (cf. Appendix A.1): for men we reveal the next entry in the preference list just before a new proposal; for women, we reveal whether a new proposal is the best one so far, only when the proposal is made.

for large enough n. If the proposal is accepted by a woman $\hat{w}$, each subsequent proposal in the chain is, independently, at least as likely to go to an unmatched woman as it is to go to $\hat{w}$. Hence, the subsequent rejection chain has a probability at least $1/2$ of terminating in an unmatched woman before there is another proposal to woman $\hat{w}$. As such, the probability that the i-th proposal will be the last proposal made by m is at least $(1/2) \cdot 1/(1.2 \log n) = 1/(2.4 \log n)$. It follows that the probability that m has to make more than $3(\log n)^2$ proposals before the MOSM is reached or failure occurs is no more than

$$\Big(1 - \frac{1}{2.4 \log n}\Big)^{3(\log n)^2} \leq \Big(\exp\Big\{-\frac{1}{2.4 \log n}\Big\}\Big)^{3(\log n)^2} \leq \exp(-1.25 \log n) = 1/n^{1.25} .$$

Combined with the probability of failure, using a union bound, the overall probability that man m makes more than $3(\log n)^2$ proposals is bounded above by $1/n^{1.25} + \mathbb{P}(\hat{\mathcal{E}}^c) \leq 1/n^{1.25} + \exp(-\Theta(n)) \leq 1/n^{1.2}$, for large enough n. Since the same bound applies to any man, we can use a union bound to find that

$$\mathbb{P}(\text{Any man makes more than } 3(\log n)^2 \text{ proposals}) \leq n \cdot 1/n^{1.2} = 1/n^{0.2} .$$

We conclude that wvhp, no man makes more than $3(\log n)^2$ proposals, establishing (ii).

We now establish the lower bound in (i), i.e., that the sum of men's rank of wives is at least $(1-\epsilon)(n+k)\log((n+k)/k)$. The proof is similar to that of the upper bound in (i). From (ii), we have that wvhp, no man makes more than $3(\log n)^2$ proposals. It follows that for each proposal that occurs during the search for the i-th unmatched woman, the probability that an unmatched woman is found is at most

$$p_i = \frac{n+k-i+1}{n+k-\min(3(\log n)^2, i-1)} .$$

It follows that the number of proposals needed to find the i-th woman stochastically dominates Geometric(p_i), conditional on what has happened. It follows that the mean total number of proposals is at least

$$\begin{aligned}\sum_{i=1}^{n} \frac{n+k-3(\log n)^2}{n+k-i+1} &= \big(n+k-3(\log n)^2\big)\Big(\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{n+k}\Big) \\ &= (n+k)\log((n+k)/k)(1+O(1/(\log n)) ,\end{aligned}$$

where we bound the error term as above and using $3(\log n)^2/(n+k) = O((\log n)^2/n) = O(1/(\log n))$. Again, routine arguments (e.g., Durrett (2010)) can be used to show that, in fact, a sum of independent Geometric(p_i) random variables for $i = 1, 2, \ldots, n$ is less than $(1-\epsilon)(n+k)\log((n+k)/k)$ with probability $\exp(-\Theta(n))$. This establishes the lower bound on the men's average rank of wives.

Now consider the women's rank of husbands. For a woman w, who has received $\nu(w)$ proposals in Part I, the rank of her husband is a random variable that depends only on $\nu(w)$, and not anything else revealed so far. We have

$$\mathbb{E}[\text{Rank}_w(\mu(w))] = \frac{n+1}{\nu(w)+1}\,.$$

Define

$$M \equiv \mathbb{E}\left[\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \text{Rank}_w(\mu(w))\right] = (n+1)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \frac{1}{\nu(w)+1}\,.$$

Using Azuma's inequality (see Durrett (2010)), we have

$$\mathbb{P}\Big((1/n)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \text{Rank}_w(\mu(w)) \le \frac{M}{n} - \Delta\Big) \le \exp\left\{-\frac{n\Delta^2}{2n^2}\right\}\,,$$

since $\text{Rank}_w(\mu(w)) \in [0, n]$. Plugging in $\Delta = n^{3/4}$ yields

$$\mathbb{P}\Big((1/n)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \text{Rank}_w(\mu(w)) \le \frac{M}{n} - n^{3/4}\Big) \le \exp\left\{-\frac{n^{1/2}}{2}\right\}\,. \tag{2}$$

Using Jensen's inequality in the definition of M, we have

$$\begin{aligned} M &\ge (n+1)n \cdot \frac{1}{1 + (1/n)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}}\nu(w)} \\ &\ge \frac{n^2}{1 + (1+\epsilon/2)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)} \quad \text{wvhp}\,, \end{aligned}$$

where we used (i) with ϵ replaced by $\epsilon/2$, i.e., $(1/n)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}}\nu(w) \le (1+\epsilon/2)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$ wvhp. Using $\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right) \le 1.1\log n$ (see the bound on T_* in the proof of (ii)), i.e., the $n^{3/4}$ is negligible in comparison to M/n, we can deduce that

$$\frac{M}{n} - n^{3/4} \ge \frac{n}{1 + (1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)} \quad \text{wvhp}\,, \tag{3}$$

for large enough n (notice that we used the $\epsilon/2$ slack).

Combining Eqs. (2) and (3), we obtain (iii). □

Lemma B.5. *Suppose $k \le n^{0.1}$. Then, wvhp, there are fewer than $n^{0.99}$ women who each receive less than $(1/2)\log n$ proposals.*

Proof. Now consider a woman w'. For each proposal, it goes to w' with probability at least $1/(n+k)$, unless the proposing man has already proposed w'. Suppose w' receives fewer than $(\log n)/2 \le (2/3)(1-2\epsilon)\log(n+k)/k$ proposals, where, for instance, we can define $\epsilon = 0.01$. Since, wvhp, each man makes at most $3(\log n)^2$ proposals (Lemma B.4(ii)), wvhp there are at most $(3/2)(\log n)^3$ proposals by men who have already proposed w'. Using Lemma B.4(i), wvhp, there are at least $(1-\epsilon)(n+k)\log((n+k)/k)$ proposals in total. It follows that, wvhp, there are at least $(1-\epsilon)(n+k)\log((n+k)/k) - (3/2)(\log n)^3 \ge (1-2\epsilon)(n+k)\log((n+k)/k)$ proposals by men who have not yet proposed w'. Using Fact E.1 (i), the probability that fewer than $(2/3)(1-2\epsilon)\log((n+k)/k)$ of these proposals go to w' is

$$\begin{aligned} &\mathbb{P}\big(\text{Binomial}((1-2\epsilon)(n+k)\log((n+k)/k), 1/(n+k)) < (2/3)(1-2\epsilon)\log(n+k)/k)\big) \\ &\le 2\exp\{-(1-2\epsilon)\log((n+k)/k)/27\} \le ((n+k)/k)^{-1/28} \le n^{-0.02}, \end{aligned}$$

for $k \le n^{0.1}$. It follows that the expected number of women who receive fewer than $(1/2)\log n$ proposals is no more than $(n+k)n^{-0.02} \le 2n^{0.98}$. By Markov's inequality, the number of women who receive fewer than $(1/2)\log n$ proposals is no more than $n^{0.99}$, with probability at least $1 - 2n^{0.98}/n^{0.99} = 1 - o(\exp\{-(\log n)^{0.4}\})$. □

B.2 Part II

Lemma B.7 below shows that by the end of Part II, the number of proposals by each man is "small", the number of proposals received by each woman is "small", and that the set S is "large". Since S will be large at the end of this part, Part III will terminate "quickly". The next lemma says that wvhp, there will not be too many proposals in Part II (this bound will be assumed in the proof of Lemma B.7).

Lemma B.6. *Part II completes in no more than $\frac{n+k}{k}(\log n)^{0.45} \le (n+1)(\log n)^{0.45}$ proposals wvhp.*

Proof. For each proposal (Step 3) in Part II, the probability of Step 4(d), which will end Part II, is the probability that the man m proposes to an unmatched woman $\frac{k}{|\mathcal{W}\backslash R(m)|} \ge \frac{k}{n+k}$. Therefore the probability that the number of proposals in part II exceeds $((n+k)/k)(\log n)^{0.45}$

is at most $\left(1-\frac{k}{n+k}\right)^{((n+k)/k)(\log n)^{0.45}} \leq \exp(-(\log n)^{0.45}) = o(\exp\{-(\log n)^{0.4}\})$, leading to the first bound. Noticing that $(n+k)/k = 1+n/k \leq 1+n$, we obtain the bound of $(n+1)(\log n)^{0.45}$. □

Lemma B.7. *Fix any $\varepsilon > 0$. At the end of Part II, the following hold wvhp:*

(i) No man has applied to a lot of women:

$$\max_{m \in \mathcal{M}} |R(m)| < n^{\varepsilon}. \tag{4}$$

(ii) The set S is large: $|S| \geq n^{(1-\varepsilon)/2}$.

(iii) No woman received many proposals

$$\max_{w \in \mathcal{W}} \nu(w) < n^{\varepsilon}. \tag{5}$$

Proof. Using Lemma B.4, we know that wvhp, there have been no more than $3n\log(n/k)$ proposals and no man has proposed more than $3(\log n)^2$ women in Part I. Assume that these two conditions hold for the rest of the proof.

We begin with (i). We say that a man m starts a **run of proposals** when m is rejected by a woman at step 4(b) or is divorced from $\hat{w}$ at step 2. We say that a *failure* occurs if a man starts more than $(\log n)^2$ runs or if the length of any run exceeds $(\log n)^3$ proposals. We associate a failure with a particular proposal t, when for the first time, a man starts his $(\log n)^2 + 1$-th run, or the proposal is the $(\log n)^3 + 1$-th proposal in the current run.

Consider the number of runs of a given man m. Man m starts at most one run at step 2. The other runs start when the proposing man $m' \neq m$ proposes to the women m is currently matched with and m' is accepted. At any proposal the probability that m' proposes to any particular woman is no more than the probability that he proposes to $\bar{\mathcal{W}}$. Now if the latter happens, Part II ends. Therefore, it follows that the number of runs man m has in part II is stochastically dominated[41] by $1 + \text{Geometric}(1/2)$. Hence, the probability that a man has more than $(\log n)^2$ runs is bounded by $\left(\frac{1}{2}\right)^{(\log n)^2 - 1} \leq 1/n^2$, showing that man m has fewer than $(\log n)^2$ runs in Part II with probability at least $1 - 1/n^2$. It follows from a union bound over all men $m \in \mathcal{M}$ that wvhp failure due to number of runs does not occur.

[41] A real values random variable with cumulative distribution F_1 is said to be stochastically dominated by another r.v. with cumulative distribution F_2 if $F_2(x) \leq F_1(x)$ for all $x \in \mathbb{R}$.

Assume failure did not occur before or at the beginning of a run of man m. The number of proposals man m accumulates until either the run ends or a failure occurs is bounded by

$$(\log n)^2 \cdot (\log n)^3 \leq n/2$$

for sufficiently large n. In each proposal in the run before failure, man m proposes to a uniformly random woman in $\mathcal{W} \backslash R(m)$. Since there were at most $4n \log n$ proposals so far, we have that

$$\nu\left(\mathcal{W} \backslash R(m)\right) \leq \frac{4n \log n}{n/2} = 8 \log n \,.$$

From Lemma B.2, we have that the probability of acceptance at each proposal is at least $\frac{1}{\nu(\mathcal{W} \backslash R(m))+1} \geq \frac{1}{8\log n+1}$. Therefore the probability of man m making $(\log n)^3$ proposals without being accepted is bounded by[42]

$$\left(\frac{1}{8 \log n + 1}\right)^{(\log n)^3} \leq \frac{1}{n^3} \,.$$

Thus, the run has length no more than $(\log n)^3$ with probability at least $1 - 1/n^3$. Now the number of runs is bounded by n^2, so we conclude that wvhp failure due to number of runs does not occur. Finally, assuming no failure,

$$|R(m)| \leq (\log n)^2 \cdot (\log n)^3 < n^{\varepsilon}$$

establishing (i).

We now prove (ii). If $k \geq n^{(1-\varepsilon)/2}$, the set S is already large enough at the beginning of Part II, and there is nothing to prove. Suppose $k < n^{(1-\varepsilon)/2}$. Consider the evolution of $|V|$ during Part II. We first provide some intuition. Part II contains about $n/k = \omega(n^{1/2})$ proposals before it ends. We start with $|V| = 0$, and $|V|$ initially builds up without any new stable matches found. We can estimate the size of $|V|$ when Step 4(c) (new stable match) occurs as follows[43]: Suppose we reach $|V| \sim N$. Consider the next accepted proposal. Ignoring factors of $\log n$, the probability that the woman who accepts is in V is $\sim |V|/n \sim N/n$. Thus, for one of the next N accepted proposals to include a woman in V, we need $N \cdot N/n \sim 1$, i.e., $N \sim \sqrt{n}$. Thus, when $|V|$ reaches a size of about $\sqrt{n}$, then an IIC forms

[42]Again this inequality holds for large enough n. We omit the explicit mention of the condition "for large enough n" when such inequalities appear subsequently in this section.

[43]This analysis is analogous to that of the birthday paradox.

over the next $\sim \sqrt{n}$ proposals, reducing the size of $|V|$. This occurs repeatedly, with $|V|$ converging to an 'equilibrium' distribution with mean of order $\sqrt{n}$, and this distribution has a light tail. Thus, when the phase ends, we expect $|V| \sim \sqrt{n}$.

We now formalize this intuition. Whenever $|V| < n^{(1-\varepsilon)/2}$, for the next proposal, the probability that:

- The proposal goes to a woman in V is less than $2/n^{(1+\varepsilon)/2}$. Such a proposal is necessary to creating an IIC.
- The proposal goes to a woman in $S = \bar{\mathcal{W}}$, is at most $2k/(n+k) \leq 2k/n \leq 2/(n^{(1+\varepsilon)/2})$. Such a proposal would terminate the phase.
- The proposal goes to a woman in $\mathcal{W}\backslash(S \cup V)$, who accepts it, is at least $1/(5\log n)$, using the fact that there have been no more than $4n \log n$ proposals so far and Lemma B.2.

Suppose we start with any $|V| < n^{(1-\varepsilon)/2}$, for instance we have $|V| = 0$ at the start of the phase, we claim that with probability at least $1 - 3k/\sqrt{n}$, we reach $|V| = n^{(1-\varepsilon)/2}$ (call this an 'escape') before there is a proposal to S and before $\sqrt{n}$ proposals occur. We prove this claim as follows: There is a proposal to S among the next $\sqrt{n}$ proposals with probability no more than $2k/\sqrt{n}$. Suppose that $|V|$ stays less than $n^{(1-\varepsilon)/2}$. Then there are $n^{\varepsilon/4}$ or more proposals to women in $|V|$ among $\sqrt{n}$ total proposals with probability no more than $2^{-n^{\varepsilon/4}}$ using Fact E.1 (ii) on $\mathrm{Binomial}(\sqrt{\mathrm{n}}, 2/\mathrm{n}^{(1+\varepsilon)/2})$. Also, the probability that there are less than $n^{1/2-\varepsilon/8}$ proposals accepted by women in $\mathcal{W}\backslash(V \cup S)$ (these women are added to V) is at most $2^{-n^{1/2-\varepsilon/8}}$, using Fact E.1 (ii) on $\mathrm{Binomial}(\sqrt{\mathrm{n}}, 1/(5\log \mathrm{n}))$, since at each proposal, such a woman is added with probability at least $(1/(5\log n))$. But if there are

- less than $n^{\varepsilon/4}$ proposals to V, each such proposal reducing $|V|$ by at most $n^{1/2-\varepsilon/2}$,
- no proposal to S, and
- at least $n^{1/2-\varepsilon/8}$ women added to V,

then we must reach $|V| = n^{(1-\varepsilon)/2}$. Thus, the overall probability of *not* reaching $|V| = n^{(1-\varepsilon)/2}$ before there is a proposal to S and before $\sqrt{n}$ proposals occur is at most $2k/\sqrt{n} + 2^{-n^{\varepsilon/2}} + 2^{-n^{1/2-\varepsilon/4}} \leq 3k/\sqrt{n}$. In particular, the probability of a failed escape is at most $3k/\sqrt{n}$.

We now bound the number of times $|V|$ reduces from a value larger than $n^{(1-\varepsilon)/2}$ to a value smaller than $n^{(1-\varepsilon)/2}$. Suppose $|V| \geq n^{(1-\varepsilon)/2}$. The probability that a proposal goes to S is at least $k/(n+k) \geq k/(2n)$. The probability that a proposal goes to one of the first $n^{(1-\varepsilon)/2}$ women in $|V|$ is at most $2n^{(1-\varepsilon)/2}/n \leq 2/n^{(1+\varepsilon)/2}$. Thus, the number of times the latter occurs is stochastically dominated by Geometric$(k/(4n^{(1-\varepsilon)/2})) - 1$. Thus, the total number of escapes needed to ensure $|V| \geq n^{(1-\varepsilon)/2}$, including the one at the start of the phase, is stochastically dominated by Geometric$(k/(4n^{(1-\varepsilon)/2}))$, which exceeds $n^{1/2-\varepsilon/4}/k$ with probability at most

$$\left(1 - k/(4n^{(1-\varepsilon)/2})\right)^{n^{1/2-\varepsilon/4}/k} \leq \exp(-n^{\varepsilon/4}/4)\,.$$

Assuming no more than $n^{1/2-\varepsilon/4}/k$ escapes are needed, one of these escapes fails with probability at most $(n^{1/2-\varepsilon/4}/k)\cdot(3k/\sqrt{n}) = 3n^{-\varepsilon/4}$. Thus, the overall probability of $|V| < n^{(1-\varepsilon)/2}$ when the phase ends is bounded by $\exp(-n^{\varepsilon/4}/4)+n^{-\varepsilon/4} = o(\exp\{-(\log n)^{0.4}\})$. Thus, wvhp, $|V| \geq n^{(1-\varepsilon)/2}$ for all phases in Part II, including the terminal phase. This establishes (ii).

Finally, we establish (iii). Again we assume in our proof that Parts I and II end in no more than $4n\log n$ proposals in total, and that (i) holds (if not, we abandon our attempt to establish (iii), but this does not happen wvhp). Fix a woman w. For each proposal, the probability that *she* receives the proposal is no more than $2/n$, using (i). Thus, the total number of proposals she receives is no more than $\mathrm{Binomial(2n\log n, 2/n)}$ which is less than n^ε, except with probability $2^{-n^\varepsilon/(4\log n)}$ by Chernoff bound (see Fact E.1 in Appendix E). Union bound over the women gives us that (iii) holds wvhp. □

Lemma B.8. *Wvhp, the number of accepted proposals in Part II is no more than $n/(2\sqrt{\log n})$ and the improvement in sum of women's rank of husbands during Part II is no more than $n^2/(2(\log n)^{3/2})$.*

Proof. If $k \geq n^{0.1}$, then we already know that wvhp the number of proposals is no more than $n^{0.95}$ using Lemma B.6.

If $k < n^{0.1}$, then we know from Lemma B.5, that, wvhp, fewer than $n^{0.99}$ women each received fewer than $\log n/2$ proposals in Part I. Further, from Lemma B.7, wvhp, no man has proposed to more than n^ε women in Parts I and II. It follows that for each proposal in Part II, it goes to a woman who has already received $\log n/2$ or more proposals with probability at least $1 - n^{-0.01}/2$. Hence, the probability that the proposal is accepted is at most $2.5/\log n$.

But the total number of proposals in Part II, wvhp, is less than $(n+1)(\log n)^{0.45}$ from Lemma B.6. It follows using Fact E.1 that, wvhp, fewer than $3(n+1)/(\log n)^{0.55} \leq n/(10\sqrt{\log n})$ proposals are accepted in Part II.

We now bound the improvement in the sum of women's rank of husbands. Using Markov's inequality, there are, wvhp, at most $n^{0.995}$ proposals to women who have received fewer than $\log n/2$ proposals so far. The maximum possible improvement in rank from these proposals is $(n+k)n^{0.995} \leq 2n^{1.995}$. The number of proposals accepted by women who have received at least $\log n/2$ proposals so far is, wvhp, at most $n/(10\sqrt{\log n})$, as we showed above. For such a proposal accepted by a woman w' who has received $\nu(w') \geq (\log n)/2$ previous proposals, the expected improvement in rank is

$$\frac{n-\nu(w')}{\nu(w')+1} - \frac{n-\nu(w')-1}{\nu(w')+2} \leq n/((\log n)/2)^2 \leq 4n/\log n\,,$$

since $\log n \geq 1$. Further the improvement in rank is in the interval $[1, n-1]$. Thus, the total improvement in rank is stochastically dominated by a sum of independent X_i, for $i = 1, 2, \ldots, n/(10\sqrt{\log n})$, with $\mathbb{E}[X_i] \leq 4n/\log n$ and $1 \leq X_i \leq n-1$. It follows using Azuma's inequality (see, e.g., Durrett (2010)) that this sum exceeds $n/(10\sqrt{\log n}) \cdot 4n/\log n + n^{1.6} \leq n^2/(2.1(\log n)^{3/2})$ with probability at most

$$2\exp\left\{-\frac{(n^{1.6})^2}{2\cdot n/(10\sqrt{\log n})\cdot n^2}\right\} = \exp\{-n^{0.1}\} = o(\exp\{-(\log n)^{0.4}\})\,.$$

Thus, the total improvement in the sum of women's rank of husbands is, wvhp, no more than $2n^{1.995} + n^2/(2.1(\log n)^{3/2}) \leq n^2/(2(\log n)^{3/2})$. □

B.3 Part III

Let S_{II} be the set S at the end of part II.

The next lemma provides upper bounds (that are achieved wvhp) on the number of proposals each man makes and the number of proposals each woman receives throughout Parts III and IV.

Let $\mathcal{E}_t$ be the event that until proposal t, no man has applied to more than $n^{0.6}$ women in total or to more than $n^{3\varepsilon}$ women in S_{II}, and no woman has received $n^{2\varepsilon}$ or more proposals. Let $\mathcal{E}_\infty$ be the event that these same conditions hold when Part IV ends.

Lemma B.9. *The event $\mathcal{E}_\infty$ occurs wvhp.*

Proof. By Lemma B.7, we know that at the end of Part II, no man has made more than n^{ε} proposals, that $|S_{\text{II}}| \geq n^{(1-\varepsilon)/2}$, and that no woman has received more than n^{ε} proposals, wvhp. We assume that all these conditions hold.

Fix a man m. We argue that if m makes a successful proposal to a woman in $S_{\text{II}} \cup \bar{\mathcal{W}}$, then he makes no further proposals in Algorithm 2: If m makes a successful proposal to a woman in S, this ends the phase making the phase a terminal one, man m goes back to the woman to whom he was matched at the beginning of the phase, and this woman becomes a member of S. Thus, if a m makes a successful proposal to a woman in S, he makes no further proposals. In particular, if m makes a successful proposal to a woman in $S_{\text{II}} \backslash \bar{\mathcal{W}}$, then he makes no further proposals.

Suppose man m is proposing in proposal t and that $\mathcal{E}_t$ holds. Then m has not yet applied to at least $3n^{(1-\varepsilon)/2}/4$ women in S_{II}. Hence the probability of applying to a woman in S_{II} is at least $n^{(1-\varepsilon)/2}/2$. Further, since no woman has received $n^{2\varepsilon}$ or more proposals, the probability of the proposal being accepted is at least $1/n^{2\varepsilon}$. Hence, the probability of the proposal going to a woman in S_{II} and being accepted is at least $n^{-3\varepsilon-1/2}$. Hence, the man makes fewer than $n^{0.6}/2$ proposals in Part IV, and proposes to fewer than $n^{3\varepsilon}/2$ additional women in S_{II}, except with probability $\exp(-n^{\varepsilon}/2)$. Using a union bound over the men, wvhp, no man has applied to more than $n^{0.6}$ women in total or to more than $n^{3\varepsilon}$ women in S_{II} until the end of Part IV.

Fix a woman w. Each time a proposal occurs, since no man has proposed to more than $n^{0.6}$ women (assuming $\mathcal{E}_t$ holds), the probability of the proposal going to w is less than $2/n$. Since there are at most $50n \log n$ proposals in total, the number of proposals received by w in Part III is more than n^{ε} with probability less than $\mathbb{P}(\text{Binomial}(50n\log n, 2/n) \geq n^{\varepsilon}) \leq 2^{-n^{\varepsilon}}$, using Chernoff bounds (see Fact E.1(ii) in Appendix E). Using a union bound over the women, wvhp, no woman has received more than n^{ε} proposals until the end of Part IV.

The result follows combining the analyses in the two paragraphs above. □

We now focus on Part III. We show (Lemma B.10) that for every phase in Part III, whp:

- the phase is a terminal phase, and
- that $|V|$ at the end of the phase is at least $n^{0.25}$.

For each such phase, $|S|$ increases by at least $n^{0.25}$. In addition, we show that phases are short, with the expected length of a phase being $O(n^{1/2+3\varepsilon})$. We infer that, wvhp, we reach $|S| \geq n^{0.7}$, i.e., the end of Part III, in $o(n^{0.47})$ phases, containing $o(n)$ proposals. Lemma B.11 below formalizes this.

Lemma B.10. *Assume $|S_{\mathrm{II}}| \geq n^{(1-\varepsilon)/2}$, cf. Lemma B.7. Consider a phase during Part III. Suppose $\mathbb{I}(\mathcal{E}_t) = 1$ at the start of the phase. Then, whp, either $\mathbb{I}(\mathcal{E}_{t'}) = 0$ at the end of the phase, or we have:*

- *The phase is a terminal phase.*
- *At the end of the phase is at least $|V| \geq n^{0.25}$.*

Proof. Assume $\mathbb{I}(\mathcal{E}_\tau) = 1$ throughout the phase (otherwise there is nothing to prove). Since we are considering a phase during Part III, we know that $|S| < n^{0.7}$. Also, $|S| \geq |S_{\mathrm{II}}| \geq n^{(1-\varepsilon)/2}$ by assumption. For each proposal, there is a probability of at least $|S| n^{-2\varepsilon}/(2n) \geq n^{-1/2-3\varepsilon}$ and at most $2|S|/n \leq 2n^{-0.3}$, that the proposal is to a woman in S and is accepted. It follows that, whp, the phase is a terminal phase, and the number of proposals in the phase is in $[n^{0.28}, n^{0.52}]$. It is easy to see that with probability at least $(1 - 2n^{0.28}/n)^{n^{0.28}} = 1 - o(1)$, all of the first $n^{0.28}$ proposals in the phase are to distinct women, meaning that there are no IICs. For each proposal, the probability of acceptance is at least $1/(1 + n^{2\varepsilon})$, since no woman has received $n^{2\varepsilon}$ proposals, so whp, there are at least $n^{0.25}$ accepted proposals among the first $n^{0.28}$ proposals, using Fact E.1 (ii) on $\mathrm{Binomial}(\mathrm{n}^{0.26}, 1/(1 + \mathrm{n}^{2\varepsilon}))$. Now, consider the first $n^{0.25}$ women in V. These women receive no further proposals during the phase with a probability at least $(1 - 2n^{0.25}/n)^{n^{0.52}} = 1 - o(1)$. Hence, whp, these women are part of V at the end of the phase, establishing $|V| \geq n^{0.25}$ at the end of the phase as needed. □

Lemma B.11. *Wvhp, Part III contains less than $n^{0.99}$ proposals.*

Proof. We first show that the next $n^{0.47}$ phases after the end of Part II complete in fewer than $n^{0.99}$ proposals. Since, wvhp, $|S_{\mathrm{II}}| \geq n^{(1-\varepsilon)/2}$, if $\mathbb{I}(\mathcal{E}_t) = 1$ then for proposal t the probability of ending the phase (due to acceptance by a woman in S) is at least $n^{-1/2-3\varepsilon}$. It follows that either $\mathbb{I}(\mathcal{E}_\infty) = 0$ or wvhp, the next $n^{0.47}$ phases after the end of Part II complete in no more than $n^{0.47+1/2+4\varepsilon} \leq n^{0.99}$ proposals, using Fact E.1 (ii) on $\mathbb{P}(\mathrm{Binomial}(\mathrm{n}^{0.97+4\varepsilon}, \mathrm{n}^{-1/2-3\varepsilon}) \geq \mathrm{n}^{0.47})$.

Now we show that wvhp, Part III contains fewer than $n^{0.47}$ phases. Suppose this is not the case, then, by our definition of Part III, at most $n^{0.45}$ of these phases increase $|S|$ by $n^{0.25}$ or more. But using Lemma B.10, either $\mathbb{I}(\mathcal{E}_\infty) = 0$, or this occurs with probability at most

$$\mathbb{P}\big(\mathrm{Binomial}(\mathrm{n}^{0.47}, 1-\varepsilon) \leq \mathrm{n}^{0.45}\big) \leq \mathbb{P}\big(\mathrm{Binomial}(\mathrm{n}^{0.47}, 1/2) \leq \mathrm{n}^{0.47}/4\big) \leq 2\exp(-\mathrm{n}^{0.47}/24)\,,$$

using Fact E.1 (i). In other words, either $\mathbb{I}(\mathcal{E}_\infty) = 0$ or, wvhp, Part III contains fewer than $n^{0.47}$ phases.

But Lemma B.9 tells us that $\mathbb{I}(\mathcal{E}_\infty) = 1$ wvhp. Combining the above, we deduce that wvhp, Part III contains fewer than $n^{0.47}$ phases and fewer than $n^{0.99}$ proposals. □

B.4 Part IV

Lemma B.12. *Suppose we are at Step 3 (time t) of Algorithm 2 during Part IV, we have $\mathbb{I}(\mathcal{E}_t) = 1$, and man m is proposing. Then, for large enough n, the probability that:*

(i) Man m proposes to S and is accepted is at least $n^{-0.31}$.

(ii) Man m proposes to $\mathcal{W}\backslash(R(m) \cup \hat{w})$ and is accepted is at least $0.9n/t$.

Proof. Note that $|S| \geq n^{0.7}$, whereas by definition of $\mathcal{E}_t$ the man has proposed to no more than $n^{0.6}$ women and no woman has received more than $n^{2\varepsilon}$ proposals. (i) follows from Lemma B.2.

Proof of (ii): Since m has not applied to more than $n^{0.6}$ women so far, we know that $|\mathcal{W}\backslash(R(m) \cup \hat{w})| \geq 0.95n$. Also, the total number of proposals so far is $t-1$. Using Lemma B.2, the probability of applying to $\mathcal{W}\backslash(R(m) \cup \hat{w})$ and being accepted is at least

$$\frac{1}{1 + (t-1)/(0.95n)} \geq \frac{0.9n}{t}\,,$$

for large enough n, using $t > n$ since Part I itself requires at least n proposals. □

Lemma B.13. *Wvhp, Part IV ends due to termination of the algorithm.*

Proof. Suppose Part IV does not end with termination (if not we are done) and that $\mathcal{E}_\infty$ occurs (Lemma B.9 guarantees this wvhp). Reveal each proposal sequentially.

For $t \leq 40n\log n$, call proposal t a 'seemingly-good' proposal when acceptance by $w' \in \mathcal{W}\backslash(R(m) \cup \hat{w})$ occurs. Denote the set of seemingly good proposals by $\mathcal{A}$. We use Lemma

B.12. For each proposal t, there is a probability at least $0.9n/t$ of it being a seemingly-good proposal, conditioned on the history so far. Define independent $X_t \sim \text{Bernoulli}(0.9n/t)$ for $t = t_0, t_0 + 1 \ldots, 40n \log n$, where t_0 is the first proposal in Part III. Then we can set up a coupling so that proposal $t \in \mathcal{A}$ whenever $X_t = 1$. Now

$$\sum_{t=4n \log n}^{40n \log n} 1/t \geq (0.99) \ln(10) \geq 2.27\,,$$

$$\Rightarrow \qquad \sum_{t=4n \log n}^{40n \log n} \mathbb{E}[X_t] \qquad \geq 2n$$

Using Fact E.1, we deduce that

$$\sum_{t=4n \log n}^{40n \log n} X_t \geq 7n/4$$

wvhp, implying

$$|\mathcal{A}| \geq \sum_{t=t_0}^{40n \log n} X_t \geq 7n/4 \tag{6}$$

wvhp, since we know that $t_0 \leq 4n \log n$ wvhp using Lemmas B.4 and B.6.

We call a seemingly-good proposal $t \leq 40n \log n$ a 'good' proposal if the following conditions are satisfied:

- During the current phase, there is no proposal to a woman in V. In particular, there are no IICs.
- The phase is a terminal phase that ends during Part IV.

We denote the set of good proposals by $\mathcal{G} \subseteq \mathcal{A}$. We now argue that

$$|S_{\text{II}}| \geq |\mathcal{G}|\,, \tag{7}$$

where S_{II} is the set S at the end of Part IV. If $w' \notin (S \cup \bar{w})$, then w' becomes part of S at the end of the phase if the proposal is a good proposal. For each terminal phase, there is exactly one good proposal to a woman in $S \cup \bar{w}$, which we think of as accounting for $\hat{w}$, which also becomes a part of S. Thus, for every good proposal, one woman joins S during Part IV, establishing Eq. (7).

Now consider any phase in Part IV that starts before the $40n \log n$-th proposal. Call such a phase an *early* phase. Using Lemma B.12, the phase contains more than $n^{0.32}$ proposals with probability at most $(1 - n^{-0.31})^{n^{0.32}} \leq \exp(-n^{0.01}) \leq 1/n^2$. But the total number of early phases is no more than $40n \log n$. It follows that using a union bound that, wvhp, there is no early phase that contains more $n^{0.32}$ proposals.

Now, the probability of a phase containing fewer than $n^{0.32}$ proposals, and containing a proposal to a woman in V is at most $n^{0.32} \cdot 2n^{0.32}/n \leq n^{-0.35}$, since $|V| \leq n^{0.32}$ throughout such a phase. Further, there are at most $40n \log n \leq n^{1.01}$ early phases. It follows, using Fact E.1 (ii) on $\mathrm{Binomial}(\mathrm{n}^{1.01}, \mathrm{n}^{-0.35})$, that the number of early phases containing a proposal to a woman in V is, wvhp, no more than $n^{0.67}$. It follows that, wvhp, no more than $n^{0.67} \cdot n^{0.32} = n^{0.99}$ proposals occur in early phases containing a proposal to V. But all proposals in $\mathcal{A} \backslash \mathcal{G}$ must occur in such phases. We deduce that, wvhp,

$$|\mathcal{A} \backslash \mathcal{G}| \leq n^{0.99} . \tag{8}$$

Combining Eqs. (6) and (8), we deduce that $|\mathcal{G}| \geq 3n/2 \geq |\mathcal{W}|$ wvhp. Plugging in Eq. (7), we obtain $|S_{\mathrm{II}}| \geq |\mathcal{W}|$ at the end of Part IV wvhp, which we interpret[44] as "With wvhp, our assumption that Part IV does not end with termination was incorrect. In other words, Part IV ends with termination wvhp". □

Lemma B.14. *The number of proposals in improvement phases and in IICs in Part IV is no more than $n^{0.99}$ wvhp.*

Proof. In the proof of Lemma B.13, we in fact showed that whp in Part IV, the number of proposals in phases that include a proposal to a woman in V is no more than $n^{0.99}$. (Actually, we showed this bound for 'early' phases, and also showed that wvhp, the algorithm terminates with an early phase, so that all phases in Part IV are early phases). But improvement phases and phases containing IICs must include a proposal to a woman in V. The result follows. □

We now establish two claims that follow from elementary calculus.

Claim B.15. *For large enough n and any $k \geq 1$ we have $k \log(1 + n/k) \geq (\log n)/2$.*

[44] Recall our initial assumption that Part IV does not end with termination. Finding that $|S| \geq n$ under this assumption simply means that Part IV did, in fact, end with termination.

Proof. We divide possible values of k into three ranges, and establish the bound for each range.

First, suppose $k \leq 10 \log n$. Then $\log(1 + n/(10 \log n)) \geq \log \sqrt{n} \geq \log n/2$ for large enough n. It follows that $k \log(1 + n/k) \geq \log n/2$ since $k \geq 1$.

Next, suppose $k \in (10 \log n, 10n]$. Now $\log(1+n/k) \geq \log(1+n/(10n)) = \log 1.1 \geq 0.09$. The bound follows by multiplying with $k \geq 10 \log n$.

Finally, consider $k > 10n$. Now, $n/k \leq 1/10$ leading to $\log(1+n/k) \geq n/k - (n/k)^2/2 \geq 0.95n/k$. It follows that $k \log(1 + n/k) \geq 0.95n \geq \log n/2$ for large enough n. □

Claim B.16. *For any* $n \geq 1$ *and any* $k \geq 1$*, we have* $(1+1/n)\log(1+n) \geq (1+k/n)\log(1+n/k) \geq 1$.

Proof. Consider $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = (1 + 1/x) \log(1 + x)$. Then

$$f'(x) = \frac{x - \log(1 + x)}{x^2} > 0 \quad \forall x > 0\,,$$

using $\log(1 + x) < x$ for all $x > 0$. It follows that for any $x > 0$, we have

$$f(x) \geq \lim_{x \to 0} f(x) = 1,$$

using $\lim_{x\to 0} \log(1+x)/x = 1$. The lower bound in the claim follows by plugging in $x = n/k$. The upper bound follows by plugging in $k = 1$, since this maximizes n/k for fixed n. □

Finally, using these lemmas and claims we give the proof of Theorem 3.

Proof of Theorem 3. Using Lemma A.3, we can calculate the sum of men's rank of wives by summing up the rank under the MOSM and the number of proposals made during improvement phases and IICs during the run of Algorithm 2. By Lemma B.6, B.11 and Lemma B.14, the total number of proposals that occur in improvement phases and IICs (in Parts II-IV) of Algorithm 2 is, wvhp, no more than $(1 + n/k)(\log(n))^{0.45} + 2n^{0.99}$. Using Lemma A.3, we get that

$$\begin{aligned} &\text{Sum of men's rank of wives(WOSM)} - \text{Sum of men's rank of wives(MOSM)} \\ &\leq (1 + n/k)(\log(n))^{0.45} + 2n^{0.99} \end{aligned} \tag{9}$$

But

$$\begin{aligned}
&\text{Sum of men's rank of wives(MOSM)} \\
&\geq 0.99(n+k)\log((n+k)/k) \\
&\geq \max(0.49(1+n/k)\log n, 0.99n) \qquad (10)
\end{aligned}$$

wvhp, from Lemma B.4 (i), along with Claims B.15 and B.16. We deduce from Eqs. (9) and (10) that

$$\frac{R_{\text{MEN}}(\text{WOSM}) - R_{\text{MEN}}(\text{MOSM})}{R_{\text{MEN}}(\text{MOSM})} \leq (\log n)^{-0.55}/0.49 + 2n^{-0.01}/0.99 \leq (\log n)^{-0.4}$$

wvhp, immediately implying Theorem 3 (iii).

The only agents whose partner changes in going from the MOSM to the WOSM are the ones who make or receive accepted proposals during improvement phases and IICs. But this number on each side of the market is, wvhp, no more than $n/(2\sqrt{\log n})$ in Part II (Lemma B.8), and no more than $n^{0.99}$ each in Part III (Lemma B.11) and Part IV (Lemma B.14, leading to a bound of $n/(2\sqrt{\log n}) + 2n^{0.99} \leq n/\sqrt{\log n}$ on the total number of agents with multiple stable partners on each side of the market, establishing Theorem 3 (ii).

By Lemma B.8, wvhp, the improvement in the sum of women's rank of husbands in Part II is at most $n^2/(2(\log n)^{3/2})$. In Parts III and IV, wvhp there are at most $2n^{0.99}$ women who obtain better husbands (since each such woman must have received a proposal during an improvement phase or IIC; see above), and the rank improves by less than n for each of these women, so the improvement in sum of ranks is less than $2n^{1.99}$. It follows that the total improvement in sum of ranks is, wvhp, less than $n^2/(\log n)^{3/2}$. On the other hand, using Lemma B.4 (iii) and the upper bound in Claim B.16, we obtain that wvhp

$$\text{Sum of women's rank of husbands(MOSM)} \geq n^2/(2\log n)\,.$$

Theorem 3 (iv) follows.

Lemma B.4 (i) and (iii) with $\epsilon' = \epsilon/2$, combined with Theorem 3 (iii) and (iv) (established above) yields Theorem 3 (i).

□

C Many-to-one matching markets

This section discusses the extension of our results to many-to-one matching markets, in which colleges are matched with more than one student. We consider many-to-one markets in which colleges have a small capacity relative to the size of the market, each student has an independent, uniformly random complete preference list over colleges, and each college has responsive preferences (Roth, 1985) and an independent, uniformly random complete preference list over individual students. In Section 4.4, we presented computational experiments demonstrating that imbalance in such markets again leads to a small core and allows the short side to approximately "choose." We now follow to describe how our theoretical results can be extended.

Assume that each college has a constant number of seats q. Students are on the short side of the market if there are fewer students than seats, and, symmetrically, colleges are on the short side of the market if there are more students than seats. We denote the extreme stable matchings by SOSM (the student-optimal stable matching) and COSM (the college-optimal stable matching). The students' average rank of their colleges is defined as before. We define the colleges' rank of students to be the average rank of students assigned to the college.

We argue that the bounds on average rank stated in Theorem 2 will deteriorate by a factor that depends on q,[45] whereas Theorem 1 will hold as stated. Thus, with high probability, the core will be small and the short side will "choose."

Our proof can be extended as follows. First, using results from Roth and Sotomayor (1989), one can decompose the many-to-one market into a one-to-one market as follows. For each seat in a college, create an "agent" that ranks students according to that college's preferences. Students rank all seats according to their preferences for the corresponding schools, and all students rank seats within a college in the same order (i.e., there is a "top" seat and a "bottom" seat in each college). A matching is stable in the original market if and only if there is a corresponding stable matching in the decomposed market. With this decomposition, all our results from Appendix A extend to the many-to-one case, allowing us to use our algorithms to calculate the extreme stable matchings via a sequence of proposals by agents on the short side.

[45]That is, there is a similar upper bound on the average rank of partners for the short side of the market, and there is a similar lower bound on the average rank for the long side of the market.

Next, the stochastic analysis can be extended to the many-to-one case as follows. Suppose there are n colleges, each with q seats. First consider the case in which there are fewer than qn students. The first part of the analysis is student-proposing DA, and we can bound the number of proposals in this stage by considering q repetitions of the coupon collector's problem. The next steps in our proof follow with slight modifications, using the fact that rejection chains have the same structure as in the one-to-one case. Whenever a seat rejects a student, the student matched with the bottom seat in the college gets rejected, and that bottom student in turn applies to a randomly drawn college that ranks the student uniformly at random. A college will accept the student if the applying student is more preferred than the q-th best student currently at the college and will reject that q-th best student if it accepts the applying student. A phase (chain), initiated by a college c rejecting a student, can terminate either with an application to a school that has not filled its seats, or with a successful application to college c. Therefore, a phase consists of a series of proposals to random colleges that in case of acceptance, always reject their lowest-ranked student. The main difference is that in order to calculate the acceptance probability, we need to calculate the probability that a (randomly ranked) proposing student is better than the q-th best proposal the college received (instead of being better than the best proposal the woman received in the one-to-one case). Bounds on this probability will be affected by at most a constant factor.

Similar arguments allow us to extend our proof to the case where there are more students than seats. The first part of the analysis is college-proposing DA, and we can again bound the number of proposals in this stage by considering the coupon collector's problem. The rest of our analysis, controlling rejection chains, is almost unchanged. Whenever a student in a college rejects a seat, the seat accepts the student matched with the next lower seat at the college, and so on until the bottom seat in the college is rejected. This seat proposes to a randomly drawn student who ranks the college uniformly at random and accepts if she prefers the college over her current match. A phase (chain) initiated by a student s rejecting a seat can either terminate with an offer to an unmatched student, or with a successful offer to student s. Overall, the analysis in this case will be almost identical to our original proof.

We note that these results do not imply that colleges cannot gain from manipulation in unbalanced matching markets, as a college can potentially manipulate even if there is a unique stable matching. Kojima and Pathak (2009) show that a college can manipulate only if a rejection of one of the students assigned to that college triggers a rejection chain that cycles back to the college. We therefore conjecture that when the imbalance is larger than a college's capacity, even colleges have a limited scope for manipulation, but this conjecture does not directly follow from our analysis, which only establishes that the core is small.

D Average rank estimates

Table 1 in Section 4 includes values of the following function for each unbalanced market.

$$\mathrm{EST} = \mathrm{EST}(|\mathcal{M}|, |\mathcal{W}|) = \begin{cases} \frac{|\mathcal{W}|}{|\mathcal{M}|}\log\left(\frac{|\mathcal{W}|}{|\mathcal{W}|-|\mathcal{M}|}\right) & \text{for } |\mathcal{W}| > |\mathcal{M}| \\ |\mathcal{W}| \Big/ \left(1 + \frac{|\mathcal{M}|}{|\mathcal{W}|}\log\left(\frac{|\mathcal{M}|}{|\mathcal{M}|-|\mathcal{W}|}\right)\right) & \text{for } |\mathcal{W}| < |\mathcal{M}| \end{cases} \tag{11}$$

The definition of EST is based on Theorem 2, as justified by the following facts.

Remark 2. *Fix any $\epsilon > 0$.*

- *For the case $|\mathcal{W}| > |\mathcal{M}|$, Theorem 2 and its proof imply that, with high probability (asymptotically in $|\mathcal{M}|$), $R_{\text{MEN}}/\mathrm{EST} \in (1-\epsilon, 1+\epsilon)$ under all stable matches (including the MOSM and WOSM). Note that the upper bound on R_{MEN} is part of the statement of Theorem 2, whereas the lower bound follows from the proof (though it is not part of the statement). Hence, we think of* EST *as a heuristic estimate for R_{MEN} in finite markets with $|\mathcal{W}| > |\mathcal{M}|$.*
- *For the case $|\mathcal{W}| < |\mathcal{M}|$, Theorem 2 implies that, with high probability (asymptotically in $|\mathcal{W}|$), $R_{\text{MEN}}/\mathrm{EST} \geq 1 - \epsilon$ under all stable matches (including the MOSM and WOSM). Hence, we think of* EST *as a heuristic lower bound for R_{MEN} in finite markets with $|\mathcal{W}| < |\mathcal{M}|$.*

E Chernoff bounds

Fact E.1 Chernoff bounds (see Durrett (2010))**.** *Let $X_i \in \{0,1\}$ be independent with $\mathbb{P}[X_i = 1] = \theta_i$ for $1 \leq i \leq n$. Let $X = \sum_{i=1}^n X_i$ and $\lambda = \sum_{i=1}^n \theta_i$.*

(i) Fix any $\delta \in (0,1)$. Then

$$\mathbb{P}(|X - \lambda| \geq \lambda\delta) \leq 2\exp\{-\delta^2\lambda/3\}. \tag{12}$$

(ii) For any $R \geq 6\lambda$, we have

$$\mathbb{P}(X \geq R) \leq 2^{-R}. \tag{13}$$