Dynamic Reserve Prices for Repeated Auctions:

Learning from Bids

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A large fraction of online advertisements are sold via repeated second price auctions. In these auctions, the reserve price is the main tool for the auctioneer to boost revenues. In this work, we investigate the following question: Can changing the reserve prices based on the previous bids improve the revenue of the auction, taking into account the long-term incentives and strategic behavior of the bidders? We show that if the distribution of the valuations is known and satisfies the standard regularity assumptions, then the optimal mechanism has a constant reserve. However, when there is uncertainty in the distribution of the valuations and competition among the bidders, previous bids can be used to learn the distribution of the valuations and to update the reserve prices. We present a simple approximately incentive-compatible and optimal dynamic reserve mechanism that can significantly improve the revenue over the best static reserve in such settings.

1. Introduction

Advertising is the main component of the monetization strategy of most Internet companies. A large fraction of online advertisements are sold via advertisement exchange platforms such as Google's Doubleclick (AdX) and Yahoo!'s Right Media.[1] Using these platforms, online publishers such as the New York Times and the Wall Street Journal sell the advertisement space on their webpages to advertisers. A large fraction of advertisement space is allocated using auctions where advertisers bid in real time for a chance to show their ads to the users. Every day, tens of billions of online ads are sold via these exchanges (Muthukrishnan 2009, McAfee 2011, Balseiro et al. 2011, Celis et al. 2014).

The second-price auction is the dominant mechanism used by the advertisement exchanges. Among the reasons for such prevalence are the simplicity of the second-price auction and the fact

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[1] Other examples of major ad exchanges include Rubicon, AppNexus, and OpenX.

that it incentivizes the advertisers to be truthful. The second price auction maximizes the social welfare (i.e., the value created in the system) by allocating the item to the highest bidder.

In order to maximize the revenue in a second price auction, the auctioneer can set a reserve price and not make any allocations when the bids are low. In fact, under symmetry and regularity assumptions (see Section 2), the second-price auction with an appropriately chosen reserve price is optimal and maximizes the revenue among all selling mechanisms (Myerson 1981, Riley and Samuelson 1981).

However, in order to set the reserve price effectively, the auctioneer requires information about distribution of the valuations of the bidders. A natural idea, which is widely used in practice, is to construct these distributions using the history of the bids. This approach, though intuitive, raises a major concern with regards to long-term (dynamic) incentives of the advertisers. Because the bid of an advertiser may determine the price he or she pays in future auctions, this approach may result in the advertisers shading their bids and ultimately in a loss of revenue for the auctioneer.

To understand the effects of changing reserve prices based on the previous bids, we study a setting where the auctioneer sells impressions (advertisements space) via repeated second price auctions. We demonstrate that the long-term incentives of advertisers play an important role in the performance of these repeated auctions by showing that under standard symmetry and regularity assumptions (i.e., when the valuations are drawn independently and identically from a regular distribution), the optimal mechanism is running a second price auction with a *constant* reserve and changing the reserve prices over time is not beneficial. However, when there is *uncertainty* in the distribution of the valuations and *competition* among the bidders, we show that there can be substantial benefit in learning the reserve prices using the previous bids.

More precisely, we consider an auctioneer selling multiple copies of an item sequentially. The item is either a high type or a low type. The type determines the distribution of the valuations of the bidders. The type of the item is not a-priori known to the auctioneer. Broadly, we show the following: when there is competition between bidders and the valuation distributions for the two types are sufficiently different from each other, there is a simple dynamic reserve mechanism that can effectively "learn" the type of the item, and thereafter choose the optimal reserve for that type.[2] As a consequence, the dynamic reserve mechanism does much better than the best fixed reserve mechanism, and in fact, achieves near optimal revenue. Importantly, the mechanism retains (approximate) incentive compatibility; implying that bidders cannot gain significantly by deviating from truthfulness; see Section 2.

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[2] On the other hand, when the valuation distributions for the two types are close to each other, the improvement from changing the reserve is insignificant.

We obtain our results using a simple mechanism called the *threshold* mechanism. In each round, the mechanism implements a second price auction with reserve. The reserve price starts at some value, and stays there until there is a bid exceeding a pre-decided threshold, after which the reserve rises (permanently) to a higher value. We also present a generalization of this mechanism where the reserve rises when sufficiently many bidders bid above the threshold.

We compare the revenue of our mechanism with two benchmarks. Our baseline is the static second price auction with the optimal constant reserve. Our upper-bound benchmark is the optimal mechanism that knows the type of the impressions (e.g., high or low) in advance. These two benchmarks are typically well separated. We show that the threshold mechanism is near optimal and obtains revenue close to the upper-bound benchmark. In addition, we present numerical illustrations of our results that show up to 23% increase in revenue by our mechanism compared with the static second price auctions. These examples demonstrate the effectiveness of dynamic reserve prices under fairly broad assumptions.

Related Work

In this section, we briefly discuss the closest work to ours in the literature along different dimensions starting with the application in online advertising.

Ostrovsky and Schwarz (2009) conducted a large-scale field experiment at Yahoo! and showed that choosing reserve prices, guided by the theory of optimal auctions, can significantly increase the revenue of sponsored search auctions. To mitigate the aforementioned incentive concerns, they dropped the highest bid from each auction when estimating the distribution of the valuations. However, they do not formally discuss the consequence of this approach.

Another common solution offered to mitigate the incentive constraints is to bundle different types of impressions (or keywords) together so that the bid of each advertiser would have small impact on the aggregate distribution learned from the history of bids. However, this approach may lead to significant estimation errors, resulting in a sub-optimal choice of reserve price.

Recently, Cesa-Bianchi et al. (2013), Mohri and Medina (2014b) study the problem of learning optimal reserve prices for repeated second-price auctions but they ignore the strategic aspects and incentive compatibility issues. To the extent of our knowledge, ours is the first work that rigorously studies the long-term and dynamic incentive issues in repeated auctions with dynamic reserves.

Iyer et al. (2011) and Balseiro et al. (2013) demonstrate the importance of setting reserve prices in dynamic setting in environments where agents are uncertain about their own valuations, and respectively, are budget-constrained. We discuss the methodology of these papers in more details at the end of Section 7. McAfee et al. (1989), McAfee and Vincent (1992) determine reserve prices in common value settings.

Our work is closely related to the literature on behavior-based pricing strategies where the seller changes the prices for a buyer (or a segment of the buyers) based on her previous behavior; for instance, increasing the price after a purchase or reducing the price in the case of no-purchase; see Fudenberg and Villas-Boas (2007), Esteves (2009) for surveys.

The common insight from the literature is that the optimal pricing strategy is to commit to a single price over the length of the horizon (Stokey 1979, Salant 1989, Hart and Tirole 1988). In fact, when customers anticipate reduction in the future prices, dynamic pricing may hurt the seller's revenue (Taylor 2004, Villas-Boas 2004). Similar insights are obtained in environments where the goal is to sell a fixed initial inventory of products to unit-demand buyers who arrive over time (Aviv and Pazgal 2008, Dasu and Tong 2010, Aviv et al. 2013, Correa et al. 2013).

There has been renewed interest in behavior-based pricing strategies, mainly motivated by the development in e-commerce technologies that enables online retailers and other Internet companies to determine the price for the buyer based on her previous purchases. Acquisti and Varian (2005) show that when a sufficient proportion of customers are myopic or when the valuations of customers increases (by providing enhanced services) dynamic pricing may increase the revenue. Another setting where dynamic pricing could boost the revenue is when the seller is more patient than the buyer and discounts his utility over time at a lower rate than the buyer (Bikhchandani and McCardle 2012, Amin et al. 2013, Mohri and Medina 2014a). See Taylor (2004), Conitzer et al. (2012) for privacy issues and anonymization approaches in this context. In contrast with these works, our focus is on auction environments and we study the role of competition among strategic bidders.

The problem of learning the distribution of the valuation and optimal pricing also have been studied in the context of revenue management and pricing for markets where each (infinitesimal) buyer does not have an effect on the future prices and demand curve can be learned with near-optimal regret (Segal 2003, Besbes and Zeevi 2009, Harrison et al. 2012, Wang et al. 2014); see den Boer (2014) for a survey. In this work, we consider a setting where the goal is to learn the optimal reserve price with (finitely many) strategic and forward-looking buyers, with multi-unit demand, where the action of each buyer can change the prices in the future.

Organization. The rest of the paper is organized as follows. We formally present our model in Section 2, followed by our lower bound and upper bound revenue benchmarks in Section 3. We describe the threshold mechanism in Section 4. We show that the mechanism is dynamic incentive compatible in Section 5. In Sections 6, we present an extension of the threshold mechanism. We provide a discussion in Section 7 and conclude in Section 8.

2.   Model and Preliminaries

A seller auctions off $T > 1$ items to $n \geq 1$ agents in T rounds of second price auctions, numbered $t = 1, 2, \ldots, T$. The items are of type high or low denoted by $s \in \{L, H\}$, where informally we think of an item of type H as being more valuable than an item of type L. The items are *all of the same type*. The type is s with probability p_s.

The valuation of agent $i \in \{1, \ldots, n\}$ for an item of type s, denoted by v_i, is drawn independently and identically from distribution F_s, i.e., the valuations are i.i.d. conditioned on s. Note that agents' valuations are identical in each round (if they participate, see below). In Section 7 we consider an extension of our model where the valuations of the agents may change over time.

Each agent participates in each auction with probability α_i exogenously and independently across rounds and agents. One can think of α_i's as throttling probabilities[3] or matching probabilities to a specific user demographic (Celis et al. 2014). Let X_{it} be the indicator random variable corresponding to the participation of agent i in auction at time t. Note that $\alpha_i = \mathrm{E}[X_{it}]$. We denote the realization of X_{it} by x_{it}. Agent i learns x_{it} at the beginning of round t. [4] In particular, our (incentive compatibility) results hold in the special case when all the agents participate in all the auctions, i.e., $\alpha_i = 1$ for all i. Participation probabilities allow us to model (ad exchange) auctions in which a small number of bidders participate in each auction (cf. Celis et al. (2014)).[5]

Information Structure. We assume T, p_L, and p_H to be common knowledge. We also assume that the type of the item s is unknown to the auctioneer, who only knows p_s, and this is common knowledge. In our motivating application, advertisers sometimes may have more information about the value of a user or an impression than the publisher. Thus, we allow for both possibilities:

• the type of the item s is unknown to the agents, only p_s is common knowledge among the agents and seller; or,

• the type of the item s is common knowledge among the agents.

We prove our results for the latter information structure since it corresponds to a stronger requirement for the incentive compatibility of the mechanism; hence, our results remain valid in the former information structure, i.e., agents have the same information as the seller about the type of the

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[3] Due to budget and bandwidth constraints or other considerations, online advertising platforms often randomly select a subset of bidders, from all eligible advertisers, to participate in the auction. This process is referred to as throttling (Goel et al. 2010, Charles et al. 2013, Chakraborty et al. 2010).

[4] In our setting, the population of the agents is fixed over time; see Said (2011), Vulcano et al. (2002) for sequential second price auctions with randomly arriving buyers.

[5] Thus, the expected number of bidders participating in each auction is $\sum_i \alpha_i$, which may be much smaller than n, for instance, in Example 1, we have $\sum_i \alpha_i = 1$ whereas $n = 20$.

item. Similarly, we assume that α_i's are common knowledge among the agents and the auctioneer. Our mechanism remains incentive compatible, as defined below, if the agents have incomplete information about the α_i's. At the start, each agent i knows his own valuation, v_i, but not the other agents' valuations (but agents may make inferences about the valuations of the other agents over time).

Let us now consider the seller's problem. The seller aims to maximize her expected revenue via a repeated second price auction.

A "generic" dynamic second price mechanism. At time 0, the auctioneer announces the reserve price function $\Omega : \mathcal{H} \to \mathbb{R}^+$ that maps the history observed by the mechanism to a reserve price. The history observed by the mechanism up to time τ, denoted by $H_{\Omega,\tau} \in \mathcal{H}$, consists of, for each round $t < \tau$, the reserve price, the agents participating in round t and their bids, and the allocation and payments at that round. More precisely,

$$H_{\Omega,\tau} = \langle (r_1, x_1, b_1, q_1, p_1), \cdots, (r_{\tau-1}, x_{\tau-1}, b_{\tau-1}, q_{\tau-1}, p_{\tau-1}) \rangle$$

where

- r_t is the reserve price at time t.
- $x_t = \langle x_{1t}, \cdots, x_{nt} \rangle$. Here x_{it} is equal to 1 if agent i participates in the auction for item t.
- $b_t = \langle b_{1t}, \cdots, b_{nt} \rangle$ where b_{it} denotes the bid of agent i at time t. We assign $b_{it} = \phi$ if $x_{it} = 0$, i.e., if agent i does not participate in round t.
- q_t corresponds to the allocation vector. Since the items are allocated via the second price auction with reserve r_t, if all the bids are smaller than r_t, the item is not allocated. Otherwise, the item is allocated uniformly at random to an agent $i^\star \in \arg\max_i \{b_{it}\}$ and we have $q_{i^\star t} = 1$. For all the agents that did not receive the item, q_{it} is equal to 0.
- p_t is the vector of payments. If $q_{it} = 0$, then $p_{it} = 0$ and if $q_{it} = 1$, then $p_{it} = \max\{\max_{j \neq i}\{b_{jt}\}, r_t\}$.

Note that in our notation, Ω includes a reserve price function for each $t = 1, 2, \ldots, T$. The length of the history $\mathcal{H}$ implicitly specifies the round for which the reserve is to be computed. Further note that the auctioneer commits beforehand to a reserve update rule Ω (cf. Devanur et al. (2014)).

An important special case is a *static mechanism* where the reserve is not a function of the previous bids and allocations.

We can now define the seller problem more formally. The seller chooses a reserve price function Ω that maximizes the expected revenue, which is equal to $\mathrm{E}\left[\sum_{t=1}^{T} \sum_{i=1}^{n} p_{it}\right]$, when the buyers play an equilibrium with respect to the choice of Ω. In order to define the utility of the agents, let H_{ik}

denote the history observed by agent i up to time t including the allocation and payments of (only) agent i. Namely,

$$H_{ik} = \langle (r_1, x_{i,1}, b_{i,1}, q_{i,1}, p_{i,1}), \cdots, (r_{t-1}, x_{i,t-1}, b_{i,t-1}, q_{i,t-1}, p_{i,t-1}) \rangle.$$

Bidding strategy $B_i : \mathbb{R} \times \mathcal{H}_i \times \mathbb{R} \to \mathbb{R}$ of agent i maps the valuation of the agent v_i, history H_{it}, and the reserve r_t at time t to a bid $b_{it} = B_i(v_i, H_{it}, r_t)$. Here $\mathcal{H}_i$ is the set of possible histories observed by agent i. [6]

Definition 1 (Best-Response) Given strategy profile $< B_1, B_2, \cdots, B_n >$, B_i is a best-response strategy to the strategy of other agents B_{-i}, if, for all s and v_i in the support of F_s, it maximizes the expected utility of agent i,

$$U_i(v_i, s, B_i, B_{-i}) = \mathrm{E}\left[\sum_{t=1}^{T} v_i q_{it} - p_{it} \right],$$

where the expectation is over the valuations of other agents, the participation variables x_{jt}'s, and any randomization in bidding strategies. Strategy B_i is an ϵ-best-response if, for all v_i in the support,

$$U_i(v_i, s, B_i, B_{-i}) \geq U_i(v_i, s, \mathrm{BR}(v_i, s, B_{-i}), B_{-i}) + T\alpha_i\epsilon,$$

where BR denotes a best response.

Note that $\alpha_i T$ is the expected number of rounds in which agent i participates. Therefore, under an ϵ-best-response, on average the agent loses at most ϵ in utility, relative to playing a best response, per-round of participation.

A mechanism is incentive compatible if, for each agent i, the truthful strategy is a best-response to the other agents being truthful. In this paper, we consider the notion of approximate incentive compatibility and assume that an agent does not deviate from the truthful strategy when the benefit from such a deviation is insignificant. This notion is appealing when characterizing, or computing, the best response strategy is challenging (Archer et al. 2004, McSherry and Talwar 2007, Nazerzadeh et al. 2013).[7] In online ad auctions, finding profitable deviation strategies requires

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[6] Similar to the common practice in ad exchanges, we assume that agents see the reserve price before they bid. Note that, similar to s and α_i's, our incentive compatibility results are still valid if the agents have incomplete information about previous and current reserve prices.

[7] Approximate incentive compatibility correspond to a special case of approximate Nash equilibrium (Daskalakis et al. 2009, Chien and Sinclair 2011, Kearns and Mansour 2002, Feder et al. 2007, Hémon et al. 2008)) where the focus is on the truthful strategy. In Section 7, we provide a comparison with the notion of mean-field equilibrium (Iyer et al. 2011, Balseiro et al. 2013, Gummadi et al. 2013).

solving complicated dynamic programs in a highly uncertain environment, so agents can plausibly be expected to bid truthfully under an approximately incentive compatible mechanism.

Definition 2 (Approximate Incentive Compatibility) A mechanism is ϵ-incentive compatible if the truthful strategy of agent i is an ϵ-best-response to the truthful strategy of other agents for all s and all v_i in the support of F_s.

In Section 5 (Definition 3) we present a stronger and dynamic notion of incentive compatibility that implies that with probability close to 1, truthfulness remains an ϵ-best-response for the agents over time. We also provide conditions under which our proposed mechanism satisfies this stronger notion.

3. Benchmarks

In this section we introduce suitable lower and upper-bound revenue benchmarks for mechanism design in our setting.

To obtain a lower bound on revenue we consider the *static mechanism*, which implements a second price auction in each round with a constant reserve r_0. The reserve is chosen at time 0 before the mechanism observes any of the bids, and does not change over time.[8] The reserve r_0 is chosen optimally based on the prior information available to the auctioneer.

For an upper-bound on revenue, we consider the optimal T-round mechanism that knows the type of the items (this is a "genie-aided" setting where a genie reveals s to the auctioneer). The benefit from doing this is two-fold: First, we avoid the intractable problem of finding the optimal T-round mechanism with s unknown.[9] Second, the optimal mechanism in the genie-aided setting turns out to be have a simple and revealing structure that later facilitates our construction of the "threshold mechanism" that achieves revenue close to the upper bound benchmark even without knowing s beforehand.

Throughout this paper, we make the following standard regularity assumption (cf. Myerson (1981)).

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[8] Since the valuations of the agents are correlated through the type of the items, finding the optimal static auction is challenging and could be computationally intractable (Papadimitriou and Pierrakos 2011). Cremer and McLean (1988) proposed a mechanism that can extract the whole surplus if the valuations are correlated by offering lotteries to the bidders; however, their mechanism is significantly different from practically used mechanisms in online ad auctions; furthermore, their mechanism is not ex-post individually rational which is a desirable property satisfied by the second price auction; also see Fu et al. (2014).

[9] In fact, the optimal single round mechanism is itself intractable. See footnote 8.

Assumption 1 (Regularity) *Distribution F_s, for $s \in \{L, H\}$, with density f_s, is regular, i.e., c.d.f. $F_s(v)$ and $\left(v - \frac{1 - F_s(v)}{f_s(v)}\right)$ are strictly increasing in v over the support of F_s.*

Examples of regular distributions include many common distributions such as the uniform, Gaussian, log-normal, etc.

If s is known and F_s is regular, then, we know from Myerson (1981) that the optimal mechanism for $T = 1$ is the second price auction with reserve price $r_s^\star$ that is the unique solution of

$$r - \frac{1 - F_s(r)}{f_s(r)} = 0 \,. \tag{1}$$

We generalize this result to T rounds.

Proposition 1 (No "dynamic" improvement with single type) *If the valuations are drawn i.i.d. from a regular distribution (i.e., s is known and F_s satisfies Assumption 1), the optimal mechanism is the second price auction with a constant reserve that is the solution of* Eq. (1) *and there is no benefit from having dynamic reserve prices.*

Proposition 1 says that even for $T > 1$, the revenue maximizing mechanism simply runs T second price auctions with a constant reserve of $r_s^\star$, obtaining an expected revenue that is T times the revenue of a single Myerson auction. We call this the *upper-bound revenue benchmark.*

We prove Proposition 1 in the appendix using a reduction argument that reduces a mechanism in the T-round setting to a mechanism for a single-round. The proposition generalizes the insights from settings with a single buyer (Stokey 1979, Baron and Besanko 1984, Salant 1989, Hart and Tirole 1988, Acquisti and Varian 2005) to auction environments with multiple buyers.

An insight obtained from Proposition 1 is that the goal of a (revenue-maximizing) mechanism should be to learn the distribution of the valuations of the agents, not their actual bids. In the next section, we present a simple mechanism (for our setting with unknown s) that learns the distribution of valuations (i.e., learns s) from the bids by exploiting competition between bidders. The mechanism extracts higher revenue than the static mechanism (our lower bound benchmark) by using a reserve of $r_s^\star$ in subsequent rounds, as in the upper-bound benchmark mechanism. In fact, for a broad class of distributions of valuations, our mechanism is approximately incentive compatible and obtains revenue close to the upper-bound benchmark.

4. The Threshold Mechanism

In this section, we present the class of threshold mechanisms and state our first result on approximate incentive compatibility and revenue optimality of threshold mechanisms.

A **threshold mechanism** is defined by three parameters and is denoted by $\mathcal{M}(\rho, r_L, r_H)$ where r_L is the initial reserve price. The reserve stays r_L until any of the agents bid above ρ, then for all subsequent rounds, the reserve price will increase to r_H. If there are no bids above ρ, the reserve stays r_L until the end.

As we demonstrate in the following, this class of mechanisms (and a generalization of it, presented in Section 6, include good candidates for boosting revenue if the modes of F_L and F_H are sufficiently well separated. The idea is to choose ρ such that the valuation of an agent is unlikely to be above ρ if $s = L$, whereas, a valuation exceeding ρ is quite likely if $s = H$. Moreover, as we establish, truthful bidding forms an approximate equilibrium, allowing the mechanism to correctly infer s for typical realizations.[10]

To convey the intuition behind our incentive compatibility and revenue optimality results, we start with the following "warm up" result.

Proposition 2 *Suppose that F_L is supported on $[0, \bar{L})$ and F_H is supported on $[\underline{H}, \infty)$ for $\bar{L} < \underline{H}$ and $\alpha_i = 1$ for at least 2 agents. Consider any $r_L < r_H$. Then, $\mathcal{M}(\rho, r_L, r_H)$, for any $\rho \in (\bar{L}, \underline{H})$, is incentive compatible and obtains revenue at least $(1 - 1/T)$ times the upper bound benchmark under truthful bidding.*

Proof: Consider agent i and round t where i participates (i.e., $x_{it} = 1$). First observe that since bidding truthfully is a (weakly) dominant strategy in the second-price auction, truthfulness is a myopic best-response in our setting.

If $s = L$, then bidding truthfully will not increase the reserve in the future rounds and truthfulness is a (weakly) dominant strategy. Now suppose $s = H$. If $r = r_H$ in round t, then again truthful bidding is a best response, since the reserve will continue to be r_H for the remaining rounds. On the other hand, if $r = r_L$ (and $s = H$), at least one other agent will participate in the auction at time t. At the equilibrium, the other agent will bid truthfully, hence above ρ, and the reserve will be r_H for the remaining rounds in any case. So bidding truthfully is a best response, since it is myopically a best response.

For each round after the first round, the reserve is $r_s^\star$ and the Myerson optimal revenue is obtained. Hence, the Myerson optimal revenue is obtained in at least $T - 1$ rounds. It follows from Proposition 1 that the overall expected revenue is at least $(T - 1)/T$ times the upper bound benchmark under truthful bidding. □

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[10] A *realization* refers to the valuations of agents and their participation across rounds, cf. the beginning of Section 5.

We now show that the threshold mechanism is approximately incentive compatible when the support of the distributions overlap and the distribution of the low type is bounded. We also provide an example that shows a significant boost in the revenue.

Theorem 3 *Let F_L be supported on $[0, \bar{L}]$, for $\bar{L} < \infty$. Let $r_s^\star$ be the solution of $r - \frac{1 - F_s(r)}{f_s(r)} = 0$. Consider any positive $\epsilon < r_H^\star - r_L^\star$. Let $\delta = \epsilon/(r_H^\star - r_L^\star)$ and $\alpha = \min_i \alpha_i > 0$. Define*

$$n_0 \equiv 1 + \frac{1.59 \log(2/\delta)}{(1 - F_H(\rho))} < \infty,$$
$$T_0 \equiv \frac{2}{\delta} \left\lceil \frac{n_0 - 1}{(n-1)\alpha} \right\rceil.$$

Consider $\rho \geq \bar{L}$ such that $(1 - F_H(\rho)) > 0$. Then, Mechanism $\mathcal{M}(\rho, r_L^\star, r_H^\star)$ is ϵ-incentive-compatible for all $n \geq n_0$ and $T \geq T_0$. In addition, for each $s \in \{L, H\}$ the expected revenue is additively $T\epsilon$ close to the upper-bound revenue benchmark under truthful bidding.

The following shows that the mechanism, under the assumptions of Theorem 3, is asymptotically optimal.

Corollary 1 *Let F_L be supported on $[0, \bar{L}]$, $\bar{L} < \infty$. If $\rho \in (\bar{L}, r_H^\star)$, treating $n \geq 1$ and $\alpha > 0$ as fixed, under truthful bidding, the ratio of the expected revenue of Mechanism $\mathcal{M}(\rho, r_L^\star, r_H^\star)$ to the upper bound revenue benchmark is bounded below by $1 - O(\log T/T) \xrightarrow{T \to \infty} 1$.*

An appealing feature of Theorem 3 is that the lower bound on number of agents, n_0, grows only as $O(\log(1/\delta))$. On the other hand, the lower bound on number of rounds T_0 grows as $O(\log(1/\delta)/\delta)$ for $n\alpha = \Omega(1)$. This is somewhat larger than n_0 for small δ but this is not a major concern since the number of identical (or very similar) impressions is often large in online advertising settings. The example below demonstrates a numerical illustration of Theorem 3.

Example 1 Suppose F_L is the normal distribution, with mean 1 and standard deviation 0.4, truncated to interval $[0, 3]$, and $F_H = \mathcal{N}(3, 0.8^2)$, i.e., normal with mean 3 and standard deviation 0.8, and that $p_L = p_H = 0.5$. These distributions are shown in Figure 4. Also, let $n\alpha = 1$, $\alpha_i = \alpha$ for all i. This means that, on average, one agent participates in each round. Note that each agent participates in $\alpha T = T/n$ rounds on average, making it non-trivial to have dynamic reserves without losing incentive compatibility if T/n is much larger than 1.

We have $r_L^\star \approx 0.796$ and $r_H^\star \approx 2.318$. Using a threshold of $\rho = 3$ gives $n_0 \approx 19.52$. Using $n = 20$, we obtain $T_0 \approx 6800$ for $\epsilon = 0.009$ so we consider $T = T_0 = 6800$. Theorem 3 guarantees that $\mathcal{M}(\rho, r_L^\star, r_H^\star)$ is ϵ-incentive compatible and that the loss relative to the upper-bound benchmark revenue is $\epsilon = .009$ at most per round. Using numerical computations, we find that the (optimal)
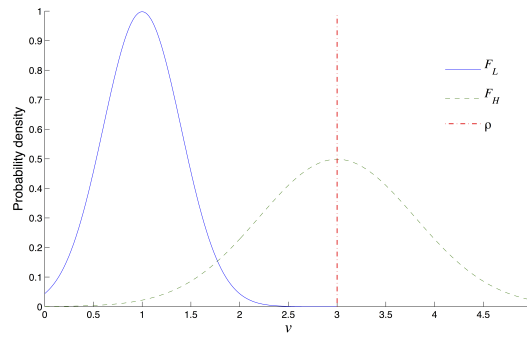
Figure 1   **Illustration of distributions in Example 1.** F_L **is the normal distribution, with mean** 1 **and standard deviation** 0.4**, truncated to interval** $[0,3]$**, and** $F_H = \mathcal{N}(3, 0.8^2)$**, i.e., normal with mean** 3 **and standard deviation** 0.8**. We use** $\rho = 3$**.**

static second price auction obtains average-revenue per-round revenue equal to 0.755 using (constant) reserve price 1.05. Mechanism $\mathcal{M}(\rho, r_L^\star, r_H^\star)$ is found to yield a per-round revenue of 0.935 in simulations, improving more than 23% over the static mechanism. The per-round revenue of the optimal mechanism that knows the type of the impressions is equal to 0.938 (the 95% confidence error in estimating the revenues is less than 0.007).

The average surplus of a buyer per round of participation, averaged over s and v_i is found to be 0.335 (with 95% confidence error 0.002). Given this, and the fact that the mechanism is (provably) ($\epsilon = 0.009$)-incentive compatible, truthful bidding appears quite plausible.

5.   Dynamic Incentive Compatibility

In this section, we show that, with high probability, no agent has a large incentive to deviate from the truthful strategy in later rounds after acquiring new information. We also relax the requirement that F_L needs to have bounded support.

We start by defining a stronger notion of incentive compatibility. By a *realization*, denoted by $(v_i, x_i^T)_{i=1}^n$, we refer to a valuation vector $(v_1, v_2, \ldots, v_n)$ along with a participation vector $(x_1^T, x_2^T, \ldots, x_n^T)$.

Definition 3 (Dynamic Incentive Compatibility) We call a realization ϵ-dynamic-incentive-compatible (ϵ-DIC) with respect to a mechanism, if truthfulness, for each agent i and in *each round* $t \in \{1, 2, \ldots, T\}$, remains an (additive) $\epsilon \alpha_i (T - t)$-best-response to the truthful strategy of the other agents. We say that a mechanism is (δ, ϵ)-*dynamic-incentive-compatible* if the probability of the realization being ϵ-DIC with respect to the mechanism is at least $1 - \delta$.

Thus, in a (δ, ϵ)-dynamic-incentive-compatible mechanism, assuming truthful bidding, with probability at least $1 - \delta$, the realization satisfies the following property: for each agent and each round

that the agent participates, the average cost of truthful bidding is at most ϵ for each future round that he may participate in, relative to any other strategy. The above definition extends the notion of (exact) interim dynamic incentive compatibility (Bergemann and Välimäki 2010) which implies that the agents will not deviate from the truthful strategy even as they obtain more information over time.

Theorem 4 *Fix $\rho \in (0, \infty)$. Let $\lambda = 1 - F_L(\rho)$. Consider any $\epsilon < r_H^\star - r_L^\star$ and let $\delta = \epsilon/(r_H^\star - r_L^\star)$ and $\alpha = \min_i \alpha_i > 0$. Then, Mechanism $\mathcal{M}(\rho, r_L^\star, r_H^\star)$ is (δ, ϵ)-dynamic incentive compatible for all $n \in [n_1, n_2]$ and $T \geq T_1$ where*

$$n_1 \equiv 1 + \frac{3.18 \log(2/\delta)}{(1 - F_H(\rho))} < \infty,$$
$$n_2 \equiv \frac{\delta(r_H^\star - r_L^\star)}{\lambda r_H^\star},$$
$$T_1 \equiv \frac{4}{\delta} \left\lceil \frac{n_1 - 1}{(n-1)\alpha} \right\rceil.$$

Further, under truthful bidding, the expected revenue of the mechanism is additively within ϵT of the upper-bound benchmark revenue.

Note that this theorem requires n to be not too large, to ensure that a valuation exceeding ρ is unlikely under $s = L$. The assumed upper bound on n can be eliminated in two different ways: In the corollary below, we assume a bounded support for F_L leading to $\lambda = 0 \Rightarrow n_2 = \infty$. Later in Section 6, we introduce a generalized threshold mechanism, which then facilitates a result similar to Theorem 4 while allowing n to be arbitrarily large (Theorem 5).

Corollary 2 (Bounded Support) *Let F_L be supported on $[0, \bar{L}]$, for $\bar{L} < \infty$. Fix $\rho \in [\bar{L}, \infty)$. Consider any $\epsilon < r_H^\star - r_L^\star$ and let $\delta = \epsilon/(r_H^\star - r_L^\star)$. Then, Mechanism $\mathcal{M}(\rho, r_L^\star, r_H^\star)$ is (δ, ϵ)-dynamic incentive compatible for all $n \geq n_1$ and $T \geq T_1$ where*

$$n_1 \equiv 1 + \frac{3.18 \log(2/\delta)}{(1 - F_H(\rho))} < \infty,$$
$$T_1 \equiv \frac{4}{\delta} \left\lceil \frac{n_1 - 1}{(n-1)\alpha} \right\rceil.$$

Further, the expected revenue of the mechanism is additively within ϵT of the upper-bound benchmark revenue under truthful bidding.

Comparing with Theorem 3, we see that the cost of the stronger notion of equilibrium in the above results is only a factor 2 loss in n_1 and a further factor 2 loss in T_1.

We prove Theorem 4 in the appendix. (Corollary 2 is immediate from Theorem 4.) We state below the main lemma and key ideas leading to a proof of dynamic incentive compatibility for $s = H$.

Let Q_t be the event that the reserve in round $t+1$ is $r_H^\star$ assuming truthful bidding (thus Q_t, here, is the event that a agent with valuation exceeding ρ participates in one of the first t rounds). Let $Q_t^{\sim i}$ be the event that the reserve in round $t+1$ would have been $r_H^\star$ assuming truthful bidding even if the bids of agent i are removed (thus $Q_t^{\sim i}$, here, is the event that a agent $j \neq i$ with valuation exceeding ρ participates in one of the first t rounds). Let

$$t_\delta = \min\{t : \exists i \text{ s.t. } \Pr\left(Q_t^{\sim i}\,\middle|\,s = H\right) \geq 1 - \delta\} \tag{2}$$

(It turns out that $t_\delta \leq \lceil \frac{n_1-1}{(n-1)\alpha} \rceil$). By definition, $Q_t \supseteq Q_t^{\sim i}$ for all i and all t. It follows that $\Pr(Q_{t_\delta}|s = H) \geq 1 - \delta$, so, in establishing (δ, ϵ)-dynamic incentive compatibility, we can ignore the trajectories under which Q_{t_δ} does not occur (these trajectories have combined probability bounded above by δ). Under Q_{t_δ}, the reserve in round $t_\delta + 1$ (and all later rounds) is already $r_H^\star$, making truthful bidding an exact best response in those rounds. Also, for agents whose valuation is less than ρ, truthful bidding is always a best response. Consider an agent i with valuation exceeding ρ. Once i has bid truthfully the first time, future bids of i have no impact on the reserve (which is already $r_H^\star$) and truthful bidding is a best response.[11] Hence, we only need to consider the first time that i participates (and if the reserve is still $r_L^\star$). Thus, it suffices to show that under $s = H$ for $t \leq t_\delta$, if an agent i participates for the first time in round t, truthful bidding is an $(\alpha(T - t_\delta)\epsilon)$-best-response to the truthful strategy of the other agents (even if the reserve is still $r_L^\star$ in round t). The following lemma establishes that this is the case (the idea is that the reserve is very likely to rise to $r_H^\star$ by the end of round τ, irrespective of the bidding behavior of agent i).

Lemma 1 *Assume that $s = H$, $n \geq n_1$, $\delta < 1$. Let t_δ be defined as in Eq. (2). For any agent with valuation exceeding ρ who participates for the first time in a round $t \leq t_\delta$, and sees a reserve $r_L^\star$, truthful bidding is (additively) $\alpha(T - t_\delta)\delta(r_H^\star - r_L^\star)$-optimal, assuming that others bid truthfully. Further, we have $t_\delta \leq \tau = \left\lceil \frac{(n_1-1)}{(n-1)\alpha} \right\rceil$ and $\Pr(Q_{t_\delta}|s = H) \geq 1 - \delta$.*

Lemma 1 is proved in Appendix A.5.

6. The Generalized Threshold Mechanism

We now present a generalization of the threshold mechanism that allows us to eliminate the upper bound on n in Theorem 4. (Alternatively, one can interpret Theorem 5 below as allowing F_L to have a larger tail.)

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[11] In the present setting with mechanism $\mathcal{M}(\rho, r_L^\star, r_H^\star)$, agent i will see a reserve that has already risen to $r_H^\star$ in each subsequent round that i participates in. (Later we will generalize the threshold mechanism in Section 6, but it will still be true that if i bids above ρ once, future bids of i will not affect the reserve, hence truthful bidding will be exactly optimal in subsequent rounds.)

The *generalized threshold mechanism* is defined by four parameters and is denoted by $\mathcal{M}(\rho, r_L, r_H, k)$ where r_L is the initial reserve price. The reserve stays r_L until k distinct agents bid above ρ (possibly in different rounds). If this occurs then for all subsequent rounds, the reserve price will increase to r_H.

Theorem 5 *Let* $\lambda = 1 - F_L(\rho)$. *Assume* $\lambda \le (1 - F_H(\rho))/18$. *Fix positive* $\epsilon < r_H^\star - r_L^\star$. *Define* $\delta = \epsilon \min(1/(r_H^\star - r_L^\star), 2/r_H^\star)$ *and* $\alpha = \min_i \alpha_i > 0$. *Let*

$$n_3 \equiv 1 + 8.48 \log(2/\delta)/(1 - F_H(\rho)) < \infty,$$

$$n_4 \equiv 0.56 \log(2/\delta)/\lambda,$$

$$\bar{n} \equiv \max(n_3, n_4),$$

$$T_1 \equiv \frac{4}{\delta} \left\lceil \frac{n_3 - 1}{(\min(n, \bar{n}) - 1)\alpha} \right\rceil.$$

We provide mechanisms that work well for any $n \ge n_3$ *and* $T \ge T_1$.

• *Suppose* $n_3 < n_4$. *For all* $n \in [n_3, n_4]$ *and* $T \ge T_1 = \frac{4}{\delta} \left\lceil \frac{n_3 - 1}{(n-1)\alpha} \right\rceil$, *the generalized threshold mechanism* $\mathcal{M}(\rho, r_L^\star, r_H^\star, 2.26 \log(2/\delta))$ *is* (δ, ϵ)-*dynamic incentive compatible, and it is additively* ϵT *close to the revenue benchmark under truthful bidding.*

• *For all* $n \ge \max(n_3, n_4)$ *and* $T \ge T_1 = \frac{4}{\delta} \left\lceil \frac{n_3 - 1}{(\bar{n} - 1)\alpha} \right\rceil$, *the generalized threshold mechanism* $\mathcal{M}(\rho, r_L^\star, r_H^\star, 4\lambda n)$ *is* (δ, ϵ)-*dynamic incentive compatible, and it is additively* ϵT *close to the upper-bound revenue benchmark under truthful bidding.*

As a remark to ease the burden of notation: note that $T_1 \le \frac{4}{\delta} \left\lceil \frac{1}{\alpha} \right\rceil$ for the n values of interest, i.e., for $n \ge n_3$. In other words, $\frac{4}{\delta} \left\lceil \frac{1}{\alpha} \right\rceil$ rounds suffice to obtain our positive results. Also, note that for $n\alpha = \Theta(1)$, we still have $n_3 = O(\log(1/\delta))$ and $T_1 = O(\log(1/\delta)/\delta)$ as was the case for Theorem 3, so our requirements on the number of agents and number of rounds continue to be reasonable.

7.   Discussion

Transient Valuations So far we assumed that the valuations of the agents are constant over time. In this section, we consider the following extension of our model: each time an agent j participates, he draws a new valuation from F_s independently with some probability β_j (our original model corresponds to the case $\beta_j = 0$), and retains his previous valuation with probability $1 - \beta_j$. As we show in Appendix A.6, our incentive compatibility results of Theorem 3 and Corollary 2 (for a bounded low type distribution) still hold in this setting. It turns out that the incentive of the agents to deviate is even smaller and the proofs work nearly as before for any β_j's in $[0, 1]$.

On the other hand, the following example shows that with transient valuations, a mechanism that is not ex-post individually rational can obtain a revenue higher than the "upper bound"

benchmark, (cf. Section 3) by charging the agents a high price in advance: Suppose there is only one agent ($n = 1$), the agent participates in all rounds ($\alpha = 1$), and the agent draws a new valuation in each round from the uniform distribution over $[0,1]$ (i.e., $\beta = 1$). It is not difficult to see that the optimal constant reserve for this setting is equal to $\frac{1}{2}$ which yields an expected revenue of $\frac{T}{4}$ since the agent will purchase the item with probability $\frac{1}{2}$. Now consider a mechanism that offers reserve price $\frac{T-1}{2} - \varepsilon$ (for an arbitrarily small ε) in the first round; if the agent accepts that price, the mechanism offers the item for free in the future rounds, whereas if the agent refuses the offer, the mechanism posts a price of 1 in each future round. Observe that the agent will accept the mechanism's offer in the first round and the revenue obtained in this case is equal to $\frac{T-1}{2} - \varepsilon$, which exceeds $T/4$ for large enough T and small ε. However, this mechanism is not ex-post individually rational in each round. For $\beta \in (0,1)$ the optimal mechanism (that does not satisfy the ex-post IR property) would take the form of contracts followed by a sequence of auctions (Battaglini 2005, Kakade et al. 2013, Pavan et al. 2014).

Robustness of the threshold mechanism The generalized threshold mechanism only considers the maximum bid submitted by each bidder thus far, and combines the highest among these maximum bids suitably to determine the reserve price for the current round. This mechanism is robust in the following ways:

• The mechanism is unaffected by the creation of fake bidders who may only bid low values with the intention of misleading the seller into believing that the type of the item is L, without winning the auction. A mechanism which relies, for instance, on the average/typical behavior of a bidder would be vulnerable to this. (Note that bidders never want to mislead the seller into thinking that the item type is H, so they would not want to create fake bidders who bid high values.)

• Bidders cannot benefit from the use of multiple identities: this can only increase the likelihood that the reserve price rises.

We expect that the principle underlying our mechanism has further robustness properties:

• If the type of the item is subject to change (slowly) over time this can be handled by only using "recent" bids to determine the reserve price, where recency is determined by using a window of appropriate length. We expect that our results will generalize to such an environment.

• We expect that even with a continuum of types, a mechanism that is similar in spirit will do well. Such a mechanism may determine the reserve as a suitable continuous function of the k highest bids submitted by distinct bidders, for appropriately chosen k. We expect that good incentive and revenue properties will again emerge for relatively small values of n and T, though rigorous analysis appears complicated.

The size of n and T needed We believe that the size of n and T required for our results (from conservative theoretical bounds) are practical (~ 10 and ~ 10000 respectively), cf. Example 1. Most advertisers participate in hundreds of thousands and sometimes millions of auctions. The pool of advertisers that are interested in certain impression types is often large. We also point out that even for large values of n, for small α (e.g., $\alpha n = 1$), the number of advertisers that participate in each auction is small (mirroring reality), therefore, the reserve prices will have significant impact on revenue.

Connection to Mean Field Equilibrium We now comment briefly on the connection between our work and the concept of mean field equilibrium. A number of recent papers study notions of mean field equilibrium, e.g., Iyer et al. (2011), Balseiro et al. (2013) study mean field equilibria in dynamic auctions, and Gummadi et al. (2013) studies mean field equilibrium in multiarmed bandit games. An agent making a mean field assumption assumes that the set of competitors (or cooperators) she faces will be drawn uniformly at random from a large pool of agents with a known distribution of types. In our work, agents' participate in a particular round of a dynamic auction independently at random, but our results do not require $n \to \infty$ and agents retain their valuation for all rounds in which they participate. In Theorem 4, one can have any fixed number of agents exceeding $n_1 = O(\log(1/\epsilon))$, and participants reason about the posterior distribution of competitors they will face in a round, given the information available to them. This posterior distribution of the valuations of competitors is in general different from the prior distribution of valuations and evolves from one round to the next, in contrast with mean field analyses.

8. Conclusion

We considered repeated auctions of items, all of the same type, with the auctioneer not knowing the type of the items a-priori. In our model, the issue of incentives is challenging because a bidder typically participates in multiple auctions, and is hence sensitive to changes in future reserve prices based on current bidding behavior. We demonstrated a fairly broad setting in which a simple dynamic reserve second price auction mechanism can lead to substantial improvements in revenue over the best fixed reserve second price auction. In fact, our threshold mechanism is approximately truthful and achieves near optimal revenue in our setting. We demonstrate a numerical illustration of our results with a reasonable choice of model parameters, and show significant improvement in revenue over the static baseline.

In future work, we would like to investigate the effects of various properties of the (joint) distributions of the valuation of the advertisers (e.g., more than two types), the characteristics of learning algorithms (beyond simple threshold mechanisms), and the effect of the rate (and manner)

in which the valuations of advertisers change over time on the equilibrium and the revenue of the auctioneer.

Appendix

A. Proofs

A.1. Proof of Proposition 1

The following is the key lemma leading to Proposition 1.

Lemma 2 (Upper-bound) *Let $\mathcal{M}_T^s$ be the optimal T-round mechanism that knows the type of the item, s. Similarly, $\mathcal{M}_1^s$ corresponds to the optimal (static) mechanism when $T = 1$. Then, the revenue of $\mathcal{M}_T^s$, denoted by $\mathrm{REV}(\mathcal{M}_T^s)$, is bounded by $T \times \mathrm{REV}(\mathcal{M}_1^s)$. Furthermore, if mechanism $\mathcal{M}_1^s$ is ex-post incentive compatible, then, $\mathcal{M}_T^s$ can be implemented by repeating mechanism $\mathcal{M}_1^s$ at each step $t = 1, \cdots, T$.*

Proof of Lemma 2.   To prove the first part of the claim, we construct mechanism $\tilde{\mathcal{M}}$ that obtains, in expectation, revenue equal to $\mathrm{REV}(\mathcal{M}_T^s)/T$. Since by definition $\mathrm{REV}(\mathcal{M}_1^s)$ is the optimal revenue that can be obtained when $T = 1$, we conclude that $\mathrm{REV}(\mathcal{M}_T^s) \le T \times \mathrm{REV}(\mathcal{M}_1^s)$.

We construct mechanism $\tilde{\mathcal{M}}$ as follows: Let $\mathcal{B} \subseteq \{1, \cdots, n\}$ be the set of agents who participate in the one-round auction. Note that each agent i knows his own x_{it} but not x_{jt} for any other agent $j \ne i$. For all agents $j \notin \mathcal{B}$, draw a (hypothetical) valuation i.i.d. from the distribution of valuations F_s. Now consider the probability space generated by simulating mechanism $\mathcal{M}_T^s$ over a T round auction by sampling X_{jt}'s in each round and emulating the (optimal) bidding strategy of the agents under $\mathcal{M}_T^s$.

Consider the distribution $\mathcal{D}_\mathcal{B}$ of $(q_\mathcal{B}, p_\mathcal{B})$ in rounds where the set of agents who participate is exactly $\mathcal{B}$, in this probability space. More precisely, we are considering not a single simulation, but the probability space of possible simulation trajectories. For each trajectory ω, the pair $(q_\mathcal{B}, p_\mathcal{B})$ for each round in which agents $\mathcal{B}$ participate contributes a weight in H proportional to the probability of trajectory ω.

To determine the payments under $\tilde{\mathcal{M}}$, draw $(q_\mathcal{B}, p_\mathcal{B})$ uniformly from distribution $\mathcal{D}_\mathcal{B}$. The mechanism $\tilde{\mathcal{M}}$ charges the agents in $\mathcal{B}$ these amounts $p_\mathcal{B}$ and allocate the items according to $q_\mathcal{B}$.

We argue that the mechanism $\tilde{\mathcal{M}}$ is truthful: It is not hard to see that the ex interim expected utility of an agent i from participating in $\tilde{\mathcal{M}}$ with bid b_i when others bid truthfully, is exactly $1/T$ times the ex interim expected utility of participating in $\mathcal{M}_T^s$ and following his equilibrium strategy for valuation b_i there if others follow their equilibrium strategies. Recall that each agent i knows his own x_{it} but not x_{jt} for any other agent $j \ne i$. It follows that truthful bidding is an equilibrium in mechanism $\tilde{\mathcal{M}}$. Further, under truthful bidding, it

is not hard to see that the expected revenue of mechanism $\tilde{\mathcal{M}}$ is $\text{REV}(\mathcal{M}_T^s)/T$, as claimed. Note that when $\alpha_i = 1$, $1 \le i \le n$, the proof would be simplified and could be argued using the revelation principle Myerson (1986).

We now prove the second part of the claim. Note that if $\mathcal{M}_1^s$ is ex-post incentive compatible, the leakage of information from one round to another does not change the strategy of the agents. For private value settings such as this, ex-post incentive compatibility implies that truthfulness is a (weakly) dominant strategy for each agent for any realizations of other agents' valuations. Therefore, repeating mechanism $\mathcal{M}_1^s$ obtains revenue $T \times \mathcal{M}_1^s$ which is the upper-bound revenue. □

Proof of Proposition 1. When F_s is regular and s is known, we know that the optimal mechanism for a single round is a second price auction with reserve $r_s^\star$. Now, this mechanism is ex-post incentive compatible. The result then follows immediately from Lemma 2. □

A.2. Proof of Theorem 3

In Appendix A.5, we prove the following lemma.

Lemma 3 *Fix any $C < \infty$. Assume $s = H$, $n \ge n_0 \equiv 1 + C \log(2/\delta)/(1 - F_H(\rho))$. Fix an agent i. With probability at least $1 - (\delta/2)^{C/1.59}$ at least one agent $j \ne i$ with valuation exceeding ρ will bid in the first $\tau = \left\lceil \frac{n_0 - 1}{(n-1)\alpha} \right\rceil$ rounds. For $k \le C \log(2/\delta)/3.18$, at least k agents different from i with valuation exceeding ρ will bid in the first $\tau = \left\lceil \frac{n_0 - 1}{(n-1)\alpha} \right\rceil$ rounds with probability at least $1 - (\delta/2)^{C/4.24}$.*

Here we have used $1/(1 - e^{-1}) < 1.59$. We now show how the lemma, for $k = 1$, implies the results.

Proof of Theorem 3. We now prove Theorem 3 by first showing that the threshold mechanism is ϵ-incentive-compatible. First assume $s = L$. Consider any agent i with valuation $v_i \in [0, \bar{L}]$ and assume that other agents are truthful always. Since $v_i \le \rho$, it is clear that truthful bidding weakly dominates any other strategy, since this is true myopically, the reserve is unaffected, and the bidding behavior of others is unaffected by the bids of agent i. In this case, the reserve remains $r_L^\star$ and the agents bid truthfully throughout, so there is no loss in revenue.

Now assume $s = H$. Agent i being truthful can cause the reserve to rise though it wouldn't otherwise have risen, leading to a loss of at most $(r_H^\star - r_L^\star)\alpha_i\tau \le \frac{\alpha_i \epsilon T}{2}$ in expected utility for the agent during the first τ rounds. If the reserve would not have risen in the first τ rounds but agent i caused it to rise, this can lead to a further loss of up to $(r_H^\star - r_L^\star)$ per round of participation, and such a loss occurs with probability at most $\epsilon/(2(r_H^\star - r_L^\star))$ from Lemma 3 (with $C = 1.59$), leading to a bound of $\frac{\alpha_i \epsilon T}{2}$ for this loss to the agent. Combining yields the overall bound of $\alpha_i \epsilon T$ on the loss incurred by agent i by being truthful relative to any other strategy for $s = H$.

Finally we bound the loss in revenue from using this mechanism if $s = H$. Recall that the optimal auction if the auctioneer knows $s = H$ beforehand is to commit and run a second price auction with reserve $r_H^\star$ in all rounds. Hence, similar to the above, the expected revenue loss to the auctioneer is bounded by $\tau(r_H^\star - r_L^\star) +$

$\frac{\epsilon}{2(r_H^\star - r_L^\star)} \cdot T(r_H^\star - r_L^\star) \le \epsilon T$ if $s = H$. Since for each possible s, the expected loss in revenue is bounded above by ϵT, the same bound holds when we take expectation over s. □

Proof of Corollary 1.   Again, the case $s = L$ is trivial so we focus on $s = H$. Consider $\tau' = C \log T$, where we will choose $C \in (0, \infty)$ appropriately. Then, with probability at least $1 - 1/T$, each of the n agents participates in at least one of the first τ' rounds (for large enough C). As such, with probability at least $1 - 1/T$, the reserve rises to $r_H^\star$ by the end of round τ' if and only if at least one agent has a valuation exceeding ρ. Now, if no agent has valuation exceeding ρ, the benchmark mechanism extracts no revenue (under the assumption $\rho \le r_H^\star$) so there is no loss in revenue from not increasing the reserve. We conclude that, with probability at least $1 - 1/T$, there is no loss in revenue relative to the benchmark after round τ'. We conclude that the ratio of the expected mechanism revenue (under truthful bidding) to the benchmark revenue is at least $1 - O(\tau'/T + 1/T) = 1 - O(\log T / T)$. □

A.3.   Proof of Theorem 4

First assume $s = L$. A simple union bound ensures that all agents have a valuation not greater than ρ with probability at least $1 - \lambda n \ge 1 - \delta(r_H^\star - r_L^\star)/r_H^\star$, in which case truthful bidding weakly dominates any other strategy. (Since $v_i \le \rho$, it is clear that truthful bidding weakly dominates any other strategy, since this is true myopically and the bidding behavior of others is unaffected by the bids of agent i.) Hence, the realization is 0-DIC with respect to the mechanism with probability at least $1 - \delta(r_H^\star - r_L^\star)/r_H^\star \ge 1 - \delta$. Further, we can easily bound the loss in expected revenue relative to the benchmark under truthful bidding: There is no loss with probability $1 - \delta(r_H^\star - r_L^\star)/r_H^\star$ (the mechanism matches the benchmark mechanism, since the reserve remains $r_L^\star$ throughout) and a loss of at most $r_H^\star T$ (due to the reserve rising to $r_H^\star$) with probability $\delta(r_H^\star - r_L^\star)/r_H^\star$. Thus, the loss in expected revenue is bounded by $\delta(r_H^\star - r_L^\star)T = \epsilon T$ as required.

Now assume $s = H$. For any agent with a valuation less than or equal to ρ, truthful bidding again weakly dominates any other strategy, since this is true myopically and the bidding behavior of others is unaffected by the bids of agent i. It remains to deal with agents whose valuation exceeds ρ, to establish that truthful bidding is (δ, ϵ)-incentive-compatible. In particular, we need to show that with probability at least $1 - \delta$, the realization is ϵ-DIC with respect to the mechanism, i.e., that each such agent i loses no more than $\alpha_i(T - t)\epsilon$ in expectation on the equilibrium path from bidding truthfully in round t, for each t that i participates in. But this follows from Lemma 1: all realizations such that Q_{t_δ} occurs are ϵ-DIC, and $\Pr(Q(t_\delta)) \ge 1 - \delta$. See the argument after the statement of Theorem 4 in Section 4 for further details.

It remains to show that the loss in revenue is no more than ϵT, assuming truthful bidding under ϵ-DIC realizations. Now, using Lemma 3 with $C = 3.18$ and $Q_t \supseteq Q_t^{\sim i}$, we have

$$\Pr(Q_\tau) \ge 1 - (\delta/2)^2 \ge 1 - \delta/2\,.$$

Under Q_τ, the mechanism matches the benchmark mechanism for rounds after τ and hence there is no loss in revenue relative to the benchmark, after the first τ rounds. In any round, the loss due to setting the

wrong reserve (under truthful bidding) is bounded by $r_H^\star - r_L^\star$. Under $\bar{Q}_\tau$ (the complement of Q_τ), the loss can be this large in each of T rounds, in worst case. It follows that the overall loss in revenue is bounded by $(r_H^\star - r_L^\star)(\tau + \Pr(\bar{Q}_\tau)T)$. But by definition, $\tau = \delta T_1/4 \le T\delta/4$ and $\Pr(\bar{Q}_\tau) \le \delta/2$, implying that the loss in revenue relative to the benchmark is at most $(r_H^\star - r_L^\star)\delta(3/4)T = (3/4)\epsilon T \le \epsilon T$ as required, using the definition of δ.

A.4. Proof of Theorem 5

We start with the first bullet. The proof for $s = H$ follows exactly the same steps as the proof of Theorem 4, except that we use of the second part of Lemma 3 (using $n \ge n_3$) with $k = 2.26 \log(2/\delta) \le (C/3.18) \log(2/\delta)$ for $C = 8.48$, instead of the first part of the lemma (corresponding to $k = 1$). Consider $s = L$. The probability of k or more agents with valuation exceeding ρ is $\Pr(\text{Binomial}(n, \lambda) \ge k)$. Since $n \le n_4$, we have the mean of the binomial $\mu = n\lambda \le \mu_0 = 0.56 \log(2/\delta)$, in particular, $k \ge 4\mu_0 \ge 4\mu$. Now, using a Chernoff bound (on $\text{Binomial}(n, \lambda_0)$ where $\lambda_0 = \mu_0/n \ge \lambda$ leading to a mean of μ_0; clearly this binomial stochastically dominates the one we care about), we infer that

$$\Pr(\text{Binomial}(n, \lambda) \ge k) \le \Pr(\text{Binomial}(n, \lambda_0) \ge k)$$

$$\le \exp\{-\mu_0 \cdot 3^2/(2+3)\} = \exp\{-0.56 \log(2/\delta) \cdot 9/5\}$$

$$\le \exp\{-1.00 \log(2/\delta)\} = \delta/2\,.$$

If all valuations are no more than ρ then such a realization is clearly 0-DIC (i.e., incentive compatible in an exact sense) with respective to the mechanism. Hence, we have shown that the probability of the realization being ϵ-DIC is at least $1 - \delta/2$, implying (δ, ϵ)-dynamic incentive compatibility for $s = L$. Further, the loss in revenue for $s = L$ under truthful bidding is bounded above by $r_H^\star$ per round for cases in which k or more agents have valuation exceeding ρ, and there is no loss otherwise. Hence, the overall loss in expected revenue is bounded above by $(\delta/2)Tr_H^\star \le \epsilon T$ (using $\delta \le 2\epsilon/r_H^\star$) as required.

Now consider the second bullet. Consider $s = H$. The threshold is $k = 4\lambda n$. Let $\bar{n} = \max\{n_3, n_4\}$. The following lemma is similar to Lemma 3, but allows for larger values of k. The proof is in Appendix A.5.

Lemma 4 *Assume $s = H$, $n \ge \bar{n} \ge n_0 \equiv 1 + C \log(2/\delta)/(1 - F_H(\rho))$ and $k \le C \log(2/\delta)n/(3.18\bar{n})$. Fix an agent i. With probability at least $1 - (\delta/2)^{C/4.24}$, at least k agents different from i with valuation exceeding ρ will bid in the first $\tau = \left\lceil \frac{n_0 - 1}{(\bar{n} - 1)\alpha} \right\rceil$ rounds.*

To use Lemma 4 we need an upper bound on $\bar{n}$. Note that using $\delta \le 1$ and $F_H(\rho) \ge 0$, we have $\log(2/\delta)/(1 - F_H(\rho)) \ge \log 2$. Hence we have

$$n_3 \le \frac{\log(2/\delta)}{1 - F_H(\rho)}(8.48 + 1/\log 2) \le \frac{10.0 \log(2/\delta)}{1 - F_H(\rho)} \le \frac{\log(2/\delta)}{1.8\lambda}$$

using $\lambda \le (1 - F_H(\rho))/18$. It follows that $\bar{n} \le \frac{\log(2/\delta)}{1.8\lambda}$. With reference to the upper bound on k in Lemma 4, we deduce that

$$8.48 \log(2/\delta)n/(3.18\bar{n}) \ge 2.26 \log(2/\delta)n/\bar{n} \ge 2.26 \cdot 1.8\lambda n \ge 4\lambda n\,.$$

Hence, using Lemma 4, we deduce that under truthful bidding, the reserve rises to r_H by the end of $\tau = \frac{4}{\delta}\left\lceil\frac{n_3-1}{(\bar{n}-1)\alpha}\right\rceil$ rounds with probability at least $1-(\delta/2)^2$. Following the argument in the proof of Lemma 1 from here and using $\delta \leq \epsilon/(r_H^\star - r_L^\star)$, we deduce (δ,ϵ)-dynamic incentive compatibility for $s = H$. We also deduce that the loss in expected revenue is small, similar to the proof of Theorem 4.

Consider the second bullet and $s = L$. The probability of $k = 4\lambda n$ or more agents with valuation exceeding ρ is $\Pr(\text{Binomial}(n,\lambda) \geq 4\lambda n)$. The mean $\mu = \lambda n \geq 0.56\log(2/\delta)$ since $n \geq \bar{n} \geq n_3$. We infer using a Chernoff bound that

$$\Pr(\text{Binomial}(n,\lambda) \geq 4\lambda n) \leq \exp\{-\mu \cdot 3^2/(2+3)\} \leq \exp\{-0.56\log(2/\delta) \cdot 9/5\}$$

$$\leq \exp\{-1.00\log(2/\delta)\} = \delta/2\,.$$

We then complete the proof of approximate dynamic incentive compatibility and revenue optimality exactly as we did for the first bullet with $s = L$.

A.5. Proofs of Lemmas

Proof of Lemma 1. Consider any agent i. By definition of t_δ, we know that for $t \leq t_\delta$, for all agents i we have

$$\Pr(\bar{Q}_{t-1}^{\sim i}) > \delta\,, \tag{3}$$

where $\bar{Q}_{t-1}^{\sim i}$ is the complement of $Q_{t-1}^{\sim i}$.

Let $\tau = \lceil\frac{n_1-1}{(n-1)\alpha}\rceil$. Note that $\tau \leq \delta T_1/4 \leq \delta T/4 \Rightarrow \tau \leq \delta(T-\tau)/2$. It follows from Lemma 3 that

$$\Pr(\bar{Q}_\tau^{\sim i}) \leq \delta^2/2 \tag{4}$$

In particular, we have $t_\delta < \tau$ and $\Pr(Q_\tau) \geq \Pr(Q_\tau^{\sim i}) \geq 1-\delta^2/2 \geq 1-\delta$, yielding the second part of the lemma.

Combining Eqs. (3) and (4) we obtain that

$$\Pr(\bar{Q}_\tau^{\sim i}|\bar{Q}_{t-1}^{\sim i}) \leq \Pr(\bar{Q}_\tau^{\sim i})/\Pr(\bar{Q}_{t-1}^{\sim i}) \leq (\delta^2/2)/\delta = \delta/2\,. \tag{5}$$

Hence, agent i who participates for the first time in round t and sees reserve $r_L^\star$, infers that the reserve will rise to $r_H^\star$ by round $\tau+1$ with probability at least $1-\delta/2$, due to the bids of other agents. Thus, we can bound the expected cost in future rounds to agent i by causing the reserve to rise by bidding truthfully:

- Under $\bar{Q}_\tau^{\sim i}$, agent i may lose at most $\alpha_i(T-t)(r_H^\star - r_L^\star)$ in future rounds (in expectation).
- Under $Q_\tau^{\sim i}$, agent i may lose at most $\alpha_i(\tau-t)(r_H^\star - r_L^\star) \leq \alpha_i\tau(r_H^\star - r_L^\star)$ in rounds only up to round τ.

Thus, the overall future cost of bidding truthfully is bounded by

$$\Pr(Q_\tau^{\sim i})\alpha_i\tau(r_H^\star - r_L^\star) + \Pr(\bar{Q}_\tau^{\sim i})\alpha_i(T-t)(r_H^\star - r_L^\star) \leq 1 \cdot \alpha_i(r_H^\star - r_L^\star)\delta(T-\tau)/2 + (\delta/2)\cdot\alpha_i(T-t)(r_H^\star - r_L^\star)$$

$$\leq \delta\alpha_i(T-t)(r_H^\star - r_L^\star)$$

as required. Here we used $\tau \leq \delta(T-\tau)/2$ and $t \leq \tau$ from the discussion above. $\square$

Proof of Lemma 3. Let $Q_\tau^{\sim i}(k)$ denote the event of interest, and $Q_\tau^{\sim i}$ be the event for $k = 1$. For each agent $j \neq i$, agent j participates in *some* round t' for $t' \leq \tau$ with probability $1-(1-\alpha_j)^\tau \geq 1-(1-\alpha)^\tau$.

Independently, agent j has a valuation exceeding ρ with probability $1 - F_H(\rho)$. Hence, we have $v_j \geq \rho$ *and agent j enters a bids before the end of round τ*, with probability at least $(1 - (1 - \alpha)^\tau)(1 - F_H(\rho))$, and this occurs independently for $j \neq i$. Note that $(1 - x)^{1/x} \leq e^{-1}$ for $x \in (0, 1)$ since $(1 - x)^{1/x}$ is monotone decreasing in x. Using this bound, we have

$$1 - (1 - \alpha)^\tau \geq 1 - \exp(-\alpha\tau)$$

$$\geq 1 - \exp(-(n_0 - 1)/(n - 1))$$

$$\geq \frac{n_0 - 1}{n - 1}(1 - \exp(-1))$$

$$\geq \frac{n_0 - 1}{1.59(n - 1)}, \tag{6}$$

where we also used the definition of τ and convexity of $f(x) = \exp(-2kx)$.

It follows that

$$\Pr(\bar{Q}_\tau^{\sim i}) \leq \Pr\left(\text{Binomial}\left(n - 1, [1 - (1 - \alpha)^\tau][1 - F_H(\rho)]\right) = 0\right)$$

$$= \left(1 - [1 - (1 - \alpha)^\tau][1 - F_H(\rho)]\right)^{n-1}. \tag{7}$$

Hence,

$$\Pr(\bar{Q}_\tau^{\sim i}) \leq \exp\left\{[1 - (1 - \alpha)^\tau][1 - F_H(\rho)](n - 1)\right\}$$

$$\leq \exp\left\{-(n_0 - 1)(1 - F_H(\rho))/1.59\right\}$$

$$\leq \exp\{-(C/1.59)\log(2/\delta)\} = (\delta/2)^{C/1.59},$$

using $n_0 \geq 1 + C \log(2/\delta)/(1 - F_H(\rho))$ and Eq. (6).

Similarly,

$$\Pr(\bar{Q}_\tau^{\sim i}(k)) \leq \Pr\left(\text{Binomial}\left(n - 1, [1 - (1 - \alpha)^\tau][1 - F_H(\rho)]\right) < k\right)$$

$$= \left(1 - [1 - (1 - \alpha)^\tau][1 - F_H(\rho)]\right)^{n-1}. \tag{8}$$

The mean of the binomial is

$$\mu = (n - 1)[1 - (1 - \alpha)^\tau][1 - F_H(\rho)] \geq (n_0 - 1)[1 - F_H(\rho)]/1.59 = C \log(2/\delta)/1.59$$

using Eq. (6). It follows using a Chernoff bound and $k \leq C \log(2/\delta)/3.18 \leq \mu/2$ that

$$\Pr(\bar{Q}_\tau^{\sim i}(k)) \leq \exp\{-\mu(1 - 1/2)^2/2\} = \exp\{-C \log(2/\delta)/4.24\} = (\delta/2)^{C/4.24}.$$

$\square$

Proof of Lemma 4. The proof is very similar to the proof of Lemma 3. Let $Q_\tau^{\sim i}(k)$ denote the event of interest. For each agent $j \neq i$, agent j participates in *some* round t' for $t' \leq \tau$ with probability $1 - (1 - \alpha_j)^\tau \geq 1 - (1 - \alpha)^\tau$. Proceeding as before, we have

$$1 - (1 - \alpha)^\tau \geq \frac{n_0 - 1}{1.59(\bar{n} - 1)}. \tag{9}$$

We have Eq. (8) for the probability of $\bar{Q}_\tau^{\sim i}(k)$ as before. The mean of the binomial is

$$\mu = (n-1)[1-(1-\alpha)^\tau][1-F_H(\rho)] \geq (n_0-1)[1-F_H(\rho)](n-1)/(1.59(\bar{n}-1))$$

$$= C\log(2/\delta)(n-1)/(1.59(\bar{n}-1)) \geq C\log(2/\delta)n/(1.59\bar{n}) \tag{10}$$

using Eq. (9) and $n \geq \bar{n}$.

It follows using a Chernoff bound and $k \leq C\log(2/\delta)n/(3.18\bar{n}) \leq \mu/2$ that

$$\Pr(\bar{Q}_\tau^{\sim i}(k)) \leq \exp\{-\mu(1-1/2)^2/2\} = \exp\{-C\log(2/\delta)/4.24\} = (\delta/2)^{C/4.24}, \tag{11}$$

using $n \geq \bar{n}$. $\qquad\square$

A.6. Transient valuations

Consider the case where, each time an agent i participates, with probability $\beta_i \in [0,1]$, he draws a new valuation from distribution F_s. We show that Theorem 3 and Corollary 2 hold also in this case (i.e., our results with bounded low type distribution continue to hold). The following lemma (similar to Lemma 3 and proved using it) is a key building block.

Lemma 5 *Fix any $C < \infty$. Assume $s = H$, $n \geq n_0 \equiv 1 + C\log(2/\delta)/(1 - F_H(\rho))$. Consider transient valuations with resampling probability β_j for agent j. Fix an agent i. With probability at least $1 - (\delta/2)^{C/1.59}$ at least one agent $j \neq i$ with valuation exceeding ρ (in that round) will bid in the first $\tau = \left\lceil \frac{n_0-1}{(n-1)\alpha} \right\rceil$ rounds.*

Proof of Lemma 5. Compare with the case of fixed valuations in Lemma 3. Couple the two systems such that the valuation of agent j when j participates for the first time with transient valuations, is the same as the valuation of j with fixed valuations. Then if the event "at least one agent $j \neq i$ with valuation exceeding ρ (in that round) will bid in the first τ rounds" occurs with fixed valuations, it also occurs with transient valuations. The result follows from the first part of Lemma 3. $\qquad\square$

Now, using this lemma, the proof of Theorem 3 goes through verbatim. The proof of Theorem 4 for $s = H$ also goes through verbatim, whereas the case $s = L$ is trivial under a bounded low type distribution as in Corollary 2, implying that Corollary 2 also holds with transient valuations.

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