Robotica (1999) volume 17, pp. 293–301. Printed in the United Kingdom © 1999 Cambridge University Press

Extension of Usable Workspace of Rotational Axes in Robot Planning

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(Received in Final Form: September 3, 1998)

SUMMARY

Singularity of a robot manipulator is one of the obstacles that influences its capabilities. This paper discusses constrained and allowable rotational motion resulting from lost translational freedom when the robot is singular. A convenient method and simple and clear expression to determine the allowable rotational axes and the subspace that they form, under Jacobian singularity, is analyzed and presented. Different configurations, reciprocal screws, and subspaces of allowable-rotational-axes are derived in a case study involving a classic robot. The result is useful in applications involving robot path planning in task space as it extends the usable workspace of rotational axes.

KEYWORDS: Robot planning; Usable workspace; Rotational axes; Singularity; Path planning.

1. INTRODUCTION

Robotic manipulators have been used in industry, but their utilization is sometimes limited by the singularity of the manipulators (i.e. the singularity of their Jacobian matrix). When it is singular, a manipulator loses some of its freedom, and the rank of its Jacobian reduces. As a result, its workspace decreases and its kinematic and dynamic properties deteriorate.

This problem has been studied extensively,[1–9] and most of the studies are based on the screw theory.[10–13] Baker analyzed the relative motion of two-degree-of-freedom mechanisms and determined the special configuration using common reciprocal screws, which intersects with all the kinematic pair axes of the mechanisms.[1,2] Sugimoto reported that a robot end effector cannot move along the direction of a reciprocal screw.[3] Waldron derived three singular conditions of some industrial robot manipulators and their corresponding reciprocal screws.[4] Hunt solved the reciprocal screw of some robot manipulators. He pointed out that there are ∞^4 twists in space belonging to five-system, and determined the twists with different pitch.[5,6] Liu in reference [7] indicated the method of pseudo inverse of Jacobian matrix and developed a differential motion approach based on the singularity-robust inverse to deal with the singularity problem. Using the method of pseudo inverse sometimes results in discontinuities in joint space variables.

Based on the screw theory and the work done by Waldron[4] and Hunt,[5,6] the first named author has shown that the constrained rotation of a body does not affect its translation while the constrained translation of an object does influence its rotation.[14,15] Specifically, when the translational freedom is constrained, the rotational freedom is also partially constrained. As a result, the selection of rotational axes during path planning reduced, i.e. the body may still have rotational freedom, but it cannot rotate about any arbitrarily chosen axes. Its allowable rotational axes form only a subspace of the original space. The practical implication is that at singular points, trajectory planning of a manipulator in the task space may still proceed, but with certain restrictions. This effectively extends the robot workspace.

This paper further discusses the allowable rotational freedom when translational freedom is constrained, and demonstrates how to conveniently determine the allowable rotation under singularity and further a usable subspace of all rotational axes using a classical robot as an example. The results presented in this paper can be implemented in manipulator path planning software for industrial applications.

2. POSSIBLE MOVEMENT UNDER CONSTRAINTS

Assume two screws $\$_1$ and $\$_2$:

$$\begin{aligned} \$_1 &= (\mathbf{S}_1;\ \mathbf{S}_{01}) = (l_1\ m_1\ n_1;\ P_1\ Q_1\ R_1) \\ \$_2 &= (\mathbf{S}_2;\ \mathbf{S}_{02}) = (l_2\ m_2\ n_2;\ P_2\ Q_2\ R_2) \end{aligned} \tag{2.1}$$

where $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_{01}$, and $\mathbf{S}_{02}$ are four vectors, $\mathbf{S}_1 \bullet \mathbf{S}_1 = 1$, $\mathbf{S}_2 \bullet \mathbf{S}_2 = 1$, $\mathbf{S}_1 \bullet \mathbf{S}_{01} = 0$, and $\mathbf{S}_2 \bullet \mathbf{S}_{02} = 0$. The reciprocal product of two screws, $\$_1 \circ \$_2$, is defined as:

$$\$_1 \circ \$_2 = \mathbf{S}_1 \bullet \mathbf{S}_{02} + \mathbf{S}_2 \bullet \mathbf{S}_{01} \tag{2.2}$$

where "$\circ$" represents the reciprocal multiplications. When two screws are said to be reciprocal screws, the physical meaning of the product is that when a constrained body moves about an instantaneous screw axis, $\$_1$, a wrench acting on a screw, $\$^r$, does not contribute to the rate at which work is being done on the body. In such a circumstance $\$_1$ and $\$^r$ can be expressed as the reciprocal product of two screws being zero, that is:

$$\$_1 \circ \$^r = 0 \tag{2.3}$$

or $l_1P^r + m_1Q^r + n_1R^r + P_1l^r + Q_1m^r + R_1n^r = 0$

Equation 2.3 can be rewritten as:

$$(h_1 + h^r)\cos\alpha - a\sin\alpha = 0 \tag{2.4}$$

where h_1 and h^r are the screw pitchs of $\$_1$ and $\$^r$, respectively, α the angle between the two screws, and a the common normal of axes of the two screws. At this situation, the robot will lose some possible movement when singularity occurs. Its degree of freedom will also decrease. If screws $\$_i(\mathbf{s}_i;\ \mathbf{s}_{01})$, $i=1, 2, 3, \ldots, n$, $\mathbf{s}_i \bullet \mathbf{s}_i = 1$ and $\mathbf{s}_i \bullet \mathbf{s}_{01} = 0$, represent the axes of the kinematic joints of the robot, and those screws are linearly independent. There exist (6-n) reciprocal screws, $\$^r$, that are reciprocal to each screw of the screw system and we have:

$$\$_1 \circ \$^r = 0 \quad (i=1, 2, 3, \ldots, n) \tag{2.5}$$

If the force does no work on possible motion, it will only be the constraint force. The end effector will lose the corresponding freedom if there is some constraint force exerted on it. Therefore, it is convenient to analyze both the constrained motion and the possible motion of the mechanism by using the reciprocal screw theory.

The screws exist in five different forms, but only three of them are basic and are to be introduced here. If the reciprocal screw acting on the object has the form ($\mathbf{0}$; $\mathbf{s}$) it is a constraint couple with infinite pitch. If there is such a couple acting on a body but doing no work, it means that the body has no rotation about the axis that is colinear with the couple. The following expression denotes the reciprocal, ($\mathbf{0}$; $\mathbf{s}$), and the constrained motion, ($\mathbf{s}, \mathbf{r}\times\mathbf{s}$):

$$(\mathbf{0};\ \mathbf{s}) \Rightarrow (\mathbf{s},\ \mathbf{r}\times\mathbf{s}) \tag{2.6}$$

If the reciprocal screw has the form ($\mathbf{s}, \mathbf{r}\times\mathbf{s}$), with zero pitch, it is a line vector, 1_1, and expresses a constraint force passing through point, r, along the s-direction (Fig. 1). If there is such a force acting on a body and doing no work, the body is constrained and has no motion along the s-direction. Furthermore, if the body rotates about a possible rotational axis (l_2, ($\mathbf{s}'$; $\mathbf{r}'\times\mathbf{s}'$)) the projections of the velocity vectors of those points, which lie on the line l_1 and belong to that body, must be zero. This can be derived from Eq. 2.3, and it means that the rotational motion of the body is constrained when that projection is not zero. The reciprocal and constrained motion can be expressed as:

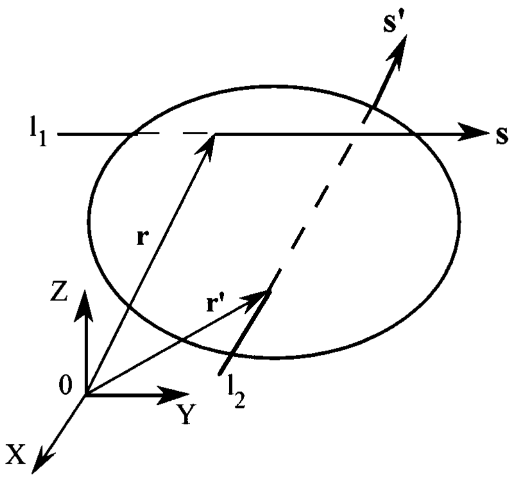


Fig. 1. Constraining force influences rotation.

$$(\mathbf{s};\ \mathbf{r}\times\mathbf{s}) \Rightarrow \begin{cases} (\mathbf{0};\ \mathbf{s}) \\ (\mathbf{s}';\ \mathbf{r}'\times\mathbf{s}'),\ (\mathbf{r}'-\mathbf{r})\times\mathbf{s}'\cdot\mathbf{s} \neq 0 \end{cases} \tag{2.7}$$

If the reciprocal is of the form ($\mathbf{s}$; $\mathbf{r}\times\mathbf{s}+\mathrm{h}\mathbf{s}$) where the pitch is not equal to zero ($h\neq 0$), it is also a constraining wrench passing through a point, r, along the s-direction. When a body is acted on by such a constraining wrench without doing any work, the body is constrained and has no twist along the s-direction passing through the same point, r, with pitch, H, not equal to $-h$. Under this condition, the moving twist with pitch $H\neq h$, including the rotational motion ($H=0$) and the translational motion ($H=\infty$). Besides, rotation about an axis passing through the other skew lines is constrained. A twist about an axis, which could be skew to that reciprocal wrench or in direct intersection with that wrench, is restricted as well. We may express the reciprocal screw and the constrained motion as

$$(\mathbf{s};\ \mathbf{r}\times\mathbf{s}+\mathrm{h}\mathbf{s}) \Rightarrow$$

$$\begin{array}{l} (\mathbf{s};\ \mathbf{r}\times\mathbf{s}+\mathrm{H}\mathbf{s}),\ \mathrm{H}\neq -\mathrm{h} \\ (\mathbf{s}';\ \mathbf{r}'\times\mathbf{s}'),\ \mathbf{s}\cdot(\mathbf{r}'\times\mathbf{s}')+\mathbf{s}'\cdot(\mathrm{r}\times\mathbf{s}+\mathrm{h}\mathbf{s})\neq 0 \\ (\mathbf{s}';\ \mathbf{r}'\times\mathbf{s}'+\mathrm{H}\mathbf{s}'),\ \mathbf{s}\cdot(\mathbf{r}'\times\mathbf{s}'+\mathrm{H}\mathbf{s}')+\mathbf{s}'\cdot(\mathbf{r}\times\mathbf{s}+\mathrm{h}\mathbf{s})\neq 0 \end{array} \tag{2.8}$$

When a body bears a reciprocal screw, the motion of the body should satisfy Eq. 2.3. This reciprocal screw, $\$^r=(\mathbf{s}^r;\ \bullet\mathbf{s}_0^r)$ acting on a body is a force where the pitch is zero when $\mathbf{s}^r \bullet \mathbf{s}_0^r = 0$. There are five degrees of freedom (DOF) including two translational DOFs, which directions are orthogonal to the force and three rotational DOFs, whose axes intersect the force. On the other hand, those lines not intersecting with the force cannot be selected as the axis of rotation. When a body has two reciprocal screws with zero pitch, that is, two forces, the body has only four DOFs. There is one translational degree in the direction orthogonal to the two reciprocal forces simultaneously, and three rotational degrees. Each rotational axis intersects with the two acting lines of the two reciprocal forces.

A reciprocal screw, $\$^r(\mathbf{s};\ \mathbf{s}_0)$, acts on a body and the pitch does not equal zero ($h^r\neq 0$) if $\boldsymbol{S}\bullet\boldsymbol{s}_0\neq 0$. The body still has five DOFs corresponding to five linearly independent motions. The twist colinear with the reciprocal screw may have the pitch negative sign to the pitch of the reciprocal. The allowed twist that is orthogonal to the axis of the reciprocal may have any pitch. It is desirable to know if the allowed movement is pure rotation or pure translation, and to be used to decompose a complex motion into several translational and rotational motions. Therefore, the allowed translational and rotational motions should be found when an object is under a reciprocal screw with pitch, $h^r\neq 0$.

When a body is under a reciprocal screw with $h^r\neq 0$, we put the coordinate frame on the body such that the Y-axis of the frame coincides with the reciprocal, as shown in Figure 2. The reciprocal screw takes the form:

$$\$^r = (\mathbf{s};\ \mathrm{h}^r\mathbf{s}) = (0\ 1\ 0;\ 0\ \mathrm{h}^r\ 0) \tag{2.9}$$

The axis, $\$^m$, of the rotational motion of the body has an angle α with the Y-axis and is located at point a. Rearranging Eq. 2.4 we have:

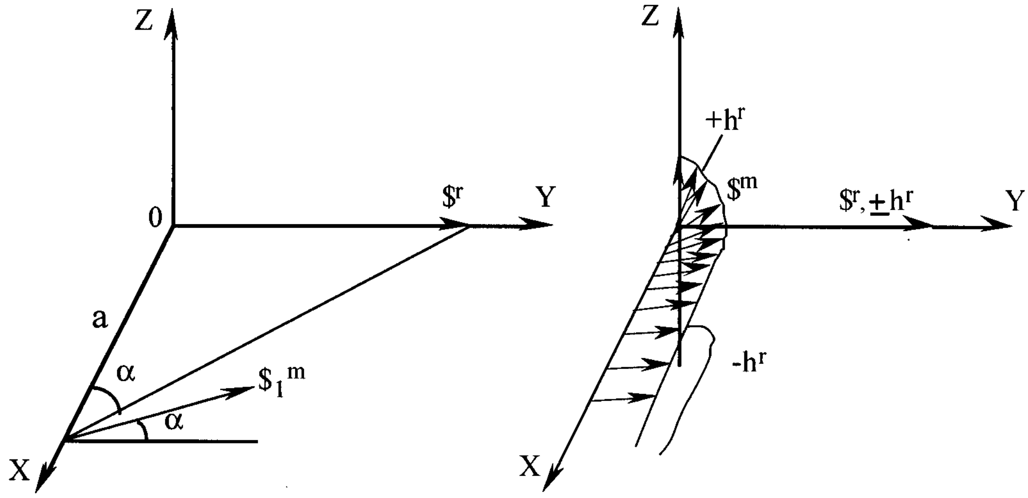


Fig. 2. Distribution of moving screws.

$$\frac{h^r}{a} = \tan \alpha \tag{2.10}$$

From Eq. 2.10, we see that when the reciprocal screw with pitch $h^r \neq 0$, acts on the body, the allowed rotational motion is skew to the reciprocal in space. The angle, α, between the reciprocal and the axis of the motion, $\$_1$, gradually becomes smaller when the perpendicular distance, a, becomes longer. When the distance, a, is zero, two lines intersect and $\alpha = 90°$. When the distance, a, is infinite ($a = \infty$), they are parallel. The function, $\alpha(a)$, is shown in Figure 2b, where the upper branch is for $h^r >$, and the lower branch for $h^r < 0$.

3. EXTENDED APPLICABLE SUBWORKSPACE OF ROBOT MANIPULATOR IN SINGULARITY

As mentioned before, there is an applicable subworkspace when a manipulator is in a singular configuration. We take the six DOF robot PUMA 560 as an example to analyze the applicable subworkspace. The PUMA robot, which has been studied extensively, is shown in Figure 3 in the D-H approach. The third local moving reference system here is selected as the mean system to set the Jacobian matrix of the manipulator. That is the z_3 axis is coincident with the axis of the third revolute joint, and x_3 is along the common normal, a_{34}, of axis 3 and 4. Those six original points of the six local reference systems are denoted by $0_1 0_2 0_3 \ldots 0_6$. Then those six axes of the robot are expressed by screws $\$_1\$_2\$_3 \ldots \$_6$ ($\$_i = (\mathbf{s}_i;\ \mathbf{s}_{0i})$), where $\mathbf{s}_i$ is the axis direction vector of the i^{th} revolute joint, and $\mathbf{s}_{0i} = \mathbf{r}_i \times \mathbf{s}_i$, where $\mathbf{r}_i$ is the position vector of one point in line vector $\mathbf{s}$. Also note that $\mathbf{s}_i \cdot \mathbf{s}_i = 1$, and $\mathbf{s}_i \cdot \mathbf{s}_{0i} = 0$. The Jacobian matrix of PUMA 560 is:[4]

$$[J] = \begin{bmatrix} -s_{2+3} & 0 & 0 & 0 & s_4 & -c_4 s_5 \\ -c_{2+3} & 0 & 0 & 1 & 0 & c_5 \\ 0 & 1 & 1 & 0 & c_4 & s_4 s_5 \\ -s_{22} c_{2+3} & a_{23} s_3 & 0 & 0 & s_{44} c_4 & s_{44} s_4 s_5 \\ s_{22} s_{2+3} & a_{23} c_3 & 0 & 0 & 0 & 0 \\ a_{23} c_2 & 0 & 0 & 0 & -s_{44} s_4 & s_{44} c_4 s_5 \end{bmatrix} \tag{3.1}$$

where s_i and c_i indicate $\sin \theta_i$ and $\cos \theta_i$, and s_{i+j} and c_{i+j} indicate $\sin(\theta_i + \theta_j)$ and $\cos(\theta_i + \theta_j)$, respectively. θ_i and θ_j are revolute angles about z_i and z_j, respectively. s_{ij} indicates the offset distance along the z_i axis. a_{ij} is the length of the common normal between axes z_i and z_j. In general, the determinant of the Jacobian matrix is not zero, therefore, the rank of the Jacobian matrix is six and the robot's hand has six DOFs, defined by three translations and three rotations. In this case, the rotational axes can be along any line in the three dimensional space.

When a robot is in a singular configuration the determinant of the Jacobian matrix is zero. As a result, the DOF of the end effector is reduced, and the kinematic and dynamic characteristics worsen. Such a configuration is normally avoided, having become unavailable for planning purposes. Using the method presented above, some extra subwork-

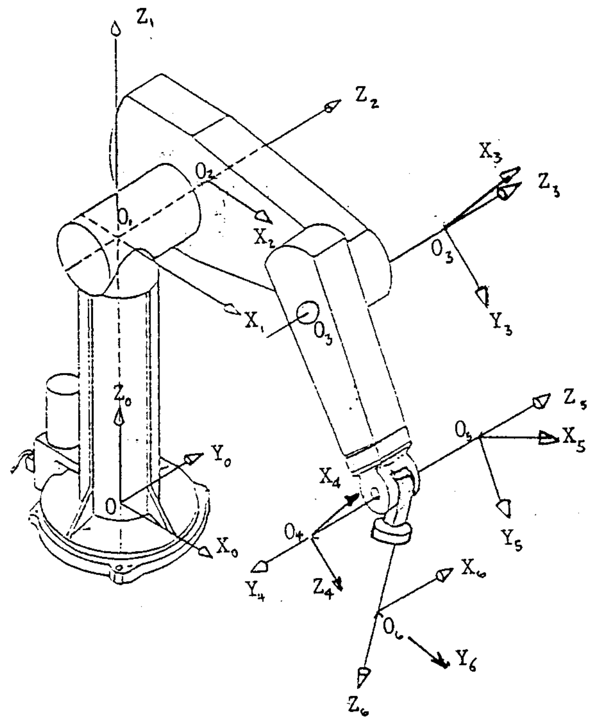


Fig. 3. PUMA 560 and its reference systems.

space may be reclaimed where the end effector may work normally. This effectively extends the applicable workspace of the robot. Let the determinant of the Jacobian matrix equal to zero:

$$\det[J] = a_{23}\, s_{44}\, c_3\, s_5 (a_{23}\, c_2 - s_{44}\, s_{2+3}) = 0 \qquad (3.2)$$

three conditions of special configuration are found, that is:

$$\begin{aligned} c_3 &= 0 \\ s_5 &= 0 \\ a_{23}c_2 - s_{44}s_{2+3} &= 0 \end{aligned} \qquad (3.3)$$

The three conditions indicate that when the forearm and reararm stretch out and coaxis, the fourth and the sixth axes are colinear, and the reference point on the wrist located on a plane determined by axis 1 and 2, respectively. Under one of the three conditions, the Jacobian matrix is singular, and the rank of the Jacobian matrix, r, is less than six. Reciprocal screws reciprocal to those screw system ($\$_1\$_2\$_3 \ldots \$_6$) exist. Suppose that the unit reciprocal screw is denoted as

$$\begin{aligned} \$_r &= (a_1, a_2, a_3; a_4, a_5, a_6) \\ a_1^2 + a_2^2 + a_3^2 &= 1 \end{aligned} \qquad (3.4)$$

Substituting Eq. 3.4 into Eq. 2.4, that is

$$\begin{aligned} & -a_1 s_{22} c_{2+3} + a_2 s_{22} s_{2+3} + a_3 a_{23} c_2 - a_4 s_{2+3} - a_5 c_{2+3} = 0 \\ & a_1 a_{23} s_3 + a_2 a_{23} a_3 + a_6 = 0 \\ & a_6 = 0 \\ & a_5 = 0 \\ & a_1 s_{44} c_4 - a_3 s_{44} s_4 + a_4 s_4 + a_6 c_4 = 0 \\ & a_1 s_{44} s_4 s_5 - a_3 s_{44} c_4 s_5 - a_4 c_4 s_5 + a_5 c_5 + a_6 s_4 s_5 = 0 \end{aligned} \qquad (3.5)$$

As the six screws $\$_1\$_2 \ldots \$_6$, are linearly dependent with the rank equal to five, the first five equations in Eq. 3.5 can be used to solve the unknown reciprocal screw. The result is:

$$\begin{aligned} a_1 &= \frac{(a_{23}c_2 - s_{44}s_{2+3})c_3 s_4}{D} \\ a_2 &= \frac{(s_{44}s_{2+3} - a_{23}c_2)s_3 s_4}{D} \\ a_3 &= \frac{s_{22}s_4 x_2 - s_{44}s_{2+3}c_3 c_4}{D} \\ a_4 &= \frac{s_{44}c_2(s_{22}s_4 - c_4 c_3 a_{23})}{D} \\ a_5 &= 0 \\ a_6 &= 0 \end{aligned} \qquad (3.6)$$

where

$$D = \sqrt{S_4^2(a_{23}c_2 - s_{44}s_{2+3})^2 + (s_{44}s_{2+3}c_3c_4 - s_{22}s_4c_2)^2} \qquad (3.7)$$

and the pitch of the reciprocal screw is:

$$h = (s \bullet s_0)/(s \bullet s) = (a_{23}c_2 - s_{44}s_{2+3})c_3 s_4 s_{44} c_2 (s_{22}s_4 - c_4 c_3 a_{23})/D^2 \qquad (3.8)$$

In Eq. 3.8, the pitch of the reciprocal screw will be zero, $h^r = 0$, if one of the following five conditions are satisfied:

$$\begin{aligned} c_2 &= 0 \\ c_3 &= 0 \\ s_4 &= 0 \\ a_{23}c_2 - s_{44}s_{2+3} &= 0 \\ s_{22}s_4 - c_4c_3a_{23} &= 0 \end{aligned} \qquad (3.9)$$

Comparing Eq. 3.3 and Eq. 3.9, we find that two of the five conditions are the same as the singularity condition. Thus, when we consider the three singular conditions and the five pitch-vanishing conditions, there are twenty-one combinations, as shown in Table I.

Among the twenty-one combinations, there are only ten different independent configurations related to the reciprocal screws and the applicable subworkspace of PUMA. They are discussed below.

Case 1: In the case of $a_{23}c_2 - s_{44}s_{2+3} = 0$, as shown in Fig. 4a, the geometric characteristic is that the reference point, 0_4, of the robot wrist sets on plane A determined by $\$_1$ and $\$_2$. The mechanism is in a special configuration, and the rank of the Jacobian matrix is five. There is one reciprocal screw, $\$^r$, with pitch equal to zero, and this reciprocal screw's axis intersects all axes of the six revolving joints. The reciprocal, $\$^r$, lies on an intersection line, a–a, of two planes, which are plane A and another plane determined by point 0_4 and axis $\$_3$. In this case, the allowed motion including two translations orthogonal to a–a, and three rotations which extend the workspace. The extended subspace considering only rotational motion can be represented by the possible rotational axes, which are all lines

Table I Two-condition combinations

	Conditions	0	I	II	III	IV	V
I	$\mathbf{s}_5 = 0$	Case 10					
II	$a_{23}c_2 - \mathbf{s}_{44}\mathbf{s}_{2+3} = 0$	Case 1	Case 1				
III	$c_2 = 0$	Case 2	Case 2	Case 8			
IV	$\mathbf{s}_4 = 0$	----	Case 3	Case 1	Case 2		
V	$\mathbf{s}_{22}\mathbf{s}_4 - c_4c_3a_{23} = 0$	----	Case 5	Case 1	Case 2	----	
VI	$c_2 = 0$	----	Case 6	Case 8	Case 8	----	----

Multi-condition combinations: (1) I III IV - Case 4
(2) I IV VI - Case 7
(3) I III IV VI - Case 9

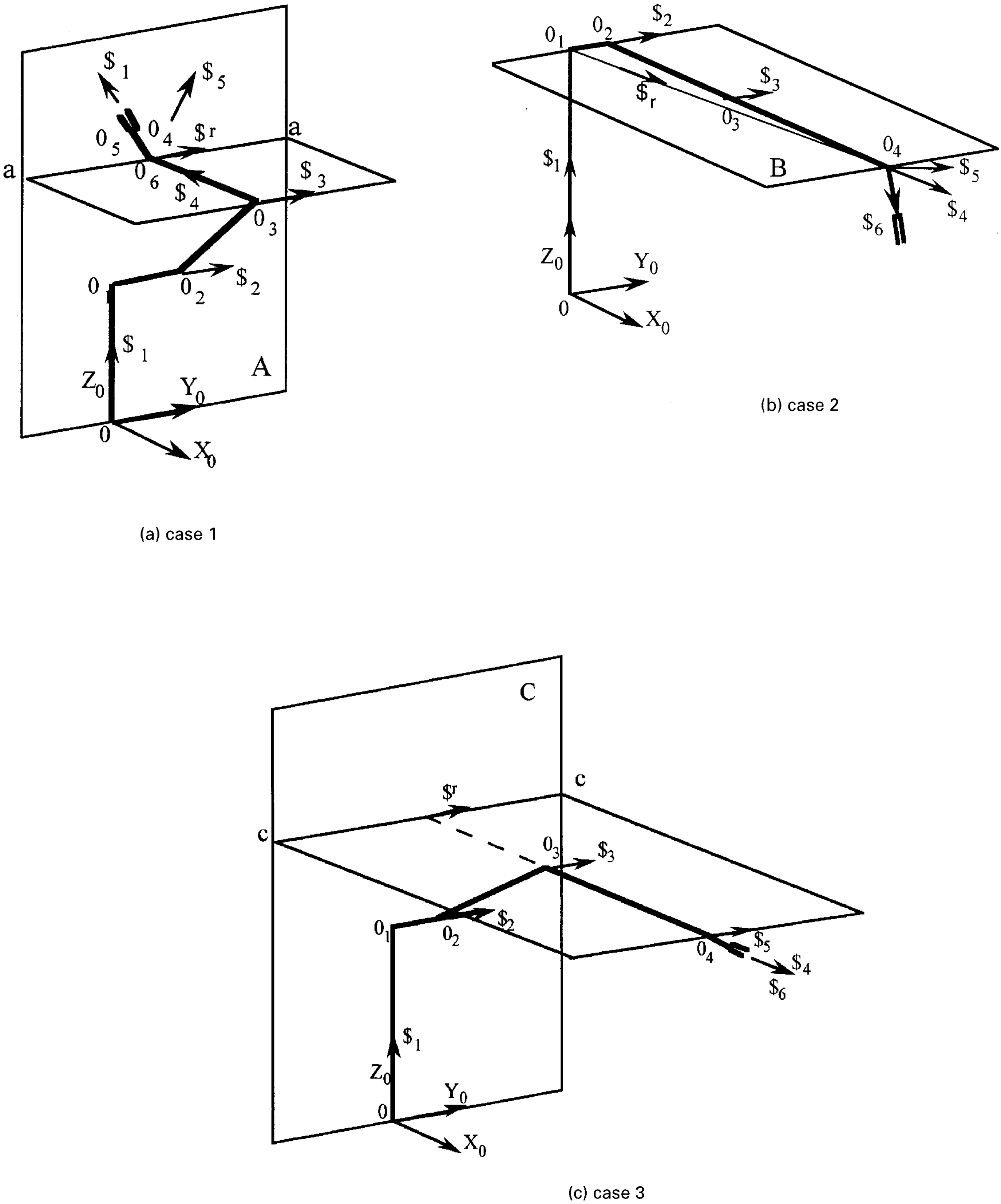
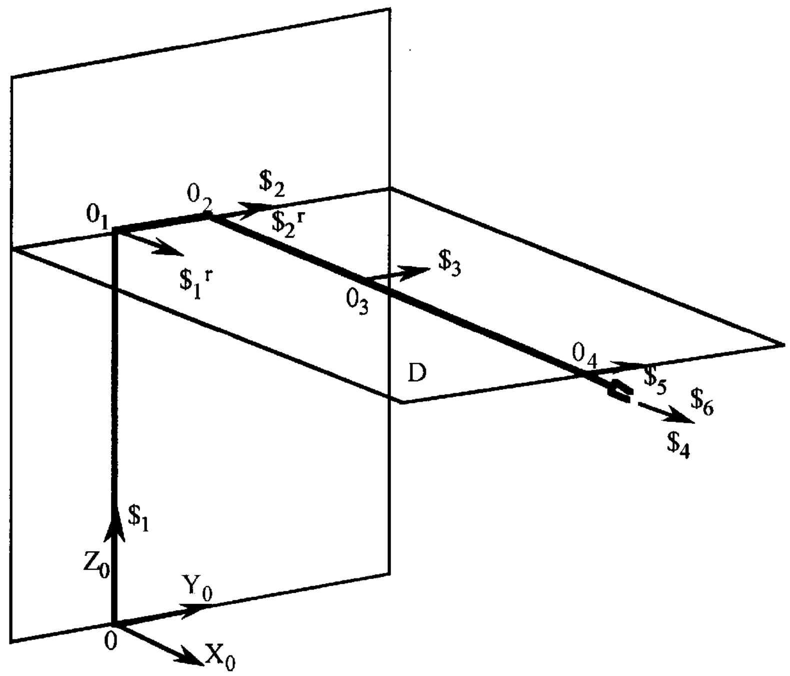
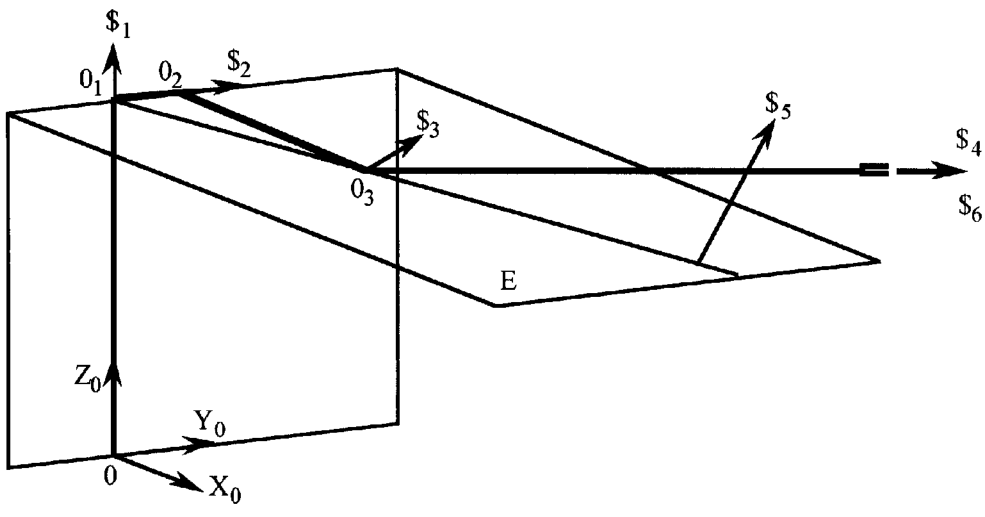


Fig. 4. Ten independent configurations related to the applicable subworkspaces of PUMA 560.

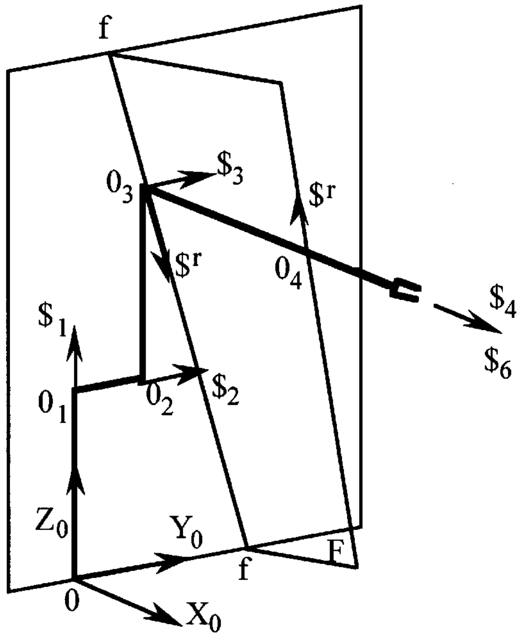


(d) case 4

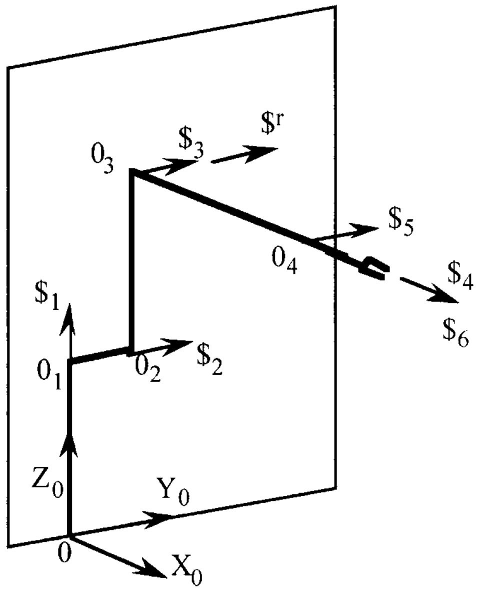


(e) case 5

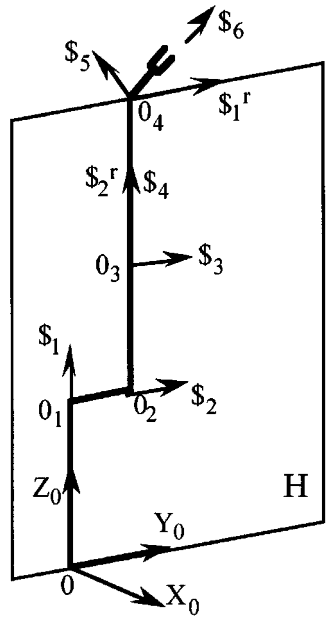
Fig. 4. Continued.



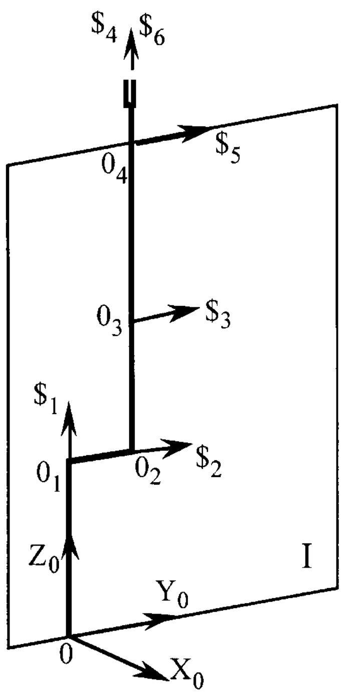
(f) case 6



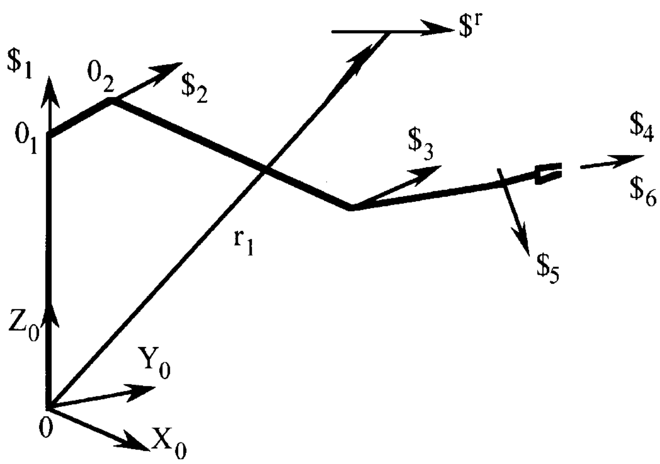
(g) case 7



(h) case 8



(i) case 9



(j) case 10

Fig. 4. Continued.

intersecting line a–a and expressed as:

$$\delta_1 = \{\text{lines} \mid \text{intersecting with line a–a}\} \qquad (3.10)$$

Case 2: In the case of $c_3 = 0$, as shown in Fig. 4b, the forearm and reararm stretch out and are coaxial. The rank of the Jacobian matrix is five. A reciprocal screw with zero pitch lies on a line determined by 0_1 and 0_4. The allowed motions include two translations orthogonal to the line $0_1 0_4$ and three revolutions whose axes are in an extended subspace expressed as:

$$\delta_2 = \{\text{lines} \mid \text{intersection with line } 0_1 0_4\} \qquad (3.11)$$

Case 3: In the case of $s_4 = 0$ and $s_5 = 0$, as shown in Fig. 4c, Axis 3 is parallel with Axis 5. Axes 4 and 6 are coaxial. The rank of the Jacobian matrix is five. A reciprocal screw, $\$^r$, with pitch $h^r = 0$, lies on a line passing through the intersecting line c–c of plane C and a plane determined by two axes, $\$_3$ and $\$_4$. The extended subspace, δ_3 of all allowed axes, which the body rotates about, is:

$$\delta_3 = \{\text{lines} \mid \text{intersecting line c–c}\} \qquad (3.12)$$

Case 4: In the case of $c_3 = 0$, $s_4 = 0$, and $s_5 = 0$, as shown in Fig. 4d, the forearm, reararm, and the effector bar are colinear and in the same direction. Axes 3 and 5 are parallel, and the five screws, $\$_2 \$_3 \ldots \$_6$, are in the same plane, D. The screws are linear dependent and the rank of the Jacobian matrix is four. Therefore, there are two reciprocal screws, $\$^r$ and $\$^{r2}$, reciprocal to all the six screws and intersecting those six axes. The two reciprocal screws are both in plane D and pass through point O_1. Thus the allowed motions are: one translation along the direction orthogonal to those two reciprocals and three rotations. The possible rotational axes intersects both reciprocal screws simultaneously. Those rotational axes can be along any line in plane D or lines not in plane D but passing through point O_1. Thus the extended subspace, δ_4, of allowed axes which the body rotates about is:

$$\delta_4 = \{\text{lines} \mid \text{in plane D} \cup \text{in space and pass through point } 0_1\} \qquad (3.13)$$

Case 5: In the case of $s_5 = 0$ and $s_{22} s_4 - c_4 c_{323} = 0$, as shown in Fig. 4e, axes 4 and 6 are coaxial and axis 5 intersects line $0_1 0_3$. The rank of the Jacobian matrix is five and a reciprocal screw with $h = 0$ exists acting along line $0_1 0_3$. Thus the extended subspace, δ_5, of allowed axes which the body rotates about is:

$$\delta_5 = \{\text{lines} \mid \text{intersecting line } 0_1 0_3\} \qquad (14)$$

Case 6: In the case of $c_2 = 0$ and $s_5 = 0$, as shown in Fig. 4f, the reararm is in a vertical position, and axes 4 and 6 are coaxial. The rank of the Jacobian matrix is five. One reciprocal screw lies in line f–f, the intersection of plane F,and a plane determined by point 0_3 and axis $\$_5$. The extended subspace is:

$$\delta_6 = \{\text{lines} \mid \text{intersecting line f–f}\} \qquad (3.15)$$

Case 7: In the case of $c_2 = 0$, $s_4 = 0$, as shown in Fig. 4g, the reararm is in a vertical position, axes 4 and 6 are coaxial and axes 3 and 5 are parallel. The rank of the Jacobian matrix is five. The reciprocal screw with zero pitch, $\$^r$, is colinear with axis 3. Thus the extended subspace is:

$$\delta_7 = \{\text{lines} \mid \text{intersecting axis line 3}\} \qquad (3.16)$$

Case 8: In the case of $c_3 = 0$ and $a_{23} c_2 - s_{44} s_{2+3} = 0$, as shown in Fig. 4h, the forearm and the reararm stretch out and are in vertical positions. The rank of the Jacobian matrix is four. There are two reciprocal screws, $\$^r_1$ and $\$^r_2$, lying in plane H and passing through point 0_4. The end effector has four degrees of freedom: one translation perpendicular to both $\$^r_1$ and $\$^r_2$ and three rotations. The rotational axes are allowed to select in the subspace, which is the extended subspace of:

$$\delta_8 = \{\text{lines} \mid \text{in plane H} \subset \text{in space and pass through point } 0_4\} \qquad (3.17)$$

Case 9: In the case of $c_2 = c_3 = s_4 = s_5 = 0$, as shown in Fig. 4i, the entire arm is stretched out and in a vertical position. Axes 3 and 5 are parallel, and six axes lines are in the same plane I. The rank of the screw system is only three. There are three reciprocal screws. Two of them are forces with $h = 0$, lying on plane I. Another is a couple with $h = \infty$, and perpendicular to plane I. The allowed motions under constraints are one translation orthogonal to plane I, and two rotations about the axes located inside plane I. Thus the subspace of the rotational axes is:

$$\delta_9 = \{\text{lines} \mid \text{inside plane I}\} \qquad (3.18)$$

Case 10: In the case of only $s_5 = 0$, as shown in Fig. 4j, the only geometrical condition is that the axes 4 and 6 are colinear. The rank of the Jacobian is five and there exists a reciprocal screw. The pitch of the reciprocal is not equal to zero, $h \neq 0$. This screw, $\$^r$, and its pitch, h^r, can both be obtained using Eq. 3.6 and Eq. 3.8. The position of the acting line of the reciprocal screw can be expressed by the line vector of the reciprocal screw. From Eq. 3.6, the reciprocal is denoted as:

$$\$^r = (\mathbf{s}^r; \mathbf{s}^r_0) = (a, b, c; d, 0, 0) \qquad (3.19)$$

The acting line of the reciprocal is:

$$\$_1 = (\mathbf{s}^r; \mathbf{s}^r_0 - h^r \mathbf{s}^r) \qquad (3.20)$$

Therefore, we have:

$$\mathbf{r}_1 \times \mathbf{s}^r = \mathbf{s}^r_0 - h^r \mathbf{s}^r \qquad (3.21)$$

From Eq. 3.21, the position vector r_1 is obtained. The constrained motions are denoted in Eq. 2.8. The allowed rotations and translations are distributed in three dimension space around the reciprocal, as shown in Fig. 2. The translations are in the directions orthogonal and intersecting to the reciprocal. The rotational motions are all skew to the reciprocal. The skew angle, α, is determined by the pitch of the reciprocal and the distance to the reciprocal of Eq. 2.10. The subspace of the rotational axes is:

$$\delta_{10} = \{\text{lines} \mid \text{satisfying Eq. 2.10}\} \qquad (3.22)$$

As there exist the above mentioned ten subworkspaces including the translational and the rotational, δ_1, δ_2, ... , δ_{10}, the workspace is effectively extended which may be useful when carrying out trajectory planing in the task space.

4. CONCLUSION

When an object has six DOF, its three-dimensional rotation has sufficient freedom. Any line in space including three coordinate axes may be chosen as a rotational axis for the body. When its Jacobian matrix is singular, the robot is in a special configuration. Its end effector often loses some translational degrees of freedom, with the reciprocal screws of the motions screw system of the serial kinematic chain as constraining forces. If so, the rotational motion may also be partially restricted. The rotating axis can be chosen so that it intersects with all the constraint force vectors acting on the same body. Under singularity the robot hand still has some utilizable rotational freedom and the subspace the rotational axis may be selected from.

PUMA 560 has three different conditions of special configuration and five conditions of screw-pitch-vanishing. By combination, there are ten different forms of reciprocal screws and ten different subworkspaces, including translational DOFs and rotational DOF subspaces. The ten subworkspaces effectively extend the usable workspace of the robot manipulator. As a result, trajectory planning in the task space may still proceed at singular points.

ACKNOWLEDGMENT

Research assistance by Mr. W.B. Zu during this work is gratefully acknowledged. Assistance in preparing this manuscript by Mr. K. Chock is also greatly appreciated.

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