Robotica (1999) volume 17, pp. 475–485. Printed in the United Kingdom © 1999 Cambridge University Press

A new closed-form kinematics of the generalized 3-DOF spherical parallel manipulator

Zhen Huang and Y. Lawrence Yao

Department of Mechanical Engineering, Columbia University, 220 S. W. Mudd Bldg. New York, NY 10027 (USA)
E-mail: yly1@columbia.edu

(Received in Final Form: February 24, 1999)

SUMMARY
This paper presents a new method to analyze the closed-form kinematics of a generalized three-degree-of-a-freedom spherical parallel manipulator. Using this analytical method, concise and uniform solutions are achieved. Two special forms of the three-degree-of-freedom spherical parallel manipulator, i.e. right-angle type and a decoupled type, are also studied and their unique and interesting properties are investigated, followed by a numerical example.

KEYWORDS: Closed-form kinematics; Spherical parallel manipulator; 3-DOF; Two types.

1. INTRODUCTION

A parallel manipulator consists of a moving platform, a basis platform, and several branches connecting both platforms through appropriate kinematic joints with actuators. Compared with the more commonly used serial manipulators, the parallel one has attractive advantages in accuracy, rigidity, capacity, and load-to-weight ratio. Six-DOF spatial parallel manipulators have been widely discussed. 3-DOF spherical parallel manipulator (SPM), however, can serve as a compact orientation unit with high stiffness. Potential usage includes as a robotic wrist, or as a mechanism for orientation of machine tool beds and workpieces, solar panels, radar antennas, telescopes, and artificial hips in biomedical engineering.

SPM has attracted the attention of researchers over the past decade. Asada and Cro Grauito in 1985 suggested to use it as a robot wrist.[1] Using it, Cox and Tesar[2] designed a robot shoulder; Kim and Tesar[3] realized a force reflecting manual controller; Yi, Freeman, and Tesar[4] studied the redundantly actuated mechanisms; Gosselin and Angeles[5–7] investigated the optimal kinematic parameters, singularity and incompletely specified tasks; Gosselin and Lavoie[8] gave a kinematic design. The closed-form kinematics of a 3-DOF SPM is relatively more complex than that of a 3-DOF serial manipulator. It is interesting to note that more work has been published on 6-DOF spatial parallel manipulators than 3-DOF SPM. Gosselin, Sefrioni, and Richard[9–10] studied the kinematics of several 3-DOF SPM's. This paper presents the kinematics of a more generalized SPM, using spherical analytical theory[11] and the more concise and uniform solution. This paper also analyzes two special types of 3-DOF SPMs which are of the right-angle type and of the decoupled type, using the screw theory[12] and the existent principle of rotatable axis.[13,14] Their special properties are investigated and illustrated further through numerical examples.

2. STRUCTURE, COORDINATE SYSTEM, AND GEOMETRY

A 3-DOF SPM consists of a fixed pyramid, $Ob_1b_2b_3$, a moving pyramid, $Om_1m_2m_3$, and three spherical serial chains, $b_ic_im_i$ $(i=1,2,3)$ as shown in Figure 1. Each of the chains in turn consists of two links, b_ic_i and c_im_i, and three revolute joints b_i, c_i, and m_i. All the axes of revolute pairs intersect at a common point O, and the nine points b_1 b_2 b_3, m_1 m_2 m_3, c_1 c_2 c_3 are all on a unit spherical surface. In general, both triangles, $b_1b_2b_3$ and $m_1m_2m_3$, are non-equilateral. The angles between adjacent joint axes in pyramids and in chains are not equal to each other, that is, $\gamma_{11}\neq\gamma_{12}\neq\gamma_{13}$, $\gamma_{21}\neq\gamma_{22}\neq\gamma_{23}$, and $\alpha_{i1}\neq\alpha_{i2}$ $(i=1,2,3)$ where

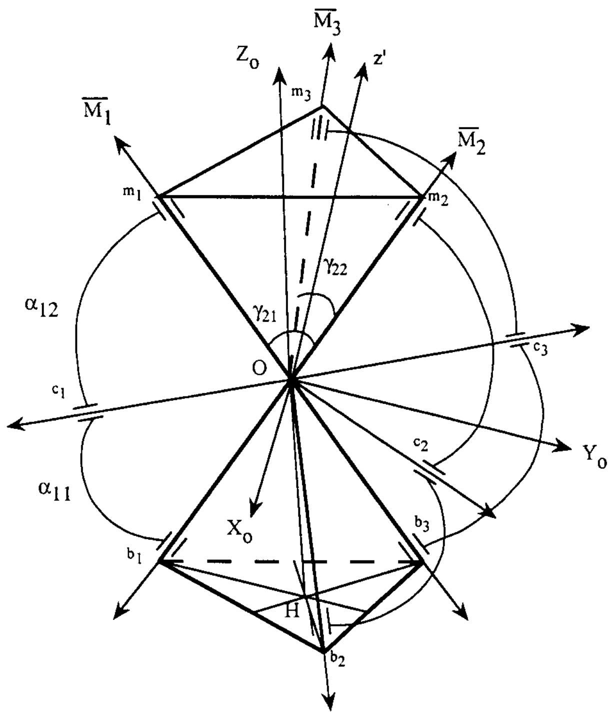


Fig. 1. A 3-DOF Spherical Parallel Manipulator.

γ_{11}, γ_{12}, and γ_{13} are the angles between Ob_1 and Ob_2, Ob_2 and Ob_3, and Ob_3 and Ob_1, respectively, γ_{21}, γ_{22} and γ_{23} the angles betweem Om_1 and Om_2, Om_2 and Om_3, and Om_3 and Om_1, respectively α_{i1} the angle between Ob_i and Oc_i, and α_{i2} the angle between Oc_i and Om_i, as shown in Figure 2. This represents a very general case. The 3-DOF spherical mechanism has three independent actuators, which are attached to the fixed pyramid and actuate the first link, b_ic_i, of each branch by an input angle, θ_i. Once three input angles, θ_1, θ_2, and θ_3, are given, the orientation of the moving pyramid is determined. Thus, this is a spherical mechanism, i.e., any point on the moving pyramid or on a link of a chain moves on a spherical surface. In addition, any link in this mechanism can only have relative rotational motion and no translational motion is permitted.

The origin O of a fixed global coordinate system, $Ox_0y_0z_0$, is located at the common intersection point of the axes of the revolute joints, passing through the edges of the pyramids. The z_0-axis is chosen to pass through the centroid H of the bottom triangle $b_1b_2b_3$. The x_0-axis is normal to the plane defined by vectors Oz_0 and Ob_1. The moving coordinate system, Oxyz, is attached to the moving pyramid. Its z-axis passes through the centroid of the upper triangle $m_1m_2m_3$. It's y-axis is inside the plane defined by vector Oz and Om_2 and its x-axis is determined by the right-hand rule.

From spherical trigonometry, three sides, γ_{11}, γ_{12}, and γ_{13} of a spherical triangle $b_1b_2b_3$ are given in Figure 2. The exterior angle of this triangle can be obtained by the cosine law, as

$$\cos b_i = \frac{\cos\gamma_{1i}\cos\gamma_{1k} - \cos\gamma_{1j}}{\sin\gamma_{1i}\sin\gamma_{1k}} \tag{1}$$

where subscripts i, j and k represent numbers 1, 2 and 3 in a cyclic manner. The corresponding interior angle is equal to π-b_i, respectively.

For the spherical triangle $b_1b_2F_2$, sides b_1b_2 and b_2F_2 are γ_{11} and $\gamma_{12}/2$, respectively. The interior angles b_1 and F_2 of the triangle $b_1b_2F_2$ can be obtained using Napier's formula

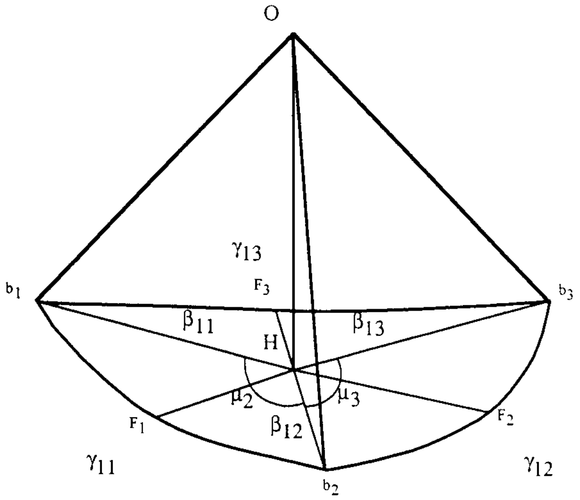


Fig. 2. A Spherical Pyramid.

$$\tan\frac{b_1+F_2}{2} = \frac{\cos\left(\dfrac{\gamma_{11}-\dfrac{\gamma_{12}}{2}}{2}\right)}{\cos\left(\dfrac{\gamma_{11}+\dfrac{\gamma_{12}}{2}}{2}\right)}\cot\left(\frac{b_2}{2}\right)$$

$$\tan\frac{b_1-F_2}{2} = \frac{\sin\left(\dfrac{\gamma_{11}-\dfrac{\gamma_{12}}{2}}{2}\right)}{\sin\left(\dfrac{\gamma_{11}+\dfrac{\gamma_{12}}{2}}{2}\right)}\cot\left(\frac{b_2}{2}\right) \tag{2}$$

The interior angle b_3 of triangle $b_2b_3F_1$, interior angle b_2 of triangle $b_1b_2F_3$, and interior angle b_2 of triangle $b_2b_3F_3$ can be solved similarly. The arcs, b_1H, b_2H, and B_3H are denoted by β_{11}, β_{12}, β_{13}, respectively, and can be solved by Napier's rule as well

$$\tan\frac{\beta_{11}+\beta_{12}}{2} = \frac{\cos\left(\dfrac{b_1-b_2}{2}\right)}{\cos\left(\dfrac{b_1+b_2}{2}\right)}\tan\left(\frac{\gamma_{11}}{2}\right)$$

$$\tan\frac{\beta_{11}-\beta_{12}}{2} = \frac{\sin\left(\dfrac{b_1-b_2}{2}\right)}{\sin\left(\dfrac{b_1+b_2}{2}\right)}\tan\left(\frac{\gamma_{11}}{2}\right) \tag{3}$$

The interior angle, μ_2, of triangle b_1b_2H can also be written as

$$\tan\frac{\mu_2}{2} = \frac{\cos\frac{1}{2}(\beta_{11}-\beta_{12})}{\cos\frac{1}{2}(\beta_{11}+\beta_{12})}\cot\frac{1}{2}(b_1+b_2) \tag{4}$$

The geometric derivation of the moving pyramid can be carried out in a similar manner.

The general formulations above can be simpliﬁced for special cases:

- In the case of $\gamma_{11}=\gamma_{12}=\gamma_{13}=\gamma_1$, and $\gamma_{21}=\gamma_{22}=\gamma_{23}=\gamma_2$, two tragopans become regular, and three exterior angles of the bottom triangle are equal, e.g., $b_1=b_2=b_3$. In addition, $\beta_{11}=\beta_{12}=\beta_{13}=\beta_1$, therefore

$$\sin\beta_1 = \frac{2\sqrt{3}}{3}\sin\frac{\gamma_1}{2} \tag{5}$$

- In the case of $\gamma_1=\gamma_2=\dfrac{\pi}{2}$, $\sin\beta_1=\sqrt{\dfrac{2}{3}}$.

- In the case of $\gamma_1 = \gamma_2 = \frac{2\pi}{3}$, the upper and lower pyramids become two planes and thre revolute joints of the same pyramid become coplanar, and thus $\sin\beta = 1$.
- In the case of $\gamma_1 = 0$, $\gamma_2 = \frac{2\pi}{3}$, tt becomes Asada's 3-DOF spherical wrist[1] with $\beta_1 = 0$ and $\beta_2 = \frac{\pi}{2}$.
- In the case of $\gamma_1 = \frac{\pi}{2}$, $\gamma_{21} = 0$, and $\gamma_{22} = \gamma_{23} = \frac{\pi}{2}$, it becomes an interesting case of a decoupled 3-DOF spherical parallel manipulator.

Some of the special cases exhibit unique and interesting characteristics which will be discussed in more details after the formulation for the general case is presented.

3. THE CLOSED-FORM INVERSE KINEMATICS

The inverse kinematics of 3-DOF spherical parallel manipulators is to solve the three unknown input angles, θ_1, θ_2, and θ_3 while the orientation of the moving platform is given. The orientation of the mobile platform is expressed by direction cosines of its three unit edge vector, $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$.

3.1 Direction cosines of vectors $\overline{C}_1$, $\overline{C}_2$, $\overline{C}_3$

In order to determine the direction cosines of the above mentioned unit vectors, the Duffy's spherical analytical theory is applied here, including both notations and formulas.[11] First, vectors $\overline{C}_1$, $\overline{C}_2$, $\overline{C}_3$ are expressed as (Figure 3).

3.1.1 Vector $\overline{C}_1$, $(C_{1x}, C_{1y}, C_{1z})^T$. Vector C_1 is in the first link of the first branch of the SPM as shown in Figure 3. Three points c_1, B_1, z_0, in the spherical surface form a spherical triangle. The direction cosine of vector $\overline{C}_1$ can be written as

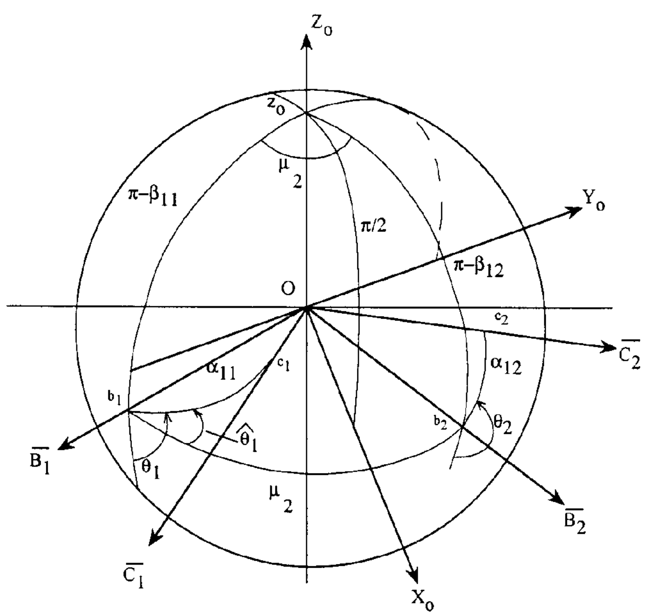


Fig. 3. Vectors $\overline{C}_1$ and $\overline{C}_2$.

$$
\begin{aligned}
C_{1x} &= \overline{X}_3 = s_{\alpha11}s_{\theta1} \\
C_{1y} &= \overline{Y}_3 = -(s_{\pi-\beta11}c_{\alpha11} + c_{\pi-\beta11}s_{\alpha11}c_{\theta1}) \\
&= c_{\beta11}s_{\alpha_{11}}c_{\theta_1} - s_{\beta_{11}}c_{\alpha_{11}} \\
C_{1z} &= \overline{Z}_3 = c_{\pi-\beta11}c_{\alpha11} - s_{\pi-\beta11}s_{\alpha11}c_{\theta1} \\
&= -c_{\beta11}c_{\alpha11} - s_{\beta11}s_{\alpha11}c_{\theta1}
\end{aligned} \tag{6}
$$

where angles θ_i, $\hat{\theta}_1$, α_{ij}, and μ_i are illustrated in Figure 3. The notations, $\overline{X}_3$, $\overline{Y}_3$, $\overline{Z}_3$, have special definitions, as seen in reference 11, and $s_{\alpha11} = \sin\alpha_{11}$, $s_{\theta1} = \sin\theta_1$, $c_{\alpha11} = \cos\alpha_{11}$, and $c_{\theta1} = \cos\theta_1$.

3.1.2 Vector $\overline{C}_2$, $(C_{2x}, C_{2y}, C_{2z})^T$. Let us define the local reference frame (x'y'z'), with z'-axis to coincide with z_0, and x'-axis to be perpendicular to the plane defined by Ob_i and Oz_0. Thus. C_{2x}, C_{2y}, and C_{2z} have the same form as that in Equation (6)

$$
\begin{pmatrix} C_{2x'} \\ C_{2y'} \\ C_{2z'} \end{pmatrix} = \begin{pmatrix} s_{\alpha12}s_{\theta2} \\ c_{\beta12}s_{\alpha12}c_{\theta2} - s_{\beta12}c_{\alpha12} \\ -c_{\beta12}c_{\alpha12} - s_{\beta12}s_{\alpha12}c_{\theta2} \end{pmatrix} \tag{7}
$$

The direction cosines of vector $\overline{C}_2$, with respect to the globe system can be found by rotating coordinate frame about z_0-axis for an angle $-\mu_2$, that is, premultiply C_2 by a transformation matrix $[R_{-\mu2}]$, where μ_i is the angle between planes b_iOZ_0 and b_iOZ_0

$$
\begin{pmatrix} C_{2x} \\ C_{2y} \\ C_{2z} \end{pmatrix} = [R_{-\mu2}] \begin{pmatrix} C_{2x'} \\ C_{2y'} \\ C_{2z'} \end{pmatrix}
= \begin{pmatrix} c_{\mu2}s_{\alpha12}s_{\theta2} - s_{\mu2}c_{\beta12}s_{\alpha12}c_{\theta2} + s_{\mu2}s_{\beta12}c_{\alpha12} \\ s_{\mu2}s_{\alpha12}s_{\theta2} + c_{\mu2}c_{\beta12}s_{\alpha12}c_{\theta2} - c_{\mu2}s_{\beta12}c_{\alpha12} \\ -c_{\beta12}c_{\alpha12} - s_{\beta12}s_{\alpha12}v_{\theta2} \end{pmatrix} \tag{8}
$$

3.1.3 Vector $\overline{C}_3$, (C_{3x}, C_{3y}, C_{3z}). The direction cosines of vector $\overline{C}_3$ can easily be written by replacing μ_2 with μ_3. Thus, the direction cosines of any one of the three vectors $\overline{C}_1$, $\overline{C}_2$, and $\overline{C}_3$ can be written in a uniform equation as

$$
\begin{pmatrix} C_{ix} \\ C_{iy} \\ C_{iz} \end{pmatrix} = \begin{pmatrix} c_{\mu i}s_{\alpha1i}s_{\theta i} - s_{\mu i}c_{\beta1i}s_{\alpha1i}c_{\theta i} + s_{\mu i}s_{\beta1i}c_{\alpha1i} \\ s_{\mu i}s_{\alpha1i}s_{\theta i} + c_{\mu i}c_{\beta1i}s_{\alpha1i}c_{\theta i} - c_{\mu i}s_{\beta1i}c_{\alpha1i} \\ -c_{\beta1i}c_{\alpha1i} - s_{\beta1i}s_{\alpha1i}v_{\theta i} \end{pmatrix} \tag{9}
$$

where $i = 1,2,3$ and $\mu_1 = 0$.

3.2 Direction cosines of unit vectors $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$

As vector $\overline{M}_2$ lies in plane YOZ as shown in Figure 4, its direction cosines with respect to moving coordinate frame $(\overline{X}\ \overline{Y}\ \overline{Z})$ are $\overline{M}_2' = (0, \sin\beta_{22}, \cos\beta_{22})$. Exterior angle m_2 of spherical triangle m_1m_2h can be similarly derived using equation (1). Edge m_1y in spherical triangle m_1m_2y is given by the cosine law

$$
\begin{aligned}
\cos(m_1y) &= \cos\gamma_{21}\cos\left(\frac{\pi}{2} - \beta_{22}\right) - \sin\gamma_{21}\sin\left(\frac{\pi}{2} - \beta_{22}\right)\cos m_2 \\
&= \cos\gamma_{21}\sin\beta_{22} - \sin\gamma_{21}\cos\beta_{22}\cos m_2
\end{aligned}
$$

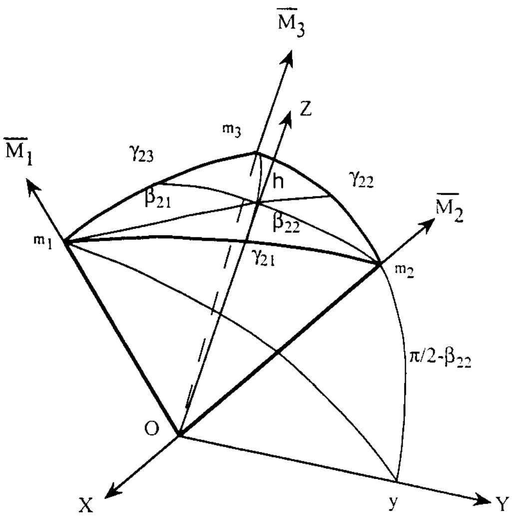


Fig. 4. Direction Cosine of $\bar{M}_1$, $\bar{M}_2$, and $\bar{M}_3$.

Similarly, one has

$$\cos(m_3y) = \cos\gamma_{22}\sin\beta_{22} - \sin\gamma_{22}\cos\beta_{22}\cos m_3$$

The other two direction cosines with respect to a moving frame can be written as

$$\begin{aligned}\bar{M}'_1 &= \left(\sqrt{1-(\cos^2\beta_{21}+\cos^2 m_1y)},\ \cos(m_1y),\ \cos\beta_{21}\right)^T \\ \bar{M}'_3 &= \left(\sqrt{1-(\cos^2\beta_{23}+\cos^2 m_3y)},\ \cos(m_3y),\ \cos\beta_{23}\right)^T\end{aligned} \tag{10}$$

When the orientation of the moving platform is given, the moving frame ($\bar{X}\ \bar{Y}\ \bar{Z}$) is known, the direction cosines of $\bar{M}_1$, $\bar{M}_2$, and $\bar{M}_3$ with respect to fixed frame $X_0Y_0Z_0$ can be written as

$$[\bar{M}_1\ \bar{M}_2\ \bar{M}_3] = [\bar{M}\ \bar{Y}\ \bar{Z}][\bar{M}'_1\ \bar{M}'_2\ \bar{M}'_3] \tag{11}$$

3.3 Inverse kinematics

As mentioned above, the inverse problem solves for the three input angles, θ_1, θ_2, and θ_3, given the orientation of the moving pyramid with respect to the fixed system. These three input angles are involved in equation (9). As there are three unknowns, three equations should be set up. Consider the center links α_{21}, α_{22}, and α_{23} connecting the moving platform and three input links α_{11}, α_{12}, and α_{13}, respectively, three constraint conditions can be written as

$$\bar{M}_i \cdot \bar{C}_i = \cos\alpha_{2i} \tag{12}$$

where $i=1, 2, 3$ and α_{21}, α_{22}, and α_{23} are known constants. Substituting equation (9) and equation (11) into equation (12), one obtains

$$\begin{aligned}\cos\alpha_{21} = {} & (c_{\mu i}s_{\alpha 1i}s_{\theta i} - s_{\mu i}c_{\beta 1i}c_{\theta i} + s_{\mu i}s_{\beta 1i}c_{\alpha 1i})M_ix \\ & + (s_{\mu i}s_{\alpha 1i}s_{\theta i} + c_{\mu i}c_{\beta 1i}s_{\alpha 1i} - c_{\mu i}s_{\beta 1i}c_{\alpha 1i})M_iy \\ & - (c_{\beta 1i}c_{\alpha li} - s_{\beta 1i}s_{\alpha 1i}s_{\theta i})M_iz\end{aligned} \tag{13}$$

where $i=1, 2, 3$. Rearranging equation (13) into the following form

$$A_i\sin\theta_i + B_i\cos\theta_i + C_i = 0 \tag{14}$$

where $i=1, 2, 3$ and

$$\begin{aligned}A_i &= c_{\mu i}s_{\alpha 1i}M_ix + s_{\mu i}s_{\alpha 1i}M_iy \\ B_i &= -s_{\mu i}c_{\beta li}s_{\alpha 1i}M_ix + c_{\mu i}s_{\beta li}s_{\alpha 1i}M_iy - s_{\beta 1i}s_{\alpha li}M_iz \\ C_i &= -s_{\mu i}s_{\beta li}c_{\alpha 1i}M_ix + c_{\mu i}s_{\beta li}c_{\alpha 1i}M_iy - c_{\beta 1i}c_{\alpha li}M_iz - c_{\alpha 2i}\end{aligned} \tag{15}$$

Further let

$$\tan\frac{\theta_i}{2} = x_i,\ \text{then}\ \sin\theta_i = \frac{2x_i}{1+x_i^2},\ \text{and}\ \cos\theta_i = \frac{1-x_i^2}{1+x_i^2} \tag{16}$$

Finally substituting equation (16) into equation (14) the following quadratic equation is arrived at

$$(C_i - B_i)x_i^2 + 2A_ix_i + (C_i + B_i) = 0 \tag{17}$$

therefore

$$x_i = \frac{A_1 \pm \sqrt{A_i^2 + B_i^2 - C_i^2}}{B_i - C_i} \tag{18}$$

where $i=1, 2, 3$.

4. THE CLOSED-FORM FORWARD KINEMATICS

The forward kinematics is to calculate the position and orientation of the output links when the three inputs, θ_1, θ_2, and θ_3 are given. Here the first branch, $b_1c_1m_1$, of the parallel mechanism is shown in Figure 5. Firstly, the direction cosine functions of vectors $\bar{M}_1$, $\bar{M}_2$, and $\bar{M}_3$ with respect to parameters θ_1, α_{11}, φ, α_{12}, and δ are derived.

4.1 Direction cosines of vector $\bar{M}_1$

The direction cosines of vector $\bar{M}_1$ can be directly set up from a spherical quadrilateral, $M_1C_1B_1Z_0$, in terms of Duffy's notation as quadrilateral 4321, as shown in Figure 5.

$$\bar{M}_1 = \begin{pmatrix} x_{32} \\ y_{32} \\ z_{32} \end{pmatrix} = \begin{pmatrix} \bar{x}_3c_2 - \bar{y}_{33}s_2 \\ c_{12}(\bar{x}_3s_2 + \bar{y}_3c_2) - s_{12}\bar{z}_3 \\ s_{12}(\bar{x}_3s_2 + \bar{y}_3c_2) + c_{12}\bar{z}_3 \end{pmatrix} \tag{19}$$

$$\begin{pmatrix} \bar{x}_3 \\ \bar{y}_3 \\ \bar{z}_3 \end{pmatrix} = \begin{pmatrix} s_{\alpha 21}s_{\varphi 1} \\ -(s_{\alpha 11}c_{\alpha 21} + c_{\alpha 11}s_{\alpha 21}c_{\varphi}) \\ c_{\alpha 11}c_{\alpha 21} - s_{\alpha 11}s_{\alpha 21}c_{\varphi} \end{pmatrix} \tag{20}$$

where notations x_{32}, y_{32}, z_{32}, $\bar{x}_3$, $\bar{y}_3$ and $\bar{z}_3$ follow.[11] Exterior angle "2" and edge "12" in equation (19) are θ_1 and π-β_{11} respectively. Substituting equation (20) into equation (19), one has

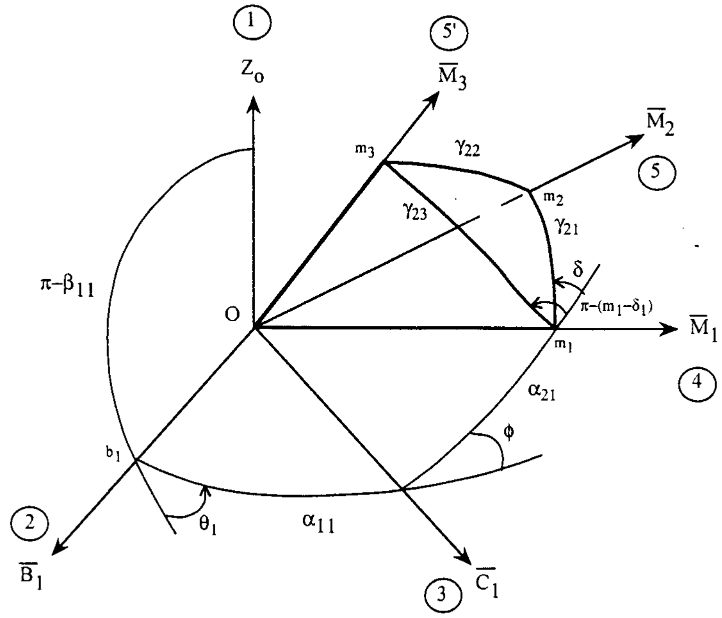


Fig. 5. Vectors $\overline{M}_1$, $\overline{M}_2$, and $\overline{M}_3$.

$$\overline{M}_1 = \begin{pmatrix} M_{1x} \\ M_{1y} \\ M_{1z} \end{pmatrix} = \begin{pmatrix} s_{\alpha21}s_\varphi c_{\theta1} + s_{\alpha11}c_{\alpha21}s_{\theta1} + c_{\alpha11}s_{\alpha21}c_\varphi s_{\theta1} \\ -c_{\beta1}s_{\alpha21}s_\varphi s_{\theta1} + c_{\beta1}s_{\alpha11}s_{\alpha21}c_{\theta1} + c_{\beta1}c_{\alpha1}s_{\alpha2}c_\varphi c_{\theta1} \\ - s_{\beta11}c_{\alpha11}c_{\alpha21} + s_{\beta11}s_{\alpha11}c_{\alpha21}c_\varphi \\ s_{\beta11}s_{\alpha11}s_\varphi s_{\theta1} - s_{\beta11}s_{\alpha11}c_{\alpha21}c_{\theta1} - s_{\beta11}c_{\alpha11}s_{\alpha21}c_\varphi c_{\theta1} \\ - c_{\beta11}c_{\alpha11}c_{\alpha21} + c_{\beta11}s_{\alpha11}s_{\alpha21}c_\varphi \end{pmatrix} \tag{21}$$

4.2 *Direction cosines of vector $\overline{M}_2$*

The direction cosines of vector $\overline{M}_2$ can directly be set up from the spherical pentagon, $M_2M_1C_1B_1Z_0$, denoted as 54321 (Figure 5).

$$\overline{M}_2 = \begin{pmatrix} x_{432} \\ y_{432} \\ z_{432} \end{pmatrix} = \begin{pmatrix} x_{43}c_2 - y_{43}s_2 \\ c_{12}(x_{43}s_2 + y_{43}c_2) - s_{12}z_{43} \\ s_{12}(x_{43}s_2 + y_{43}c_2) + c_{12}z_{43} \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} x_{43} \\ y_{43} \\ z_{43} \end{pmatrix} = \begin{pmatrix} \overline{x}_4c_3 - \overline{y}_4s_3 \\ c_{32}(\overline{x}_4s_3 + \overline{y}_4c_3) - s_{32}\overline{z}_4 \\ s_{32}(\overline{x}_4s_3 + \overline{y}_4c_3) + c_{32}\overline{z}_4 \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} \overline{x}_4 \\ \overline{y}_4 \\ \overline{z}_4 \end{pmatrix} = \begin{pmatrix} s_{\gamma1}s_\delta \\ -(s_{\alpha21}c_{\gamma21} + c_{\alpha21}s_{\gamma21}c_\delta) \\ c_{\alpha21}c_{\gamma21} - s_{\alpha21}s_{\gamma21}c_\delta \end{pmatrix} \tag{24}$$

where X_{432}, Y_{432}, and Z_{432} follow Duffy's notations[11] and $c_3 = c_\varphi$, $s_3 = s_\varphi$, $c_{32} = c_{\alpha11}$, and $s_{32} = s_{\alpha11}$. Substituting equation (23) and equation (24) into equation (22) one obtains

$$\overline{M}_2 = \begin{pmatrix} M_{2x} \\ M_{2y} \\ M_{2z} \end{pmatrix}$$

$$= \begin{pmatrix} s_{\gamma21}s_\delta c_\varphi c_{\theta1} + s_{\alpha21}c_{\gamma21}s_\varphi c_{\theta1} + c_{\alpha21}s_{\gamma21}c_\delta s_\varphi c_{\theta1} \\ - c_{\alpha11}s_{\gamma21}s_\delta s_\varphi s_{\theta1} \\ + c_{\alpha11}s_{\alpha21}c_{\gamma21}c_\varphi s_{\theta1} + c_{\alpha11}c_{\alpha21}s_{\gamma21}c_\delta c_\varphi s_{\theta1} \\ + s_{\alpha11}c_{\alpha21}c_{\gamma21}s_{\theta1} - s_{\alpha11}s_{\alpha21}s_{\gamma21}c_\delta s_{\theta1} \\ \\ - c_{\beta11}s_{\theta1}s_{\gamma21}s_\delta c_\varphi - c_{\beta11}s_{\theta1}s_{\alpha21}c_{\gamma21}s_\varphi \\ - c_{\beta11}s_{\theta1}c_{\alpha21}s_{\gamma21}c_\delta s_\varphi - c_{\beta11}c_{\theta1}c_{\alpha11}s_{\gamma21}s_\delta s_\varphi \\ + c_{\beta11}c_{\theta1}c_{\alpha1}s_{\alpha2}c_{\gamma21}c_\varphi + c_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma21}c_\delta c_\varphi \\ + c_{\beta11}c_{\theta1}s_{\alpha11}c_{\alpha21}c_{\gamma21} \\ - c_{\beta11}c_{\theta1}s_{\alpha11}s_{\alpha21}s_{\gamma21}c_\delta - s_{\beta11}s_{\alpha11}s_{\gamma21}s_\delta s_\varphi \\ + s_{\beta11}s_{\alpha11}s_{\alpha21}c_{\gamma21}c_\varphi \\ + s_{\beta11}s_{\alpha11}c_{\alpha21}s_{\gamma21}c_\delta c_\varphi - s_{\beta11}c_{\alpha11}c_{\alpha21}c_{\gamma21} \\ + s_{\beta11}c_{\alpha11}s_{\alpha21}s_{\gamma21}c_\delta \\ \\ s_{\beta11}s_{\theta1}s_{\gamma21}s_\delta c_\varphi + s_{\beta11}s_{\theta1}s_{\alpha21}c_{\gamma21}s_\varphi \\ + s_{\beta11}s_{\theta1}c_{\alpha21}s_{\gamma21}c_\delta s_\varphi + s_{\beta11}c_{\theta1}c_{\alpha11}s_{\gamma21}s_\delta s_\varphi \\ - s_{\beta11}c_{\theta1}c_{\alpha11}s_{\alpha21}c_{\gamma21}c_\varphi - s_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma21}c_\delta c_\varphi \\ - s_{\beta11}c_{\theta1}s_{\alpha11}c_{\alpha21}c_{\gamma21} \\ + s_{\beta11}c_{\theta1}s_{\alpha11}s_{\alpha21}s_{\gamma21}c_\delta - c_{\beta11}s_{\alpha11}s_{\gamma21}s_\delta s_\varphi \\ + c_{\beta11}s_{\alpha11}s_{\alpha21}c_{\gamma21}c_\varphi \\ + c_{\beta11}s_{\alpha11}c_{\alpha21}s_{\gamma21}c_\delta c_\varphi - c_{\beta11}c_{\alpha11}c_{\alpha21}c_{\gamma21} \\ + c_{\beta11}c_{\alpha11}s_{\alpha21}s_{\gamma21}c_\delta \end{pmatrix} \tag{25}$$

4.3 Direction cosines of vector $\overline{M}_3$

Considering another spherical pentagon, $m_3m_1c_1b_1z_0$ (5'4321), there are four edges, γ_{23}, α_{21}, α_{11} and π-β_{11} and three exterior angles, θ, φ, and δ' (Figure 5). The angle δ' is between sides α_{21} and γ_{23} and therefore $\delta' = \pi - (m_1 - \delta)$, where m_1 is the exterior angle in vertex m_1 of the spherical triangle $m_1m_2m_3$ and can be derived using the cosine law similar to equation (1). Thus,

$$\begin{aligned} \sin\delta' &= \sin m_1 \cos\delta - \cos m_1 \sin\delta \\ \cos\delta' &= -\cos m_1 \cos\delta - \sin m_1 \sin\delta \end{aligned} \tag{26}$$

Similarly, the direction cosines of vector $\overline{M}_3$ can also be expressed from the spherical pentagon, 5'4321,. In fact, this can be done by only replacing angles γ_{21} and δ in equation (25) by γ_{23} and δ'.

$$\begin{aligned} M_{3x} = {} & s_{\gamma23}c_{\varphi}c_{\theta1}s_{m1}c_{\delta} - s_{\gamma23}c_{\varphi}c_{\theta1}c_{m1}s_{\delta} + s_{\alpha21}c_{\gamma23}s_{\varphi}c_{\theta1} \\ & - c_{\alpha21}s_{\alpha23}s_{\varphi}c_{\theta1}c_{m1}c_{\delta} - c_{\alpha21}s_{\gamma23}s_{\varphi}c_{\theta1}s_{m1}s_{\delta} \\ & - c_{\alpha11}s_{\gamma23}s_{\varphi}s_{\theta1}s_{m1}c_{\delta} + c_{\alpha11}s_{\gamma23}s_{\varphi}s_{\theta1}c_{m1}s_{\delta} \\ & + c_{\alpha11}s_{\alpha21}c_{\gamma23}c_{\varphi}s_{\theta1} - c_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}s_{\theta1}c_{ml}c_{\delta} \\ & - c_{\alpha11}c_{\alpha21}s_{\gamma23}s_{\varphi}s_{\theta1}s_{m1}s_{\delta} + s_{\alpha11}c_{\alpha21}c_{\gamma23}s_{\theta1} \\ & + s_{\alpha11}c_{\alpha21}c_{\gamma23}c_{\theta1}c_{ml}c_{\delta} + s_{\alpha11}s_{\alpha21}s_{\gamma23}s_{\theta1}s_{ml}s_{\delta} \end{aligned}$$

$$\begin{aligned} M_{3y} = {} & - c_{\beta11}s_{\theta1}s_{\gamma23}c_{\varphi}s_{ml}c_{\delta} + c_{\beta11}s_{\theta1}s_{\gamma23}c_{\varphi}c_{ml}s_{\delta} \\ & - c_{\beta11}s_{\theta1}s_{\alpha21}s_{\gamma23}s_{\varphi} + c_{\beta11}s_{\theta1}c_{\alpha21}s_{\gamma23}s_{\varphi}c_{ml}c_{\delta} \\ & + c_{\beta11}s_{\theta1}c_{\alpha21}s_{\gamma23}s_{\varphi}s_{ml}s_{\delta} - c_{\beta11}c_{\theta1}c_{\alpha11}s_{\gamma23}s_{\varphi}s_{ml}c_{\delta} \\ & + c_{\beta11}c_{\theta1}c_{\alpha11}s_{\gamma23}s_{\varphi}c_{ml}s_{\delta} + c_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}c_{\gamma23}c_{\varphi} \\ & - c_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}c_{ml}c_{\delta} - c_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}s_{ml}s_{\delta} \\ & + c_{\beta11}c_{\theta1}s_{\alpha11}s_{\alpha21}c_{\gamma23} + c_{\beta11}c_{\theta1}s_{\alpha11}s_{\alpha21}s_{\gamma23}c_{ml}c_{\delta} \\ & + c_{\beta11}c_{\theta1}s_{\alpha11}s_{\alpha21}s_{\gamma23}s_{ml}s_{\delta} - s_{\beta11}s_{\alpha11}s_{\gamma23}s_{\varphi}s_{ml}c_{\delta} \\ & + s_{\beta11}s_{\alpha11}s_{\gamma23}s_{\varphi}c_{ml}s_{\delta} \\ & + s_{\beta11}s_{\alpha11}s_{\alpha21}s_{\gamma23}c_{\varphi} - s_{\beta11}s_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}c_{ml}c_{\delta} \\ & - s_{\beta11}s_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}s_{ml}s_{\delta} \\ & - s_{\beta11}c_{\alpha11}c_{\alpha21}c_{\gamma23} - s_{\beta11}c_{\alpha11}c_{\alpha21}s_{\gamma23}c_{ml}c_{\delta} \\ & - s_{\beta11}c_{\alpha11}c_{\alpha21}s_{\gamma23}s_{ml}s_{\delta} \end{aligned}$$

$$\begin{aligned} M_{3z} = {} & s_{\beta11}s_{\theta1}s_{\gamma23}c_{\varphi}s_{ml}c_{\delta} - s_{\beta11}s_{\theta1}s_{\gamma23}c_{\varphi}c_{ml}d_{\delta} \\ & + s_{\beta11}s_{\theta1}s_{\alpha21}s_{\gamma23}s_{\varphi} - s_{\beta11}s_{\theta1}s_{\alpha21}s_{\gamma23}s_{\varphi}c_{ml}c_{\delta} \\ & - s_{\beta11}s_{\theta1}c_{\alpha21}s_{\gamma23}s_{\varphi}s_{ml}s_{\delta} + s_{\beta11}c_{\theta1}c_{\alpha11}s_{\gamma23}s_{\varphi}s_{ml}c_{\delta} \\ & - s_{\beta11}s_{\theta1}c_{\alpha11}s_{\gamma23}s_{\varphi}c_{ml}c_{\delta} \\ & - s_{\beta11}c_{\theta1}c_{\alpha11}s_{\alpha21}c_{\gamma23}c_{\varphi} + s_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}c_{ml}c_{\delta} \\ & + s_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}s_{ml}s_{\delta} \\ & - s_{\beta11}c_{\theta1}s_{\alpha11}c_{\alpha21}c\gamma - s_{\beta11}c_{\theta1}s_{\alpha11}s_{\alpha21}s_{\gamma23}c_{ml}c_{\delta} \\ & - s_{\beta11}c_{\theta1}c_{\alpha11}c_{\alpha21}s_{\gamma23}s_{ml}s_{\delta} \\ & - c_{\beta11}s_{\alpha11}s_{\gamma23}s_{\varphi}s_{ml}c_{\delta} + c_{\beta11}s_{\alpha11}s_{\gamma23}s_{\varphi}c_{ml}s_{\delta} \\ & + c_{\beta11}s_{\alpha11}s_{\alpha21}c_{\gamma23}c_{\varphi} - c_{\beta11}s_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}c_{ml}c_{\delta} \\ & - c_{\beta11}s_{\alpha11}c_{\alpha21}s_{\gamma23}c_{\varphi}s_{ml}s_{\delta} - c_{\beta11}c_{\alpha11}c_{\alpha21}c_{\gamma23} \\ & - c_{\beta11}c_{\alpha11}s_{\alpha21}s_{\gamma23}s_{ml}c_{\delta} - c_{\beta11}c_{\alpha11}s_{\alpha21}s_{\gamma23}s_{ml}s_{\delta} \end{aligned} \tag{27}$$

4.4 Compensative equations and resolution

Equations (21), (25), and (27) specify the relationship between the direction cosines of vectors $\overline{M}_1$, $\overline{M}_2$, and $\overline{M}_3$ and geometrical parameters, β_{11}, α_{11}, α_{21}, γ_{21}, and kinematic parameters, θ_1, φ, and δ. While β_{11}, α_{11}, α_{21}, γ_{21}, and θ_1 are known, the angles φ and δ are still unknown. Therefore it is necessary to solve unknown angle φ and δ as follows: Two compensative equations are generally given

$$\overline{M}_i \cdot \overline{C}_i = \cos\alpha_{2i} \qquad i = 2, 3. \tag{28}$$

In these equations, α_{22} and α_{23} are known constants. $\overline{C}_2$ and $\overline{C}_3$ expressed by equation (9) are functions of known geometric quantities, μ_i, α_{1i}, β_{1i}, and inputs θ_2 and θ_3. Therefore the two equations can be used to solve two unknown angles, φ and δ. Substituting equations (9), (25), and (27) into equation (28) and rearranging, the two equations in equation (28) now take the following form

$$A_i \sin\varphi + B_i \cos\varphi + C_i = 0 \qquad i = 1, 2 \tag{29}$$

where the coefficients, A_i, B_i, and C_i contain the unknown angle, δ as well as all known parameters. The full expression of the coefficients is not written out here due to space constraint but in order to eliminate one of the unknowns from equation (29), one has

$$\begin{aligned} \sin\varphi &= \frac{C_2B_1 - C_1B_2}{A_1B_2 - A_2B_1} \\ \cos\varphi &= \frac{C_2A_1 - C_1A_2}{A_1B_2 - A_2B_1} \end{aligned} \tag{30}$$

Using the identity, $\sin^2\varphi + \cos^2\varphi = 1$, one obtains

$$\begin{aligned} & - B_1^2A_2^2 + C_1^2A_2^2 + 2A_1A_2B_1B_2 - A_1^2B_2^2 + C_1^2B_2^2 \\ & - 2A_1A_2C_1C_2 - 2B_1B_2C_1C_2 + A_1^2C_2^2 + B_1^2C_2^2 = 0 \end{aligned} \tag{31}$$

This equation contains only one unknown, δ but it is not in the form of $\sin\delta$ and $\cos\delta$.

Let $x = \tan\dfrac{\delta}{2}$ then $\sin\delta = \dfrac{2x}{1+x^2}$, and $\cos\delta = \dfrac{1-x^2}{1+x^2}$ and substitute $\sin\delta$ and $\cos\delta$ into equation (31), an eighth-order polynomial equation is arrived at

$$\sum_{i=1}^{8} E_i x^i = 0 \tag{32}$$

where E_i are functions of known quantities only. x and then the unknown, δ, can be solved. As a result, coefficient A_1, B_i, and c_i will become known. The last unknown, φ, can be solved from equation 30 which is rewritten as

$$\tan\varphi = \frac{C_2B_1 - C_1B_2}{C_2A_1 - C_1A_2} \tag{33}$$

Vectors $\overline{M}_1$, $\overline{M}_2$, and $\overline{M}_3$ of the moving pyramid can be then solved using equations (21), (25), and (27).

5. A SPECIAL SPHERICAL PARALLEL MECHANISM OF RIGHT-ANGLE TYPE

Figure 6a shows a special case of the 3-DOF spherical parallel mechanism where $\gamma_1 = \gamma_2 = \pi/2$ and $\alpha_1 = \alpha_2 = \pi/2$. $Ob_1b_2b_3$ is the basis pyramid, and $Om_1m_2m_3$ the moving one. Three spherical dyads, $b_1c_1m_1$, $b_2c_2m_2$, and $b_3c_3m_3$ connect two pyramids. b_1c_1, b_2c_2, and b_3c_3 are the three input links.

Three input angles defined by the angle between the input link b_ic_i and the plane b_iOb_j are denoted by $\hat{\theta}_i$, $i = 1,2,3$, which are equal to $\theta_i - \pi/4$. Initially, they are all set zero as shown in Figure 6a, with b_ic_i coplanar with plane b_iOb_j. This case can be conveniently solved by directly substituting the special geometric conditions into the formulas laid out in the preceeding sections. The rest of this section, however, will be devoted to discussions of its special kinematics which exhibits some unique and interesting characteristics. The discussions are based on a method presented in references 12 and 13.

Figure 6a shows the initial configuration of the manipulator. At this configuration three input angles $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ are all zero. This means that three input links, b_1c_1, b_2c_3, are coplanar with planes b_iOb_2, b_2Ob_3, and b_3Ob_1, respectively. In addition, they are coplanar with planes m_2Om_3, m_3Om_1, and m_1Om_2, respectively. Three center links, c_1m_1, c_2m_2, and c_3m_3 are also coplanar with planes m_3 Om_1, m_1Om_2, and

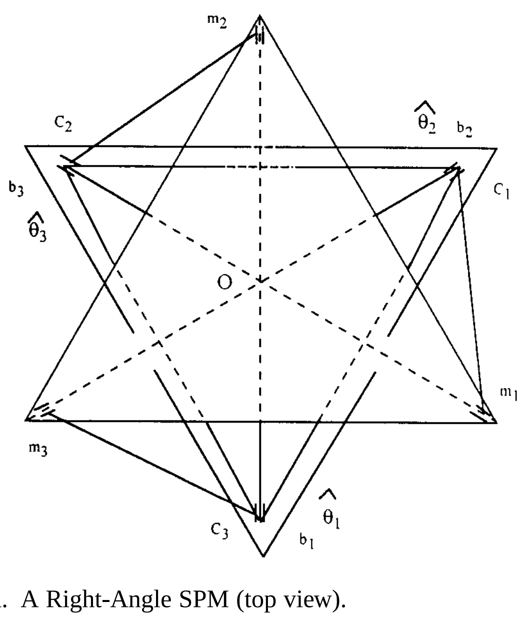


Fig. 6a. A Right-Angle SPM (top view).

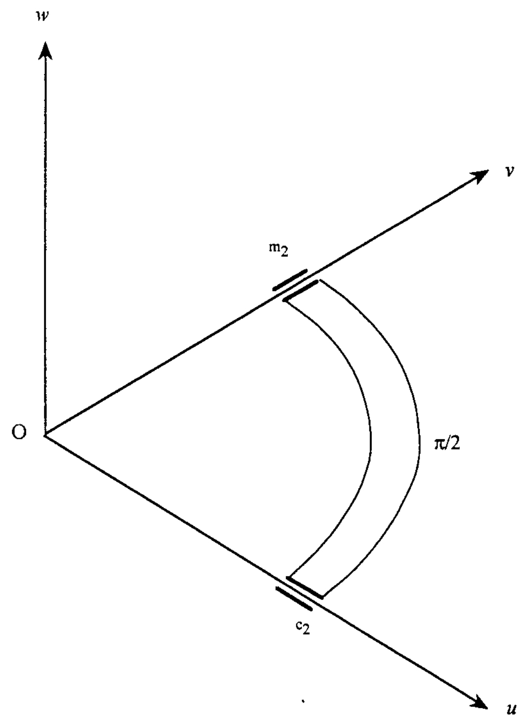


Fig. 6b. A Right-Angle Link.

m_2Om_3, respectively. In addition, the three segments b_1O, b_2O, and b_3O are colinear with Om_2, Om_3, and Om_1, respectively.

Firstly, let the input link, b_1c_1, rotate alone about axis Ob_1. Input angle $\hat{\theta}_1$ increases gradually from zero, while the other two input links are kept stationary, that is, $\hat{\theta}_2 = \hat{\theta}_3 = 0$. In this case, the 3-DOF mechanism becomes a 1-DOF six-bar spherical mechanism. Three branches of moving pyramid are different. The first branch, $b_1c_1m_1$, still is a two-link chain with links b_1c_1 and c_1m_1. The other two branches which are passive chains, have only one moving link each, that is, c_2m_2 or c_3m_3. In order to analyze the kinematic characteristics of the pyramid in the six-bar mechanism, it needs to find which axis the pyramid can rotate about when only b_1c_1 is input.

It is necessary to conduct a constraint analysis now. Take the first branch, $b_1c_1m_1$, which has three non-coplanar revolute joints b_1, c_1 and m_1. If all the joint axes are denoted as screws with zero pitch,[12] the three axes in the first branch comprises a three-system screw. Reciprocal screws, $\$^r$, of the three moving screws for an appropriate coordinate system may be written as

$$\begin{aligned} \$_1^r &= (1\ 0\ 0;\ 0\ 0\ 0) \\ \$_2^r &= (0\ 1\ 0;\ 0\ 0\ 0) \\ \$_3^r &= (0\ 0\ 1;\ 0\ 0\ 0) \end{aligned} \tag{34}$$

Therefore, the end pyramid loses three translational degrees of freedoms. It is well known that, as a result, only three-rotational motions are now possible, whose rotation axes pass through the common point O.

The second or third branches only has one moving link, that is, c_2m_2 or c_3m_3. Take link c_2m_2 for analysis. The revolute joints in uvw system as shown in Figure 6b, may be written as

$$\$_{c2} = (1\ 0\ 0;\ 0\ 0\ 0)$$
$$\$_{m2} = (0\ 1\ 0;\ 0\ 0\ 0) \quad (35)$$

Therefore, for link c_2m_2 there are four reciprocal screws constraining the motion of the pyramid, namely

$$\$_1^r = (1\ 0\ 0;\ 0\ 0\ 0)$$
$$\$_2^r = (0\ 1\ 0;\ 0\ 0\ 0)$$
$$\$_3^r = (0\ 0\ 1;\ 0\ 0\ 0) \quad (36)$$
$$\$_4^r = (0\ 0\ 0;\ 0\ 0\ 1)$$

Excluding those three constrained translations, there is a constraining wrench with infinite-pitch $\$_4^r$ along the w-axis which is perpendicular to the plane determined by the axes Oc_2 and Om_2. Therefore the end effector can only rotate about the axis which lies in the plane c_2Om_2 and passes point O. In other words, the rotatable axis can be anyline of the planar pencil determined by the two moving joint axes of the link c_2m_2 with a central point O.

For the six-bar spherical mechanism, the branch c_2m_2 limits the selection of the rotatable axis of the pyramid in the plane pencil c_2m_2. Branch c_3m_3 limits that in plane pencil c_3Om_3. The only possible rotatable axis of the pyramid is the intersecting line of the two planar pencils. At the initial position, that is, $\hat{\theta}_2 = \hat{\theta}_3 = 0$, and $\hat{\theta}_1$ is the active input angle, the intersecting line of the two planar pencil, c_2Om_2 and c_3Om_3 is line b_1Om_2 as shown in Figure 6a, which is just the revolute axis of the pyramid. When $\hat{\theta}_2 = \hat{\theta}_3 = 0$ and $\hat{\theta}_1 \neq 0$, the pyramid rotates about axis b_1Om_2, link c_2m_2 is kept stationary, and c_3m_3 rotates about Ob_1 while input link b_1c_1 rotates about Ob_1. Thus, the intersecting line b_1Om_2 of two planar pencils c_2Om_2 and c_3Om_3 keeps invariable throughout rotation of link b_1c_1.

In order to determine the quantity of rotated angle of the pyramid about b_1Om_2, a four-bar spherical linkage, $Ob_1c_1m_1m_2O$, may be analyzed. The segments b_1O and Om_2 keep invariably colinear as mentioned before. The angle $\hat{\theta}_1$ is the input of the four-bar linkage, and the rotated angle of the pyramid is the output of the four-bar linkage. The input-output equation of the four-bar linkage as shown in Figure 7 is quoted[15]

$$c_c = c_ac_bc_f + (s_ac_bc_{\varphi'} - c_as_bc_\psi)s_f + s_as_b(s_{\varphi'}s_\psi + c_{\varphi'}c_\psi c_f) \quad (37)$$

where angles a, b, c, f, φ', and ψ are shown in figure 7.

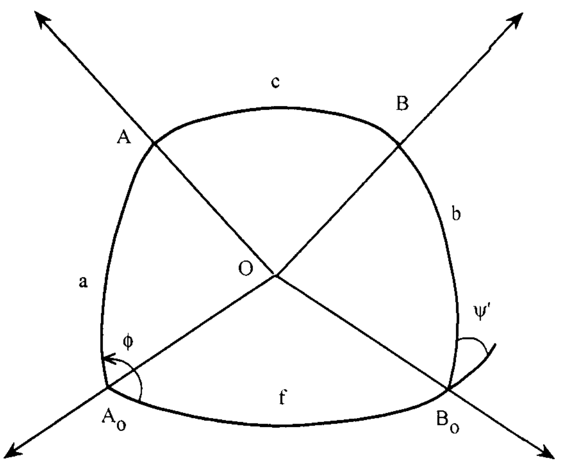


Fig. 7. A Spherical Four-Bar Linkage.

Considering the given geometrical condition, $a = b = c = \pi/2$. In the case of $f = \pi$, b_1O and Om_2 become colinear and one has

$$\cot\varphi' = \tan\psi \quad (38)$$

where ψ is input angle between planes b_1Oc_2 and b_1Ob_2. φ' is output angle between planes m_2Om_3 and b_1Ob_2. If $\varphi = \pi/2 - \varphi'$, then equation (38) becomes $\tan\varphi = \tan\psi$.

It means that the rotated angle of the pyramid is just equal to the input angle. Thus, one can say that for this kind of 3-DOF spherical parallel manipulator, the pyramid rotates about one of the edges, m_2O, if there is only one input angle, $\hat{\theta}_1$ while the other two input angles are kept at zero, that is, $\hat{\theta}_2 = \hat{\theta}_3 = 0$. The same holds if input angle is $\hat{\theta}_2$ and $\hat{\theta}_3 = \hat{\theta}_1 = 0$, the pyramid will rotate about Om_3. Similarly, the pyramid will rotate about Om_1 when $\hat{\theta}_1 = \hat{\theta}_2 = 0$, and $\hat{\theta}_3$ is input. In all three cases, the rotated angle of the pyramid is just equal to the input angle.

In the case of $\hat{\theta}_1 \neq 0$ and $\hat{\theta}_3 = 0$, $\hat{\theta}_2$ is the input angle. The only colinear two segments are c_1O and Om_3. Plane C_1Om_1 and Om_1m_3 are still coplanar so that the permitted kinematic axis which the pyramid rotates about is the intersecting line Om_3 of two plane pencils c_1Om_1 and c_3Om_3. The intersecting line is invariable throughout when $\hat{\theta}_2$ is variable. In this case, $f \neq \pi$, the rotated angle of the pyramid is found by using equation (37)

$$\tan\psi = -\cot\varphi' \cos f \quad (39)$$

In the case of $\hat{\theta}_1 \neq 0$ and $\hat{\theta}_2 = 0$, $\hat{\theta}_3$ is the input angle. The permitted rotational axis of the pyramid should be the intersecting line of planes c_1Om_1 and c_2Om_2. The intersecting line is not the line Om_1. After an input angle $\hat{\theta}_1$ is chosen, c_2m_2 is not coplanar with plane m_1Om_2. Om_1 is not coplanar with c_2Om_2. The edge of the pyramid itself is not a rotatable axis any more. The intersecting line of planar pencils c_2Om_2 and c_1Om_1 is another line in planar pencil c_1Om_1, and its position in that planar pencil is variant throughout when $\hat{\theta}_3$ is input.

In the case of $\hat{\theta}_1 \neq 0$ and $\hat{\theta}_2 \neq 0$, $\hat{\theta}_3$ is the input angle. None of the three pyramids edges is the rotatable axis of the moving platform. As the intersecting line of the two pencils c_1Om_1 and c_2Om_2 is not the edge Om_1 of the end effector. Another line inside the pencil c_1Om_1 is the intersecting line and its position is variant throughout. It is only an instantaneous rotatable axis when $\hat{\theta}_3$ is input.

6. AN ENTIRELY DECOUPLED 3-DOF SPHERICAL PARALLEL MANIPULATOR

Another special case, i.e., an entirely decoupled three-dimensional spherical mechanism is shown in Figure 8. Its basis pyramid $Ob_1b_2b_3$ is of condition $\gamma_{11} = \gamma_{12} = \gamma_{13} = \gamma_1 = \pi/2$. Its moving pyramid takes the following special values: $\gamma_{21} = 0$, $\gamma_{22} = \gamma_{23} = \pi/2$ and thus becomes a right-angle bar m_1Om_3. It also has three kinematic chains ab, cd, and ef connecting the base and the moving pyramid. The angle α of every link is $\pi/2$. The major advantage of this mechanism is that its three-dimensional movement is decoupled. The input b_1 or b_2 determines the orientational angle, pitch or yaw, of the output link independently, and the input b_3 determines the output angle, roll, of the end effector alone.

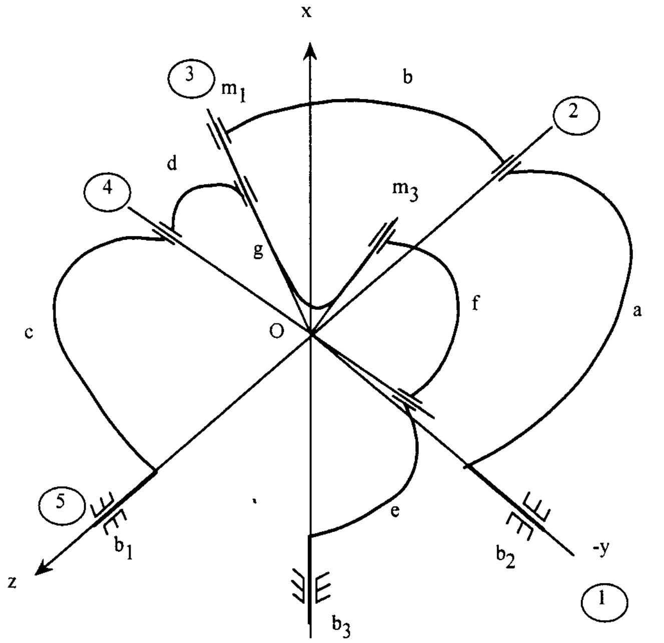


Fig. 8. A Decoupled SPM.

The roll can be more than 2π. As a result, the kinematics for this special case becomes very straight forward.

This 3-DOF spherical parallel manipulator can also be decomposed as two submechanisms: a five-bar spherical mechanism, abcd, and a four-bar spherical mechanism, efg. The latter is analogous to that of a universal joint. The spherical analytical theory[11] will be used here to analyze the five-bar mechanism and to show in an alternative way that this special case is entirely decoupled. The cosine law of a spherical pentagon 12345 showin in Figure 8, yields

$$\bar{Z}_3 = Z_{51} \tag{40}$$

where

$$\begin{aligned} \bar{Z}_3 &= c_{23}c_{34} - s_{23}s_{34}c_3 = -c_3 \\ Z_{51} &= s_{12}(\bar{X}_5 s_1 + \bar{Y}_5 c_1) + c_{12}\bar{Z}_5 \\ \bar{X}_5 &= s_{45}s_5 = s_5 \\ \bar{Y}_5 &= -s_{51}c_{45} - c_{51}s_{45}c_5 = 0 \\ \bar{Z}_5 &= c_{51}c_{45} - s_{51}s_{45}c_5 = -c_5 \end{aligned} \tag{41}$$

where Z_{51} and others follow Duffy's notations.[11] Substituting equation 41 into equation 40 and considering all the γ and α are equal to $\pi/2$, one obtains

$$c_3 = -s_5 s_1 \tag{42}$$

Let $x_2 = \tan\theta_2/2$, then from the half-angle identity[11] one obtains

$$\chi_2 = \frac{X_{51} - \bar{X}_3}{Y_{51} - \bar{Y}_3} \tag{43}$$

$$\begin{aligned} \chi_2 &= \bar{X}_5 c_1 - \bar{Y}_5 s_1 = s_5 c_1 \\ Y_{51} &= c_{12}(\bar{X}_5 s_1 + \bar{Y}_5 c_1) - s_{12}\bar{Z}_5 = c_5 \\ \bar{X}_3 &= s_{34}s_3 = s_3 \\ \bar{Y}_3 &= -(s_{23}c_{34} + c_{23}s_{34}c_3) = 0 \end{aligned} \tag{44}$$

Substituting equation 42 into equation 44 and then into equation 43 we get

$$\chi_2 = (s_5c_1 - s_3)/c_5 \tag{45}$$

From the trigonometric function one obtains

$$\begin{aligned} s_2 &= \frac{2\chi_2}{1+\chi_2^2} \\ c_2 &= \frac{1-\chi_2^2}{1+\chi_2^2} \\ s_3 &= \pm\sqrt{1-c_3^2} \end{aligned} \tag{46}$$

The direction cosines of unit vector Om_1 are

$$\overline{Om}_1 = \begin{pmatrix} X_{21} \\ Y_{21} \\ Z_{21} \end{pmatrix} = \begin{pmatrix} \bar{X}_2 c_1 - \bar{Y}_2 s_1 \\ c_{15}(\bar{X}_2 s_1 + \bar{Y}_2 c_1) - s_{15}\bar{Z}_2 \\ s_{15}(\bar{X}_2 s_1 + \bar{Y}_2 c_1) + \theta_{15}\bar{Z}_2 \end{pmatrix} = \begin{pmatrix} s_2 c_1 \\ c_2 \\ s_2 s_1 \end{pmatrix} \tag{47}$$

where

$$\begin{aligned} \bar{X}_2 &= s_{23}s_2 = s_2 \\ \bar{Y}_2 &= s_{12}c_{23} + c_{12}s_{23}c_2 = 0 \\ \bar{Z}_2 &= c_{12}c_{23} - s_{12}s_{23}c_2 = -c_2 \end{aligned}$$

Finally, substituting equation (46) into equation (47) and considering $s_3 = -1$, the direction cosines of vector Om_1 can be easily expressed as a function of input angles θ_1 and θ_5. When $\theta_1 = 0$, $\overline{Om}_1 = (c_5 - s_5\ 0)^T$. When $\theta_5 = 0$, $Om_1 = (c_1\ 0\ s_1)^T$. This is obviously decoupled, which can also be shown by using the method described in section 5.

7. NUMERICAL EXAMPLES

The following 3-DOF spherical mechanism is used in a numerical example

$$\gamma_{11} = \gamma_{12} = \gamma_{13} = \gamma_{21} = \gamma_{22} = \gamma_{23} = \pi/2$$
$$\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha_{31} = \alpha_{32} = \pi/2$$

Using the formulas given in sections 2 to 4, the direct kinematics are solved for four cases:

Case 1: $\hat{\theta}_1$ is input angle, $\hat{\theta}_1 = 0$ to 80°, $\hat{\theta}_2 = 0$, $\hat{\theta}_3 = 0$

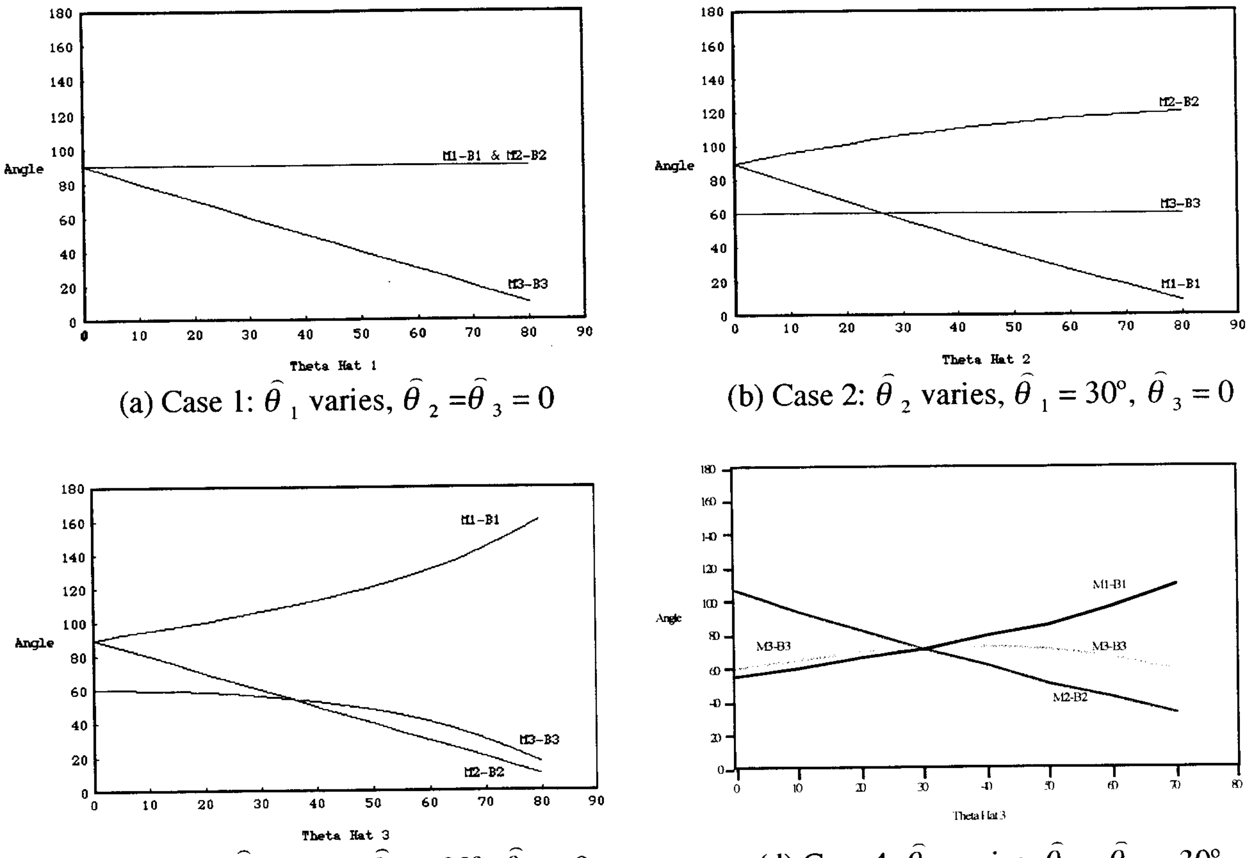
Case 2: $\hat{\theta}_1 = 30°$, $\hat{\theta}_2$ is input angle, $\hat{\theta}_2 = 0$ to 80°, $\hat{\theta}_3 = 0$

Case 3: $\hat{\theta}_1 = 30°$, $\hat{\theta}_2 = 0$, and $\hat{\theta}_3$ is input angle, $\hat{\theta}_3 = 0$ to 80°

Case 4: $\hat{\theta}_1 = 30°$, $\hat{\theta}_2 = 30°$, and $\hat{\theta}_3$ is input angle, $\hat{\theta}_3 = 0$ to 80°

where $\hat{\theta}_i = \theta_i - 45°$

The results are shown in Figure 9 and Figure 10. Figure 9 illustrates the change of angles between motive vectors M_i and stationary vectors B_i (I = 1, 2, 3), i.e., angles $M_1 - B_1$, $M_2 - B_2$, and $M_3 - B_3$ with the respective input angle. It is seen in Figure 9a that angles $M_1 - B_1$ and $M_2 - B_2$ remain constant and angle $M_3 - B_3$ decreases linearly when input angle $\hat{\theta}_1$ changes from 0 to 80°. These trends are consistent with physical understanding of the mechanism movement. Similarily, $M_3 - B_3$ remains constant and $M_1 - B_1$ decreases linearly for case 2 in Figure 9B in agreement with predictions based on physical understanding of the mechanism. Figure 10 illustrates the moving pyramid in 3-dimensional space. It is clear that, in case 1 (Figure 10a),



(a) Case 1: $\hat{\theta}_1$ varies, $\hat{\theta}_2 = \hat{\theta}_3 = 0$

(b) Case 2: $\hat{\theta}_2$ varies, $\hat{\theta}_1 = 30°$, $\hat{\theta}_3 = 0$

(c) Case 3: $\hat{\theta}_3$ varies, $\hat{\theta}_1 = 30°$, $\hat{\theta}_2 = 0$

(d) Case 4: $\hat{\theta}_3$ varies, $\hat{\theta}_1 = \hat{\theta}_3 = 30°$

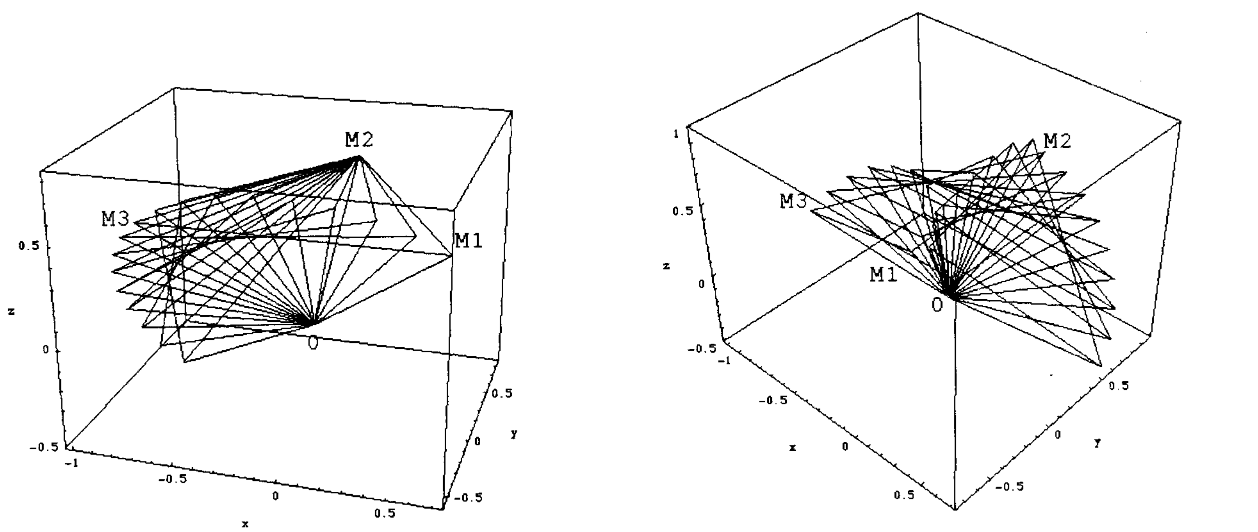
Fig. 9. Numerical Results of Angles Between Motive Vectors M_i and Stationary Vectors B_i ($i = 1,2,3$).

the moving pyramid rotates about OM_2. In case 3 (Figure 10b), none of the three edges of the moving pyramid serves as a stationary rotating axis for the moving pyramid.

8. CONCLUSIONS

In this paper a new method to derive the kinematics of a generalized 3-DOF spherical parallel manipulator is presented. This method is based on Duffy's theory and spherical trigonometry[11] which enables the derivation be more concise and uniform.

The kinematics of a special 3-DOF SPM with $\gamma_1 = \gamma_2 = \pi/2$ and $\alpha_1 = \alpha_2 = \pi/2$ is studied and three types of kinematic patterns are identified. In the case of any two of the three input angles are kept zero while the third one varies, the



(a) Case 1: $\hat{\theta}_1$ varies from 0 to 80°, $\hat{\theta}_2 = \hat{\theta}_3 = 0$

(b) Case 3: $\hat{\theta}_3$ varies from 0 to 80°, $\hat{\theta}_1 = 30°$, $\hat{\theta}_2 = 0$

Fig. 10. Numerical Results of Movement History of the Moving Pyramid $OM_1M_2M_3$.

moving pyramid can rotate about one of its three orthogonal edges, and the rotated angle of the end effector will be just equal to the input angle. In the case of one of the input angles being kept at a non-zero position, one at the zero position, while the third varies, two possibilities exist. The edge Om_3 can be a rotatable axis when the first input link, b_1c_1, is kept at the non-zero position, the third input link, b_3c_3, at the zero position, and the second one, b_2c_2, varies. On the contrary, none of the three pyramid's edges can be a rotatable axis when b_1c_1 is kept at a non-zero position, b_2c_2 at a zero position, and b_3c_3 varies. In the case of two input angles being kept at non-zero positions, none of the three pyramid's edge can be the rotatable axis.

Another special 3-DOF SPM with $\gamma_1 = \pi/2$, $\gamma_{21} = 0$, $\gamma_{22} = \gamma_{23} = \pi/2$, $\alpha_1 = \alpha_2 = \pi/2$ is also studied. It is an entirely decoupled 3 = DOF SPM. Two of the three inputs determine the orientation of the output Om_1 in a three-dimensional space, and the third one alone determines the roll angle of the moving pyramid, which can be over 2π.

Acknowledgment

The support provided by NNSFC is gratefully acknowledged. Thanks are also due to Mr. Zaidie Abdulmajid for his assistance in numerical studies and support from Columbia University.

References

1. H. Asada and J.A. Cro Granito, "Kinematic and static characterization of wrist joint and their optimal design" *Proc. of IEEE International Conf. on Robots and Automation* (1985) pp. 244–250.
2. D.J. Cox and D. Tesar, "The dynamic model of a 3-DOF parallel robotic shoulder module" *Proc. of 4th International Conf. on Advanced Robotics* (1989) pp. 475–487.
3. Kim and D. Tesar, "Study on structural design of force reflecting manual controllers" *Proc. of ASME 21st Biennial Mech. COnf.* (1990) pp. 481–488.
4. B.J. Yi, R.A. Freeman and D. Tesar, "Force and stiffness transmission in a redundantly actuated mechanism: the case for a spherical shoulder mechanism" *ASME*, **DE-Vol. 45**, 163–172 (1992).
5. C. Gosselin and J. Angeles, "The optimum kinematic design of a spherical 3-DOF parallel manipulator" *ASME Journal of Mechanism, Transmissions, and Automation in Design* **111**, 202–207 (1989).
6. C. Gosselin and J. Angeles, "Singularity analysis of closed-loop kinematic chains" *IEEE Trans. of Robotics and Automation* **6**(3), 281–290 (1990).
7. C. Gosselin and J. Angeles, "Kinematic inversion of parallel manipulators in the presence of incompletely specified tasks" *Journal of Mechanical Design* **112**, 494–500 (1990).
8. C. Gosselin and E. Lavoie, "On the kinematic design of spherical 3-DOF parallel manipulators" *Int. J. Robotic Research* **12**(4), 394–402 (1993).
9. C. Gosselin, J. Sefrioni and M.J. Richard, "On the direct kinematics of spherical 3-DOF parallel manipulators with a coplanar platform" *Journal of Mechanical Design* **116**(2) 587—593 (1994).
10. C. Gosselin, J. Sefrioni and M.J. Richard, "On the direct kinematics of spherical 3-DOF parallel manipulators of general architecture" *Journal of Mechanical Design* **116**(2), 594–598 (1994).
11. J. Duffy, *Analysis of Mechanisms and Robot Manipulators* (John Wiley and Sons, New York. 1980).
12. K.H. Hunt, *Kinematic Geometry of Mechanisms* (Clarendon Press, Oxford, 1978).
13. Z. Huang, W.S. Tao and Y.F. Fang, "Study on the kinematic characteristics of 3-DOF inparallel actuated platform mechanisms" *Mechanism and Machine Theory* **31**(8), 999–1008.
14. Z. Huang and Y.F. Fang, "Kinematic characteristics analysis of 3-DOF in-parallel actuated pyramid mechanism" *Mechanisms and Machine Theory* **31**(8) 1009—1018 (1996).
15. C.H. Chiang, *Kinematics of Spherical Mechanisms* (Cambridge University PRess, Cambridge, 1988).