Optimal Process Planning for Laser Forming of Doubly Curved Shapes

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Abstract

There has been a considerable amount of work carried out on two-dimensional laser forming. In order to advance the process further for industrial applications, however, it is necessary to consider more general cases and especially their process planning aspect. This paper presents an optimal approach to laser scanning paths and heating condition determination for laser forming of doubly curved shapes. Important features of the approach include the strain field calculation based on principal curvature formulation and minimal strain optimization, and scanning paths and heating condition determination by combining analytical and practical constraints. The overall methodology is presented first, followed by more detailed descriptions of each step of the approach. Two distinctive types of doubly curved shape, pillow and saddle shapes are focused on and the effectiveness of the proposed approach is validated by forming experiments.

1. Introduction

Compared with conventional forming techniques, laser forming (LF) of sheet metal does not require hard tooling or external forces and hence, can increase process flexibility and reduce the cost of the forming process when low- to medium-volume production or prototyping is concerned. To apply the LF process to real world problems, however, the inverse problem needs to be addressed, that is, to design process parameters (laser scanning paths and heating condition) given a desired shape. For general three-dimensional shapes, determining laser scanning paths and heating condition is not obvious since they are not directly related to the shapes.

Ueda, et. al [1] investigated the development of computer-aid process planning system for plate bending by line heating. They computed the strains using large deformation elastic finite element method (FEM) and decomposed strains into in-plane and bending components. They then chose regions with large in-plane strains as heating zones and selected heating direction normal to the principal strain. Their work in line heating provides relevant information for the LF process design. However, their approach to heating path determination is not well explained. Furthermore, they did not deal with how to determine the heating condition. Yu and Patrikalakis [2] presented algorithms for optimal development of a smooth continuous curved surface into a planar shape. The development process is modeled by in plane strain from the curved surface to its planar development. The distribution of the appropriate minimum strain field is obtained by solving a constrained nonlinear programming problem. The optimal developed planar shape is obtained by solving an unconstrained nonlinear programming problem. The algorithms presented are illuminating, however, they do not provide an explicit method on how to determine laser paths nor on how to determine heating conditions. Focusing on a class of shapes, Liu and Yao [3] proposed a response surface methodology based optimization method for LF design. The propagation of error technique is built into the design process as an additional response to be optimized via desirability function and hence make the design robust. This method is successful for a class of shapes; however, they are not directly applicable to general 3D shapes.

This study presents an optimal process planning strategy to determine scanning paths and heating condition for laser forming of general doubly curved shapes. The overall methodology is presented first, followed by more step-by-step descriptions of the proposed strategy. Two distinctive types of doubly curved surfaces, pillow and saddle shape are studied and the overall methodology is validated by experiments. This paper is concerned with thin plates and therefore the analysis of strains can be reduced to the analysis of their middle surface only.

2. Problem Description

In engineering applications, surfaces are often classified as singly and doubly curved surfaces. A singly curved surface has zero Gaussian curvature at all points and therefore can be formed by bending strain only. A doubly curved surface has non-zero Gaussian curvature and generally requires both in-plane and bending strains to form. Surfaces of many engineering structures are commonly fabricated as doubly curved shapes to fulfill functional requirements such as hydrodynamic, aesthetic, or structural. For thin plates, in-plane strain is usually much larger than the bending strain and therefore only the former is considered in this paper.

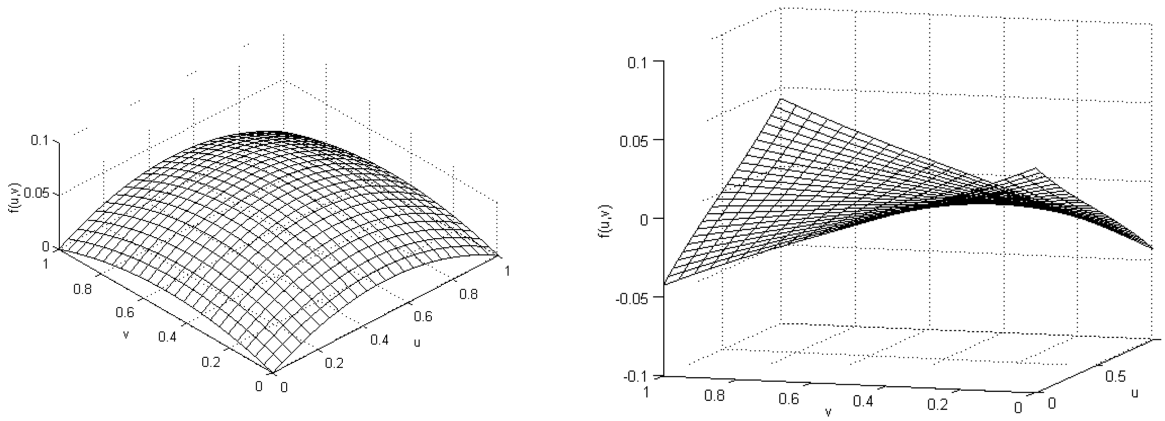
Two distinctive doubly curved surfaces are chosen as desired shapes in this study. They are a pillow shape, which has positive Gaussian curvature over the entire surface, and a saddle shape, which has negative Gaussian curvature over the entire surface. Both surfaces can be given in the form of Bezier surface. For a bicubic Bezier patch, sixteen points are required to determine a surface patch, the point on the surface patch can be expressed in the matrix form[4],

$$\mathbf{r}(u,v) = [(1-u)^3 \quad 3u(1-u)^2 \quad 3u^2(1-u) \quad u^3]\mathbf{Q}[(1-v)^3 \quad 3v(1-v)^2 \quad 3v^2(1-v) \quad v^3]^T \qquad (1)$$

where **Q** is a 4 by 4 matrix containing the sixteen control points. The control points for the pillow shape are:[(0,0,0), (0,1/3,0.025), (0,2/3,0.025), (0,1,0);(1/3,0,3.0.042), (1/3,1/3,0.08), (1/3,2/3,0.08), (1/3,1,0.042);(2/3,0,0.042), (2/3,1/3,0.08), (2/3,2/3,0.08), (2/3,1,0.042);(1,0,0), (1,1/3,0.025), (1,2/3,0.025), (1,1,0)]. For the saddle shape, the control points are:[(0,0,0.042), (0, 1/3,0.017), (0,2/3,-0.017), (0,1,-0.042);(1/3,0,3.0.017), (1/3,1/3,0.008), (1/3,2/3,-0.008), (1/3,1,-0.017);(2/3,0,-0.017), (2/3,1/3,-0.008), (2/3,2/3,0.008), (2/3,1,0.017);(1,0,-0.042), (1,1/3,-0.017), (1,2/3,0.017), (1,1,0.042)].

These two shapes are shown in Fig. 1 (a) and (b), respectively. As noticed in Fig. 1(a), the adjacent sides of the pillow shape are not identical, with the one along the *u* direction curves slightly higher. The saddle shape in Fig. 1 (b) bends up at a pair of opposite corners and down at the other pair.

The overall optimal planning strategy consists of three stages. In stage one, the principal curvature directions are calculated based on the first and second fundamental form coefficients of a given doubly curved surface. The desired doubly curved surface is then developed into a planar shape to obtain the required strain field. Based on the strain field and the coefficients of the first fundamental form of the curved surface, the initial planar shape is obtained by solving an unconstrained nonlinear problem. In stage two, the planning of laser paths is carried out on the planar developed surface and perpendicular to the principal curvature directions solved from the stage one. Segmentation along the heating paths is performed and heating condition is determined in the final stage. The required strain between adjacent scanning paths is first lumped together, a database, which is obtained from finite element method concerning the relationship between principal strains and laser power levels and scanning velocities is then consulted. The three stages are described in more details in the following three sections.



(a) (b)

Fig 1 (a) Desired pillow shape (b) Desired saddle shape.

3. Determination of Strain Field

In geometric modeling, shapes are commonly expressed in terms of parametric equations. A parametric representation of a surface can be denoted by $\mathbf{r}=\mathbf{r}(u,v)$ where $(u,v)\in[0,1]$ forms a parametric space. Its derivatives at point P on the surface are represented by $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$, $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$. The first fundamental coefficients E, F and G and the second fundamental coefficients L, M and N are defined as [5]

$$E = \mathbf{r}_u \cdot \mathbf{r}_u,\ F = \mathbf{r}_u \cdot \mathbf{r}_v \text{ and } G = \mathbf{r}_v \cdot \mathbf{r}_v$$

$$L = -\mathbf{r}_u \cdot \mathbf{N}_u,\ M = -\frac{1}{2}(\mathbf{r}_u \cdot \mathbf{N}_v + \mathbf{r}_v \cdot \mathbf{N}_u), \text{ and } N = -\mathbf{r}_v \cdot \mathbf{N}_v \qquad (2)$$

where $\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_v \times \mathbf{r}_v|}$, $\mathbf{N}_u = \frac{\partial \mathbf{N}}{\partial u}$ and $\mathbf{N}_v = \frac{\partial \mathbf{N}}{\partial v}$.

The normal curvature of the surface at P can be expressed as a function of first and second fundamental coefficients , namely,

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \qquad (3)$$

Principal directions are the two perpendicular directions at which the value κ_n take on maximum and minimum values. The value of principal curvatures and principal directions at a point *P* on a surface can be obtained by taking derivative of Eq. 3 with respect to $\lambda=dv/du$ and letting it equal to zero, that is,

$$(FN-MG)\lambda^2+(EN-LG)\lambda+(EM-LF)=0 \qquad (4)$$

where λ is a principal direction at *P*. Substituting λ into Eq. 3 yields

$$(EG-F^2)\,\kappa_n^2-(EN+GL-2FM)\kappa_n+(LN-M^2)=0 \qquad (5)$$

Gaussian curvature is the product of the roots of Eq. 5, κ_{min} and κ_{max} that is

$$K=\kappa_{min}\kappa_{max}=\frac{LN-M^2}{EG-F^2} \qquad (6)$$

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature. It follows that a curve is a line of curvature if and only if at each point the direction of its tangent satisfies Eq. 4 for some path $\mathbf{r}=\mathbf{r}(u,v)$.

3.2 Strain Field for Planar Development

Assume the strain field due to changing from a given curved surface to its planar shape is represented by $\varepsilon^s(u,v)$ and $\varepsilon^t(u,v)$ where (s,t) denote the principal curvature directions. An approach based on minimization of the total strain energy after adding the strain field to the given doubly curved surface, which maps to a planar shape on which Gaussian curvature K_{ps} is zero everywhere [2], is expressed as

$$\min\iint_D\{(\varepsilon^s)^2+(\varepsilon^t)^2\}|\mathbf{r}_u\times\mathbf{r}_v|dudv=\min\iint_D\{(\varepsilon^s)^2+(\varepsilon^t)^2\}\sqrt{EG-F^2}\,dudv \qquad (7)$$

subject to $K_{ps}=0$, $\varepsilon^s(u,v)\geq 0$ and $\varepsilon^t(u,v)\geq 0$. $(u,v)\in D$ and 3*D* is the parametric space.

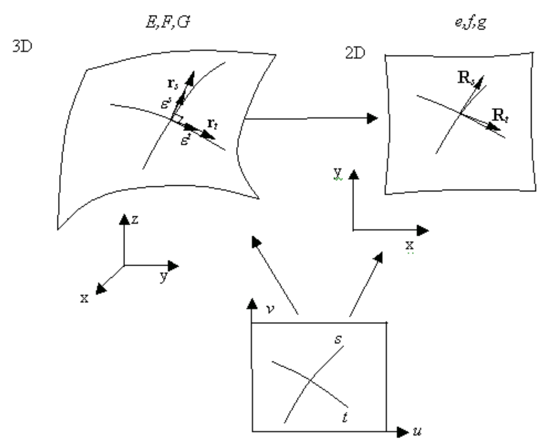


Fig. 2 Doubly curved surface, its planar development and the strain definition.

For the constraint $K_{ps}=0$, Eq. A1 in Appendix will be used by replacing *e, f,* and *g* for *E, F,* and *G*, where *e, f,* and *g* are the first fundamental form coefficients of the developed planar surface $\mathbf{R}(u,v)$ (Fig. 2). It is more convenient to use the form shown in Eq. A1 than in Eq. 6 because only the first fundamental form coefficients and their derivatives are needed. The relationships between *e, f,* and *g* and the given curved surface parameters as well as the unknowns $\varepsilon^s(u,v)$ and $\varepsilon^t(u,v)$ are written in A2. The non-negativity constraints $\varepsilon^s\geq 0$ and $\varepsilon^t\geq 0$ are imposed for the following reason. During a laser forming process, it is known that the strains generated to form a planar shape to a curved surface are mostly compressive, that is, $e^s\leq 0$, and

$e^t \le 0$. As a result, the strains required to develop the curved surface to its planar shape are tensile, that is, $\varepsilon^s \ge 0$ and $\varepsilon^t \ge 0$.

Eq. 7, representing a constrained nonlinear optimization problem, is disretized by using the trapezoidal rule of integration and central difference method for partial derivatives in the constraints. NAG e04ucf routine [6], which implements a sequential quadratic programming method (SQP) is used to solve the constrained nonlinear optimization problem. The strains $\varepsilon^s(u,v)$ and $\varepsilon^t(u,v)$ are on the curved surface and to determine scanning paths on the planar developed surface, they need to be transformed onto the planar surface $e^s(u,v)$ and $e^t(u,v)$. Since $\varepsilon^s(u,v)$ and $\varepsilon^t(u,v)$ represent the strains due to changing from curved surface to its planar development, an infinitesimal length $|\mathbf{r}_s ds|$ changes to $(1+\varepsilon^s)|\mathbf{r}_s ds|$ and $|r_t dt|$ changes to $(1+\varepsilon^t)|\mathbf{r}_t dt|$ (Fig. 1). Therefore, $|\mathbf{R}_s|=(1+\varepsilon^s)|\mathbf{r}_s|$, and $|\mathbf{R}_t|=(1+\varepsilon^t)|\mathbf{r}_t|$, where $\mathbf{r}$ and $\mathbf{R}$ represent the given curved surface and its planar development, and (s,t) represent the principal curvature directions. The strains $e^s(u,v)$ and $e^t(u,v)$ can be expressed in terms of $\varepsilon^s(u,v)$ and $\varepsilon^t(u,v)$ as

$$e^s = -\frac{\varepsilon^s}{1+\varepsilon^s}, \text{ and } e^t = -\frac{\varepsilon^t}{1+\varepsilon^t} \qquad (8)$$

The results are shown in Figs. 3 (a) and (b) for the pillow and saddle cases, respectively. The length of the bars represents the magnitude and the orientation of the bars the direction of the strains.

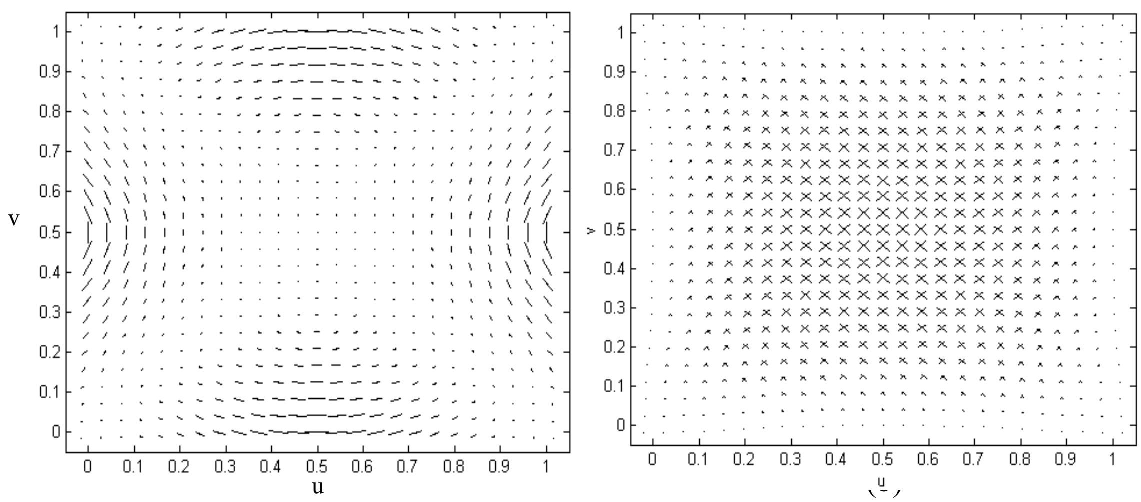


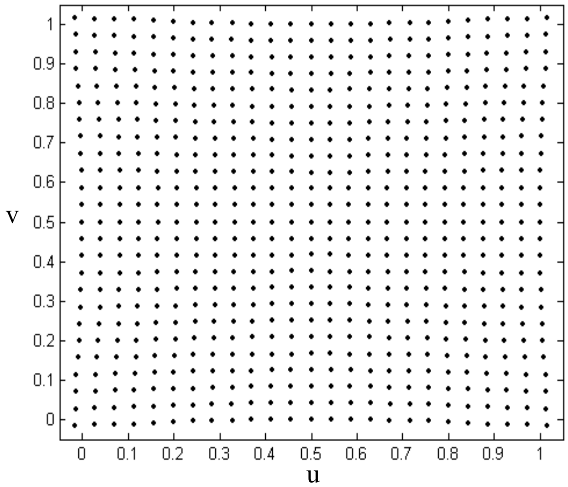
Fig. 3 Strains along the principal curvature directions.(a) pillow case (b) saddle case (the segment length represents the magnitude and the orientation represents the strain direction).

3.3 Planar Developed Shape

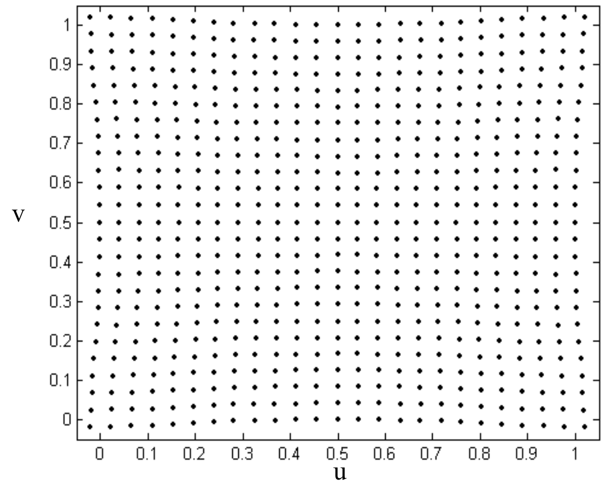
After obtaining the strain field ($\varepsilon_{ij}^{\ s}$ and $\varepsilon_{ij}^{\ t}$) at all grid points, the planar coordinates (X_{ij}, Y_{ij}) of the grid points corresponding to the planar developed shape can be determined. The decision variables to be determined are the $2mn$ coordinates of the grid points $\mathbf{R}_{ij}=(X_{ij}, Y_{ij})$. As a result, Eq. 8 gives $3mn$ known conditions with e, f and g are determined based on ($\varepsilon_{ij}^{\ s}$ and $\varepsilon_{ij}^{\ t}$) and other information of the desired curved surface (Eq. A2). To solve this over-determined system of nonlinear polynomial equations, the following unconstrained least square error minimization problem is solved by using quasi-Newton method in NAG C routine e04fcc.

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} (\mathbf{R}_u \cdot \mathbf{R}_u |_{ij} - e_{ij})^2 + (\mathbf{R}_u \cdot \mathbf{R}_v |_{ij} - f_{ij})^2 + (\mathbf{R}_v \cdot \mathbf{R}_v |_{ij} - g_{ij})^2 \qquad (9)$$

where $\mathbf{R}_u$ and $\mathbf{R}_v$ are expressed in terms of (X_{ij}, Y_{ij}) using the finite difference method. The results are shown in Figs. 4 (a) and (b) for the pillow and saddle cases, respectively.



(a)



(b)

Fig. 4 (a) Planar developed shape of the pillow case, (b) Planar developed shape of the saddle case. (distortion from the original parametric space D: $(u,v) \in [0,1]$ is magnified by a factor of 5 for viewing clarity).

4. Scanning Paths

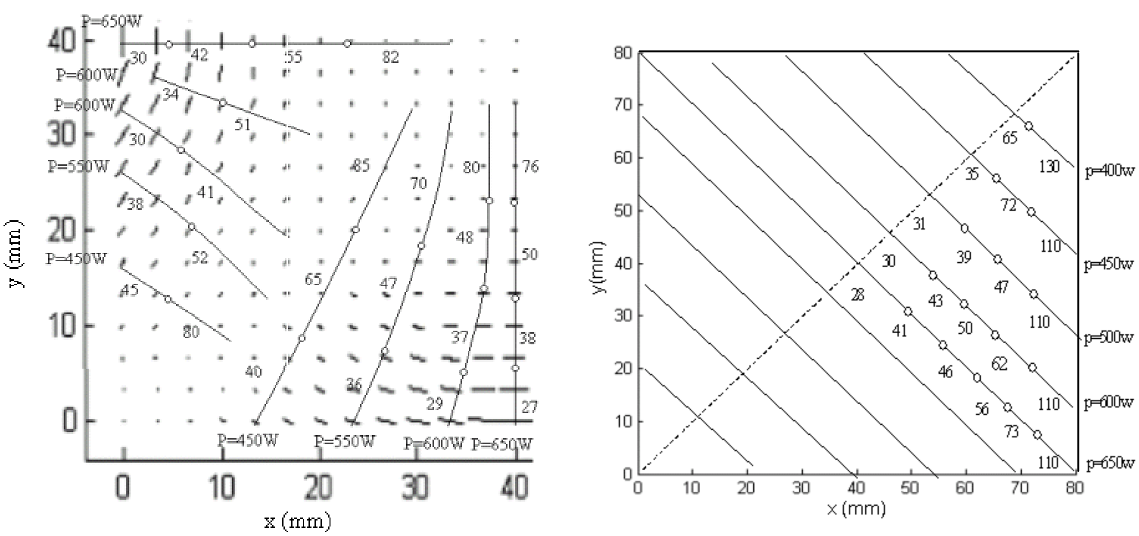
After a strain field required to develop a desired curved shape to its planar shape is determined, scanning paths are designed. It is well known that, in the laser forming process, the highest compressive strains occur in a direction perpendicular to a scanning path and in-plane within workpiece, therefore a scanning path should be perpendicular to the direction of the principal strain. Since the strains developed along the principal curvature directions (ε^s, ε^t) in the preceding section are perpendicular to each other and no shear strain is involved in the process, they can be regarded as the in-plane principal strains, and therefore a scanning path should be perpendicular to the direction of the principal curvature at every point on the path.

For laser forming based the temperature gradient mechanism [7], the target bends towards the heating source. To form a curved surface by laser forming is to superpose these positive curvatures on the appropriate regions of the plate. If the Gaussian curvatures along the lines of curvature are positive, the scanning paths should be placed on one side of the plate, otherwise, they should be placed at both sides of the plate. In the pillow shape, the Gaussian curvatures are positive all over the plate while in the saddle shape, the Gaussian curvatures are all negative.

In determining the spacing of adjacent scanning paths, the following guidelines are followed. In general, the smaller the spacing, the more precise the desired shape can be formed. However, the adjacent paths cannot be too close since they will no longer be independent with each other while independence is a requirement assumed in determining the heating condition in the next section. In addition, too small spacing implies more scanning and thus longer time to form a part.

The strain distribution over the entire plate should also be considered in determining the spacing of adjacent paths. The regions of a shape that has larger strains need to be scanned with denser paths. As a rule of thumb, spacing between two adjacent paths, D_{paths}, should be equal to strain generated by laser forming, ε_{laser}, multiplied by laser beam spot size, d_{laser} and divided by the average principal strain over the spacing. Another consideration is where to not place a scanning path or where to terminate a scanning path. If a region has strains smaller than a particular value of the maximum strain in the plate, say 5%, no scanning paths are placed there. Similarly, if the strain along a scanning is smaller than a particular value, the scanning path is terminated at that point.

Figs. 5 (a) and (b) show the scanning paths determined based on the above principles and guidelines in order to form the pillow shape and saddle shape, respectively. As seen from Fig. 5 (a) (only a quarter of the plate is shown due to symmetry), the scanning paths are particular to the principal curvature directions. This is also the case for Fig. 5(b) where the directions (shown in Fig. 3 (b)) are not shown for viewing clarity. In the pillow case the regions around the mid edges have larger strains, while regions at the corners and the center of the plate have smaller strains. Based on previous discussions, the laser paths should be positioned denser around the mid edges of the plate. On the other hand, in the saddle shape (only a quarter of paths are shown due to symmetry), the strains are larger at the center of the plate and become smaller towards the edges (Fig. 3 (b)). Therefore, laser paths should be denser at the center of the plate. Regions with strains less than 5% of the maximal strain of the plate, no paths are placed. A laser path terminates where the corresponding strain is less than 5% of the maximum strain along the path. As seen, the determination of the scanning paths involves certain practical considerations which are and the solution is obviously not unique.



(a) (b)

Fig 5(a) Optimal planning of laser paths and heating condition (power shown in watts and scanning speed shown in numbers with unit of mm/s) for the (a) pillow shape, (b) saddle shape.

5. Heating Condition

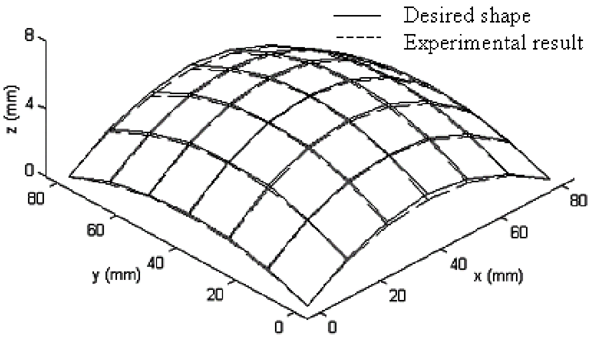
The final stage of 3D laser forming design is to determine the heating condition. If the plate dimension and the laser spot size are given, the heating condition that needs to be determined includes laser power and laser scanning velocity and laser power. In this study, the selected samples are 1010 steel coupons with dimension of 80 by 80 by 0.89 mm. The principal minimum strain averaged over a beam spot size as a function of laser power and scanning velocities determined via finite element analysis of single straight-line (independent) laser scanning of a plate with the above material and dimension. Detailed FEM modeling description can be found in Liu and Yao [3]. There are many possible combinations of power and velocity to realize a given strain. The strategy to determine the heating condition is summarized below. First, the in-plane strains along a scanning path determined in the section 4 are averaged and checked with the FEM database and a laser power level is chosen. This step is to ensure that the power level chosen is readily available in the existing laser forming equipment. The laser power is kept constant for the laser path. The next step is to determine the velocities along the path. In general, the strain distribution along the path is not uniformly distributed; therefore the velocity ideally should vary with the strain along the path. Practically, a laser path is broken into several segments so that in each segment the strain variation is not larger than, say, 1/5 of the maximum strain along the path. A constant velocity is prescribed for each segment based on the average strain of the segment, predetermined laser power, and the existing relationship from FEM database. The average strain of the segment is obtained by averaging the strain along the segment, followed by lumping the strain between adjacent paths.

The heating condition for the two desired shape is decided following the above strategy and superposed in Figs. 5 (a) and (b), respectively. Generally, regions requiring high strains get higher power levels and lower velocities prescribed simply because more energy input is required. For example for the path at x=40mm in Fig. 5 (a), where the highest strain is required, the power is highest at 650W for the path. At locations near y=0 on the path, the strains are larger, therefore the velocity is lowest (v=27mm/s). As the path moves toward the center, the velocity increases sharply, due to the quick drop of strain. Since the strain gradient along the path slows down towards the center of the plate, the segment spacing becomes larger accordingly. On the contrary, the principal strains are lager at the center of the plate for the saddle case shown in Fig. 5 (b), and therefore the spacing of the adjacent paths are denser. But the strain gradients are larger towards edges and corners of the plate, the segment spacing is therefore smaller towards the edges and corners.

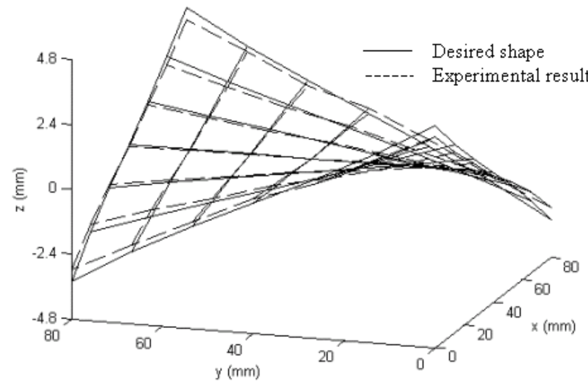
6. Experimental Validation

Laser forming experiments were conducted on 1010 steel coupons of dimension of 80 by 80 by 0.89 mm, the same as used in simulation. The scanning paths and heating condition in the experiments are shown in Figs. 16 and 17 for the pillow and saddle cases, respectively. The laser system used is a PRC-1500 CO_2 laser, which has a maximum output power of 1,500W. Laser beam spot size used is 4 mm. Workpiece movement was controlled by Unidex MMI500 motion control system, which allows convenient specifications of variable velocities along a path with smooth transitions from segment to segment.

Fig. 6 (c) shows the formed pillow and saddle shape under these conditions. A coordinate measuring machine (CMM) is used to measure the geometry of the formed shapes. Figs. 6 (a) and 6 (b) compare the geometry of formed shape under the determined conditions and desired shape. Only the geometry of top surface of the plate is measured and a general agreement can be seen from the figures. There is about 10% error for the saddle case especially at the corners seen from Fig. 6 (b). Possible sources of error include the average and lumped method used to determine strains of path segments, and the approximate method used to determine the laser power and scanning velocity based on independent scans. After all, a strain field required to develop a curved shape to its planar shape is continuous in nature while the laser forming process uses a discrete number of paths to approximate the strain field.



(a)



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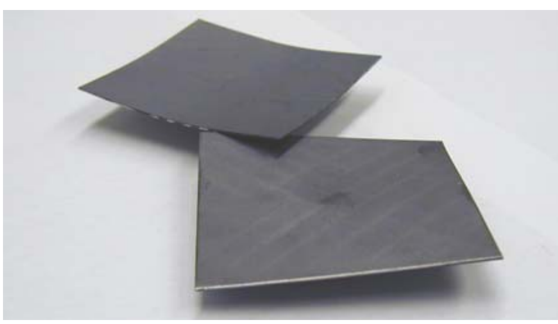

(c)

Fig. 6 Comparison of desired and measured shapes of the (a) pillow case and (b) saddle case. (an array of 7 by 7 points measured by CMM). (c) Laser formed pillow and saddle shapes using the process plans shown in Fig. 5(a) and (b).

Conclusions

The concept of determining a strain field required to develop a given curved shape to its planar shape as the first step of process planning for laser forming is of merit. Placing scanning paths perpendicular to the principal curvature directions, namely along the line of curvatures of a desired shape proves to be unambiguous and easy to implement. Practical constraints need to be combined with analytical ones in determining path spacing and heating condition. There is a trade-off between forming accuracy and efficiency by carefully choosing density of scanning paths and path segments.

Acknowledgement

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Appendix

Eq. 6 shows Gaussian curvature *K* is expressed in terms of the first and second fundamental form coefficients of a surface. In practice, *K* can be alternatively expressed as a function of the first fundamental form coefficients and their derivatives [8].

$$K = \{E(E_v G_v - 2F_u G_v + G_u^2) + F(E_u G_v - E_v G_u - 2E_v F_v + 4E_u F_v - 2F_u G_u) + G(E_u G_u - 2E_u F_v + E_v^2) - 2(EG - F^2)(E_{vv} - 2F_{uv} + G_{uu})\} / 4(EG - F^2)^2 \qquad \text{(A1)}$$

The first fundamental coefficients on the planar shape (*e, f, g*) and desired shape (*E, F, G*) have the following relationship,

$$e = \frac{v_t^2[Eu_s^2 + 2Fu_s v_s + Gv_s^2](1+\varepsilon^s)^2 + v_s^2[Eu_t^2 + 2Fu_t v_t + Gv_t^2](1+\varepsilon^t)^2}{(v_s u_t - u_s v_t)^2}$$

$$f = -\frac{u_t v_t[Eu_s^2 + 2Fu_s v_s + Gv_s^2](1+\varepsilon^s)^2 + u_s v_s[Eu_t^2 + 2Fu_t v_t + Gv_t^2](1+\varepsilon^t)^2}{(v_s u_t - u_s v_t)^2} \qquad \text{(A2)}$$

$$g = \frac{u_t^2[Eu_s^2 + 2Fu_s v_s + Gv_s^2](1+\varepsilon^s)^2 + u_s^2[Eu_t^2 + 2Fu_t v_t + Gv_t^2](1+\varepsilon^t)^2}{(v_s u_t - u_s v_t)^2}$$

For a given curved surface, *E,F,*and *G* as well as u_s ,u_t ,v_s,and v_t can be calculated, and therefore if ε^s and ε^t are calculated, *e, f* and *g* can be calculated accordingly.

Meet the Authors

Chao Liu is a PhD candidate in Mechanical Engineering from Columbia. Y. Lawrence Yao is a Professor of Mechanical Engineering and Director of Manufacturing Research Lab at Columbia University, where his research group works on laser forming, laser shock peening, and laser micro machining. Yao has a Ph.D. from the University of Wisconsin-Madison.