Homework 0

100 points

S1201, Summer 2022
Due July 7, 23:59PM EST

Notes: This is a "calibration" homework, which you may use to self-assess whether this course is suitable for you. More specifically, you can get some sense of how much math you need to know before taking this course. If you can get 60+ in total and 35+ in Problems 2, 4 and 5, it means you have enough preliminary math knowledge to proceed with this course.

(i) Please submit a **single** PDF file on courseworks. You may scan your written solutions or directly write the solutions on your tablet.

(ii) This HW will NOT be graded. You will get full marks as long as you turn it in on courseworks.

(iii) Late submission will lead to some penalty: 20 points off for within 24 hours late, 40 points off for over 24 hours late, all points off for submission after the solutions are posted on courseworks.

(iv) You can discuss the problems with others. But a direct plagiarism will lead to zero point for this assignment, and this will be reported to the university.

1. **10+10=20 points** Answer the following questions:

   (a) Solve the following quadratic equation: $x^2 + 4x + 3 = 0$;

   (b) Find the maximum of function $f(x) = -2x^2 + x$;

Solution: (a) $x^2 + 4x + 3 = (x+3)(x+1) = 0$

$\Rightarrow \quad x = -3$ or -1

(b) $f(x) = \underbrace{-2(x - \frac{1}{4})^2}_{\leq 0} + \frac{1}{8}$

So the maximum is $\frac{1}{8}$, when $x = \frac{1}{4}$.

Or: we can take the derivative of $f(x)$:

let $f'(x) = -4x + 1 = 0$

$\Rightarrow \quad x = \frac{1}{4}$

$\Rightarrow \quad f(x)_{max} = f(\frac{1}{4}) = \frac{1}{8}$. $\square$

2. **5+5+5+5=20 points** Calculate the derivative for the following functions

   (a) $x^6 + 3x^3 + 2$;

   (b) $2x^{-1} + \sqrt{x}$, where $x > 0$;

   (c) $-4 \log x + e^{3x}$, where $x > 0$;

   (d) $x^2 \log x$, where $x > 0$;

Solution:

(a) $6x^5 + 9x^2$

(b) $-\frac{2}{x^2} + \frac{1}{2\sqrt{x}}$ $(x>0)$

(c) $-\frac{4}{x} + 3e^{3x}$ $(x>0)$

(d) $2x\log x + x$ $(x>0)$ (by chain rule). □

3. (Cauchy-Schwarz inequality) **5+5+5+5=20 points**[1]

   (a) $\sqrt{2\times 3}$ and $\frac{2+3}{2}$, which one is larger? What about $\sqrt{2\times 1}$ and $\frac{2+1}{2}$? How about $\sqrt{2\times 2}$ and $\frac{2+2}{2}$?

   (b) Can you guess the relationship between $\sqrt{x_1x_2}$ ("the geometric mean") and $\frac{x_1+x_2}{2}$ ("the arithmetic mean") in general (which should hold for any $x_1, x_2 \geq 0$)?

   (c) When will them equal to each other in part (*b*) (just guess based on the intuition you get from part (a) and (b))?

   (d) Prove your conjecture in (b) and (c) rigorously.

Solution: (a) ① $(\sqrt{2\times 3})^2 = 6$, $\left(\frac{2+3}{2}\right)^2 = 6.25$, the latter is bigger

② $(\sqrt{2\times 1})^2 = 2$, $\left(\frac{2+1}{2}\right)^2 = 2.25$. the later is bigger

③ They are equal

(b) $\sqrt{x_1 x_2} \leq \frac{x_1+x_2}{2}$, for any $x_1, x_2 \geq 0$.

(c) When $x_1 = x_2$.

(d) $(RHS)^2 - (LHS)^2 = \left(\frac{x_1+x_2}{2}\right)^2 - (\sqrt{x_1x_2})^2$

$= \frac{1}{4}(x_1^2 + x_2^2 + 2x_1x_2) - x_1x_2$

$= \frac{1}{4}(x_1^2 + x_2^2 - 2x_1x_2)$

$= \frac{1}{4}(x_1 - x_2)^2 \geq 0$

$\Rightarrow$ RHS $\geq$ LHS. $\square$

[1]We call the inequality in part (b) as the Cauchy-Schwarz inequality. We will use it when we discuss the minimum sample size needed to achieve certain confidence interval precision in Bernoulli sampling (week 4).

The right-hand side of inequality in (b)

The left-hand side ...

4. **5+5+5+5=20 points** Calculate the following integrals:

   (a) $\int_0^1 1dx$;

   (b) $\int_{-1}^3 (x^2+1)dx$;

   (c) $\int_0^5 (2e^{-2x} + x^{-2})dx$;

   (d) $\int_0^4 f(x)dx$, where $f(x) = \begin{cases} 2x^3, & 0 \le x \le 1 \\ 0, & 1 < x < 3 \\ x, & 3 \le x \le 4 \end{cases}$;

Solution: (a) 1.

(b) It equals to $\left(\frac{1}{3}x^3 + x\right)\Big|_{x=-1}^{x=3}$

$= \left(\frac{1}{3}\times 3^3 + 3\right) - \left(\frac{1}{3}\times(-1)^3 + (-1)\right)$

$= \frac{40}{3}$

(c) It equals to $\left(-e^{-2x} - x^{-1}\right)\Big|_{x=0}^{x=5}$

$= \left(-e^{-10} - 1/5\right) - \left(-1 - 0^{-1}\right)$

$+\infty$

$= +\infty$

(I did not mean to make it infinity. Should have changed the lower endpoint to 1. Sorry about that :))

(d) $\int_0^4 f(x)\,dx = \int_0^1 2x^3\,dx + \int_1^3 0\,dx + \int_3^4 x\,dx$

$= \frac{1}{2}x^4\Big|_{x=0}^{x=1} + 0 + \frac{1}{2}x^2\Big|_{x=3}^{x=4}$

$= \frac{1}{2} + \frac{1}{2}\times(4^2 - 3^2)$

$= 4$ □.

5. **10+10=20 points** Define function $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\{-x^2/2\}$. Suppose we know that $\int_{-1}^{1} \phi(x)dx = 0.6827$ and $\int_{-\infty}^{+\infty} \phi(x)dx = 1$. Answer the following questions.

   (a) What's the value of $\int_{-1}^{0} \phi(x)dx$?

   (b) What's the value of $\int_{-\infty}^{-1} \phi(x)dx$?

Solution: (a) Notice that $\phi(x)$ is symmetric around 0 (This means $\phi(x) = \phi(-x)$ for any x).

$$\Rightarrow \int_{-1}^{0} \phi(x)dx = \frac{1}{2}\int_{-1}^{1} \phi(x)dx = \frac{1}{2} \times 0.6827 = 0.34135$$

(b) By the symmetry of ϕ:

$$\int_{-\infty}^{0} \phi(x)dx = \int_{0}^{+\infty} \phi(x)dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{-1} \phi(x)dx = \int_{-\infty}^{0} \phi(x)dx - \int_{-1}^{0} \phi(x)dx$$

$$= \frac{1}{2} - 0.34135$$

$$= 0.15865. \quad \square.$$