Lecture 4: Probability Basics (II)

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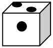
Recap: concepts

Experiment, outcomes, sample space and events:

- A **(random) experiment** is any action or process whose outcome is uncertain. E.g. rolling a die 
- An **outcome** is the result of a random experiment. E.g. the die lands with face 4 up
- The **sample space** of an **experiment**, denoted by Ω, is the **set** of all possible outcomes of that experiment. E.g.: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- An **event** is a set (collection) of outcomes. E.g. consider event $A = \{1\}$, event $B = \{3, 5, 6\}$

Event operations and relationship:

- Union, intersection, difference and complement
- Mutually exclusive events
- Venn diagram

Recap: probability

Definition:

- Given an experiment and a sample space Ω, we would like to know the likelihood that an event can occur, which is defined as **probability**.
- The objective of probability is to assign to each event A a number $\mathbb{P}(A)$, called the probability of the event A
  From this perspective: $\mathbb{P}(\cdot)$: $A \in$ a set of all events $\rightarrow$ a number $\mathbb{P}(A)$ (a mapping/function which maps an event to a number)

Three axioms:

- $0 \leq \mathbb{P}(A) \leq 1$ for any event $A \subseteq \Omega$
- $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$
- If A_1, A_2, A_3, ..., A_n is a collection of disjoint (mutually exclusive) events, then

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Recap: probability calculation

<u>**Theorem**</u>: In an experiment consisting of N outcomes **with equal probability**, denote the number of outcomes that event A contains as $N(A)$. Then for any event A,

$$\mathbb{P}(A) = \frac{N(A)}{N}.$$

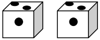
For such experiments, calculating probability reduces to counting the number of outcomes included in the event!

Today's goal

Understand the counting techniques (permutations and combinations) and apply them into probability calculation in the experiment with **equally likely** outcomes

From Product Rule to Permutations

Recall the last example

Toss two fair dice .

Die 2

Die 1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$N = |\Omega| = 6 \times 6 = 36$

- What's the probability that one die is 4 and the other is 1?
  $A = \{(4,1),(1,4)\}$, $\mathbb{P}(A) = 2/N = 2/36 = 1/18$
- What's the probability that the sum of two dice is more than 9?
  $B = \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}$
  $\mathbb{P}(B) = 6/N = 6/36 = 1/6$

Good... But can we always list all outcomes?

An "infeasible" example

Our statistics department has 55 Ph.D. students.

- In how may ways can the first, second and third honors (one student for each honor) be awarded to the students?
- If the honors are given randomly to students (although it's impossible in practice), what's the probability that three students Navid, Arnab and Ye are awarded?
- If the honors are given randomly to students, what's the probability that Ye gets the first award and Arnab gets the second award?

Impossible to list all possible cases... 😅

Product rule

If there are k steps to finish a task, and:

- There are n_1 ways to do the first step
- Given any choice in the first step, there are n_2 ways to do the second step
- Given any choices in the first and second steps, there are n_3 ways to do the third step
- ...
- Given any choices in the first $(k-1)$ steps, there are n_k ways to do the k-th step

Then finally there are $n_1 n_2 \cdots n_k$ ways to finish this task.

Example: course schedule

Tuesday 8:40am class

- Music BC1002
- French UN2102
- Statistics UN1201
- English BC1211
- Humanities UN1123

Tuesday 10:10am class

- Classical civilization UN3220
- Anthropology UN2003
- Chemistry S1404
- Italian UN1102

Question: How many different choices do we have to take two classes?

Answer: By product rule: $5 \times 4 = 20$

Example: course schedule

"Tree diagram":

Tuesday 8:40am class

- Music BC1002
- French UN2102
- Statistics UN1201
- English BC1211
- Humanities UN1123

Tuesday 10:10am class

- Classical civilization UN3220
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A special case: permutations

Any **ordered** sequence of k objects taken from a set of n distinct objects is called a **permutation**. The total number of permutations equals

$$P_{k,n} = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!},$$

where $m! = m(m-1)\cdots 1$ is the "m factorial".

A simple proof by product rule: We take k objects by k steps, and each step we take one object from the remaining objects (that haven't been chosen yet).

- Step 1: n objects $\to$ n choices
- Step 2: $(n-1)$ remaining objects $\to$ $(n-1)$ choices
- Step 3: $(n-2)$ remaining objects $\to$ $(n-2)$ choices
- ...
- Step k: $(n-k+1)$ remaining objects $\to$ $(n-k+1)$ choices

Finally there are $n(n-1)\cdots(n-k+1)$ ways in total.

The previous example

Our statistics department has 55 Ph.D. students.

(1) In how may ways can the first, second and third honors (one student for each honor) be awarded to the students?

(2) If the honors are given randomly to students (although it's impossible in practice), what's the probability that three students Navid, Arnab and Ye are awarded?

(3) If the honors are given randomly to students, what's the probability that Ye gets the first award and Arnab gets the second award?

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(1) By product rule:

- Step 1 (award first prize): 55 choices
- Step 2 (award second prize): 54 choices
- Step 3 (award third prize): 53 choices

Thus $55 \times 54 \times 53$ ways in total

Or by permutations, $P_{3,55} = 55 \times 54 \times 53$.

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(2) By (1), the number of outcomes $N = 55 \times 54 \times 53$. Denote event $A = \{$Navid, Arnab and Ye are awarded$\}$.

$N(A) = ?$ Is it 1? Is it 3? NO guessing! Let's use permutations again (or use product rule).

Is it equivalent to ask "How many different **ordered** sequences are there taken from three **distinct** students?"

$N(A) = P_{3,3} = 3! = 6$

$\Rightarrow \mathbb{P}(A) = \frac{N(A)}{N} = 6/(55 \times 54 \times 53) \approx 3.81 \times 10^{-5}$

The previous example

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(1) In how may ways can the first, second and third honors (one student for each honor) be awarded to the students?

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(3) If the honors are given randomly to students, what's the probability that Ye gets the first award and Arnab gets the second award?

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(3) By (1), the number of outcomes $N = 55 \times 54 \times 53$. Denote event $A = \{$Ye gets the first award and Arnab gets the second award$\}$.
By product rule: $N(A) = 1 \times 1 \times 53 = 53$
$\Rightarrow \mathbb{P}(A) = \frac{N(A)}{N} = 53/(55 \times 54 \times 53) \approx 3.37 \times 10^{-4}$

More examples

A homeowner doing some remodeling requires several kitchen appliances and the services of both a plumbing contractor and an electrical contractor.
The kitchen appliances will all be purchased from the same dealer, and there are 5 dealers in the area. There are 12 plumbing contractors and 9 electrical contractors available in the area.

Question:

(1) How many choices does the homeowner have?

(2) If we randomly choose one way, what's the probability that dealer Joe will be considered?

Solution: (1) Decompose the task into three steps:

- Step 1: choose one from 5 dealers
- Step 2: choose one from 12 plumbing contractors
- Step 3: choose one from 9 electrical contractors

Therefore by product rule, we have $N = 5 \times 12 \times 9 = 540$ ways in total.

More examples (con't)

A homeowner doing some remodeling requires several kitchen appliances and the services of both a plumbing contractor and an electrical contractor.
The kitchen appliances will all be purchased from the same dealer, and there are 5 dealers in the area. There are 12 plumbing contractors and 9 electrical contractors available in the area.
Question:

(1) How many ways can the contractors be chosen?

(2) If we randomly choose one way, what's the probability that dealer Joe will be considered?

Solution: (2) By (1), $N = 5 \times 12 \times 9 = 540$. Denote event $A = \{\text{dealer Joe is selected}\}$. Again by product rule, $N(A) = 12 \times 9$.
Then $\mathbb{P}(A) = \frac{N(A)}{N} = (12 \times 9)/(5 \times 12 \times 9) = 1/5$.
Remark: Is there a faster way to get the answer $1/5$?

Combinations

A modified version of the previous example

Our statistics department has 55 Ph.D. students. There is a course that nobody wants to be the TA for. So the department decides to randomly select four students to do it.

- In how may ways can the department assign the TA?
- What's the probability that Navid, Arnab, Ye and Collin are selected?

Here the order is NOT important. All that matters is which four are selected.

Combinations

Any of k objects (**unordered**) taken from a set of n distinct objects is called a **combination**. The number of combinations obtained when selecting k objects from n distinct objects to form a group is given by

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 1}.$$

A simple proof: Here we don't care about the order of k objects. Therefore all orderings of the same k objects are **equivalent**.
By permutations, we have $P_{k,n}$ permutations, where different orderings are considered as different ways. For any specific k objects, there are $P_{k,k} = k!$ orderings. Therefore the number of combinations equals $P_{k,n}/k!$.

A modified version of the previous example

Our statistics department has 55 Ph.D. students. There is a course that nobody wants to be the TA for. So the department decides to randomly select four students to do it.

- In how may ways can the department assign the TA?
- What's the probability that Navid, Arnab, Ye and Collin are selected?

Solution:

- $\binom{55}{4} = \frac{55\times 54\times 53\times 52}{4!}$
- $N = \binom{55}{4}$. Denote event
  $A = \{$Navid, Arnab, Ye and Collin are selected$\}$. Then $N(A) = 1$ (this is a combination problem and the order doesn't matter!).
  Hence $\mathbb{P}(A) = N(A)/N = 1/\binom{55}{4}$.

Example: soda

(1) One can is randomly selected. What's the probability that it is a Pepsi?

Answer: $2/11$

(2) Two cans are randomly selected. What's the probability that they are both Sprites?

Answer: $N = \binom{11}{2} = 55.$

Choose 2 Sprites from 4 $\Rightarrow \binom{4}{2} = 6$ ways

Therefore the probability $= \frac{6}{55} \approx 0.109.$

Example: soda

(3) Two cans are randomly selected. What's the probability that they are the same?

Answer: $N = \binom{11}{2} = 55$.

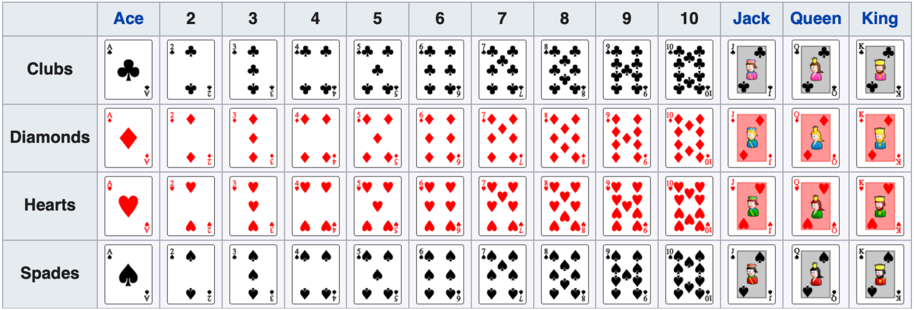
How many basic outcomes does event "Two cans are the same" include?

- Choose 2 from 3 Cokes $\Rightarrow \binom{3}{2} = 3$ ways
- Choose 2 from 2 Pepsis $\Rightarrow \binom{2}{2} = 1$ way
- Choose 2 from 2 Mountain Dews $\Rightarrow \binom{2}{2} = 1$ way
- Choose 2 from 4 Sprites $\Rightarrow \binom{4}{2} = 6$ ways

In total, there're $3 + 1 + 1 + 6 = 11$ outcomes.
Therefore the probability $= \frac{11}{55} \approx 0.2$.

A more complicated example

13 cards are selected at random from a 52-card deck.

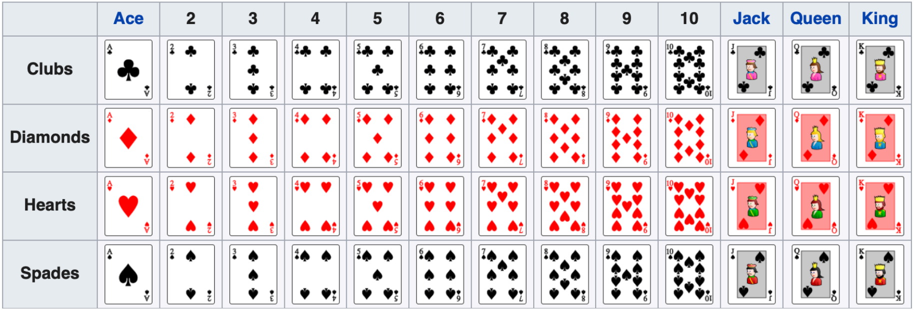


(1) How many possible hands are there?

Answer: $\binom{52}{13} = 635013559600$

A more complicated example

13 cards are selected at random from a 52-card deck.



(2) What is the probability of getting a hand consisting entirely of spades (♠) and clubs (♣) with at least one card of each suit?

Answer: By (1), $N = \binom{52}{13}$.

Choose 13 from 26 spades and clubs $\Rightarrow \binom{26}{13}$ ways

Subtract two cases where we get 13 ♠ or 13 ♣ $\Rightarrow \left[\binom{26}{13} - 2\right]$ ways

Hence the probability $= \left[\binom{26}{13} - 2\right] / \binom{52}{13} = 1.64 \times 10^{-5}$

A more complicated example

13 cards are selected at random from a 52-card deck.

(3) What is the chance of getting a hand consisting of exactly two suits?

Answer: product rule + combinations

- Step 1: choose 2 from 4 suits $\Rightarrow \binom{4}{2} = 6$ ways
- Step 2: Given 2 specific suits, from (2) we know that there are $\left[\binom{26}{13} - 2\right]$ ways

Thus the probability $= \frac{6 \times \left[\binom{26}{13} - 2\right]}{\binom{52}{13}} = 9.84 \times 10^{-5}$

(4) What is the chance of getting a hand of $4 \times \spadesuit$-$4 \times \heartsuit$-$3 \times \diamondsuit$-$2 \times \clubsuit$?

Answer: product rule + combinations

By (1), $N = \binom{52}{13}$

- Step 1: Choose 4 $\times\spadesuit$ from 13 $\times\spadesuit$ $\Rightarrow \binom{13}{4}$ ways
- Step 2: choose 4 $\times\heartsuit$ from 13 $\times\heartsuit$ $\Rightarrow \binom{13}{4}$ ways
- Step 3: choose 3 $\times\diamondsuit$ from 13 $\times\diamondsuit$ $\Rightarrow \binom{13}{3}$ ways
- Step 4: choose 2 $\times\clubsuit$ from 13 $\times\clubsuit$ $\Rightarrow \binom{13}{2}$ ways

Hence the probability $= \frac{\binom{13}{4}\binom{13}{4}\binom{13}{3}\binom{13}{2}}{\binom{52}{13}}$

Reading list (optional)

- "Probability and Statistics for Engineering and the Sciences" (9th edition):
  - Chapter 2.3

Many thanks to

- Joyce Robbins
- Yang Feng
- Chengliang Tang
- Owen Ward
- Wenda Zhou
- And all my teachers in the past 25 years