Lecture 5: Conditional Probability

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Recap: product rule and permutations

Product rule: If there are k steps to finish a task, and:

◦ There are n_1 ways to do the first step

◦ Given any choice in the first step, there are n_2 ways to do the second step

◦ ...

◦ Given any choices in the first $(k-1)$ steps, there are n_k ways to do the k-th step

Then finally there are $n_1 n_2 \cdots n_k$ ways to finish this task.

Permutations: Any **ordered** sequence of k objects taken from a set of n distinct objects is called a **permutation**. The total number of permutations equals

$$P_{k,n} = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}.$$

Recap: combinations

Any of k objects (**unordered**) taken from a set of n distinct objects is called a **combination**. The number of combinations obtained when selecting k objects from n distinct objects to form a group is given by

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 1}.$$

Recap: permutation problem or combination problem?

(1) Choose 5 people from 100 to do a job

(2) Choose the winner of the first prize and second prize from 20 people

(3) Choose two winners of the first prize from 20 people

(4) Draw 4 cards from a 52-card deck (the order doesn't matter)

(5) Draw 4 cards from a 52-card deck (the order matters)

(6) Figure out a way to put 5 people in a line to get the vaccine

Recap: probability calculation

<u>**Theorem**</u>: In an experiment consisting of N outcomes **with equal probability**, denote the number of outcomes that event A contains as $N(A)$. Then for any event A,

$$\mathbb{P}(A) = \frac{N(A)}{N}.$$

Counting techniques help us to calculate $N(A)$ and N.

Today's goal

- Understand conditional probabilities
- Know the multiplication rule and rule of total probability
- Understand Bayes Theorem and its statistical meanings
- Use all techniques we learn to calculate the probability

Conditional Probability

Conditional probability

- We have learned that given the experiment and the sample space Ω, for an event A, $\mathbb{P}(A)$ represents the probability that A occurs, where $\mathbb{P}(\cdot)$ is a function mapping any event $A \in \Omega$ to a number between 0 and 1.
- Here we want to discuss how the information "an event B has occurred" affects the probability assigned to A
- Many events are actually **related** and the event that has happened can provide information about the event that hasn't occurred yet.

Examples:

- Two Columbia students, Claire and Jack, are sampled to do a Covid test.
  - ▹ Claire feels good while Jack had a fever last night.
  - ▹ Who is more likely to test positive?
- I randomly sample a person on the street in NYC and ask whether he/she knows me or not.
  - ▹ Will the probability that this guy knows me be very high?
  - ▹ What if I know this guy is a Columbia statistics Ph.D student?

Conditional probability

Definition: **Conditional probability** of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Remark:

- It represents the probability that A occurs given that B has occurred
- We require $\mathbb{P}(B) > 0$ to make it useful
- **Fixing an event** B, we can see $\mathbb{P}(\cdot|B)$ as a probability function, which satisfies all properties of probability we described before. For example:
  - $0 \leq \mathbb{P}(A|B) \leq 1$ for any event A, $\mathbb{P}(\emptyset|B) = 0$, $\mathbb{P}(\Omega|B) = 1$
  - $\mathbb{P}(A \cup C|B) = \mathbb{P}(A|B) + \mathbb{P}(C|B) - \mathbb{P}(A \cap C|B)$ for any A and C
  - $\mathbb{P}(\cup_{i=1}^n A_i|B) = \sum_{i=1}^n \mathbb{P}(A_i|B)$ if $A_1, \ldots, A_n$ are mutually exclusive
  - $\mathbb{P}(\cdot|B)$ is a function mapping any event $A \in \Omega$ to a number between 0 and 1.
- Ω is the sample space when we study $\mathbb{P}(A)$, and $\Omega \cap B = B$ can be seen as the **new sample space** when we study $\mathbb{P}(A|B)$!

Example

Conditional probability of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Example: Suppose that of all individuals buying a certain digital camera:

- 60% include an optional memory card in their purchase
- 40% include an extra battery
- 30% include both a card and battery.

Consider randomly selecting a buyer, what's the probability that the buyer purchased the memory card given that he/she purchased the battery?

Solution: Let $A = \{\text{memory card purchased}\}$ and $B = \{\text{battery purchased}\}$. We want to know $\mathbb{P}(A|B)$.
Therefore, $\mathbb{P}(A) = 0.6$, $\mathbb{P}(B) = 0.4$ and $\mathbb{P}(A \cap B) = 0.3$. Hence,

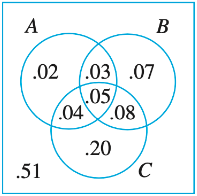
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.3}{0.4} = 0.75.$$

More examples

A news magazine publishes three columns entitled "Art" (A), "Books" (B), and "Cinema" (C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	.14	.23	.37	.08	.09	.13	.05

What's the probability that *the selected individual regularly reads the Art column given that he or she regularly reads at least one of the other two columns*?



Solution:

$$\mathbb{P}(A|B \cup C) = \frac{\mathbb{P}(A \cap (B \cup C))}{\mathbb{P}(B \cup C)} = \frac{0.04+0.03+0.05}{0.2+0.04+0.08+0.05+0.03+0.07} = 0.255.$$

The multiplication rule

Multiplication rule:

- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$
- $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A \cap B)$
- $\mathbb{P}(\cap_{i=1}^{n} A_i) =$
  $\mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \ldots\ldots A_{n-1}).$

Remark:

- This is very useful when calculating the probability of intersection of events
- Intuition: process information "step by step"

Example

A box contains 10 tickets. Three of them are winning tickets. Pick three at random **without replacement**.

- What is the chance that the first three randomly picked tickets are the winning ones?
- What is the chance that the second ticket is a winning one?

- **Solution 1:** Let A and B be the event of interest in two subproblems.
  - View it as an **unordered** problem: Sample space $\Omega = \{$all ways to draw 3 from 10 tickets$\}$. We draw 3 tickets at the same time and the order doesn't matter.
    $N = N(\Omega) = \binom{10}{3} = 120$
    $N(A) = \binom{3}{3} = 1$. Hence, $\mathbb{P}(A) = \frac{N(A)}{N} = 1/120$.
  - View it as an **ordered** problem: Sample space $\Omega = \{$all ways to draw 3 from 10 tickets in 3 steps$\}$. In each of three steps, we draw one ticket.
    By product rule: $N = N(\Omega) = 10 \times 9 \times 8 = 720$
    $N(A) = 3 \times 2 \times 1 = 6$. Hence, $\mathbb{P}(A) = \frac{N(A)}{N} = 1/120$.

Example

A box contains 10 tickets. Three of them are winning tickets. Pick three at random **without replacement**.

- What is the chance that the first three randomly picked tickets are the winning ones?
- What is the chance that the second ticket is a winning one?

- **Solution 2:** Let W_i be the event that the i-th draw is a winning ticket.

$$\begin{aligned}\mathbb{P}(W_1 \cap W_2 \cap W_3) &= \mathbb{P}(W_1)\mathbb{P}(W_2|W_1)\mathbb{P}(W_3|W_1 \cap W_2) \\ &= \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} \\ &= \frac{1}{120}\end{aligned}$$

Think about how we obtain the values of $\mathbb{P}(W_1)$, $\mathbb{P}(W_2|W_1)$, $\mathbb{P}(W_3|W_1 \cap W_2)$.

Example

A box contains 10 tickets. Three of them are winning tickets. Pick three at random **without replacement**.

- What is the chance that the first three randomly picked tickets are the winning ones?
- What is the chance that the second ticket is a winning one?

- **<u>Solution 1:</u>** We have to view it as an **ordered** problem. Denote the event as B. Sample space $\Omega = \{$all possible ways to select 3 from 10 tickets in 3 steps$\}$. In each of three steps, we draw one ticket.
  By product rule: $N = N(\Omega) = 10 \times 9 \times 8 = 720$
  Event B contains all outcomes in the form "XWX", where X can be any tickets while W is the winning one.
  $N(B) = \underbrace{P_{1,3}}_{\text{the 2nd ticket}} \times \underbrace{P_{2,10-1}}_{\text{the 1st, 3rd ticket}} = 3 \times 9 \times 8$. Hence,
  $\mathbb{P}(B) = \frac{N(B)}{N} = 3/10$.

Example

A box contains 10 tickets. Three of them are winning tickets. Pick three at random **without replacement**.

- What is the chance that the first three randomly picked tickets are the winning ones?
- What is the chance that the second ticket is a winning one?

- **Solution 2:** Let W_i be the event that the i-th draw is a winning ticket.

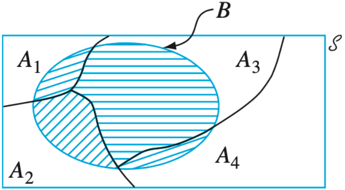
$$\begin{aligned}\mathbb{P}(W_2) &= \mathbb{P}(W_1 \cap W_2) + \mathbb{P}(W_1^c \cap W_2) \\ &= \mathbb{P}(W_1)\mathbb{P}(W_2|W_1) + \mathbb{P}(W_1^c)\mathbb{P}(W_2|W_1^c) \\ &= \frac{3}{10} \times \frac{2}{9} + \frac{7}{10} \times \frac{3}{9} \\ &= \frac{3}{10}\end{aligned}$$

Think about how we obtain the values of $\mathbb{P}(W_1)$, $\mathbb{P}(W_2|W_1)$, $\mathbb{P}(W_2|W_1^c)$.

Rule of total probability

Rule of total probability: If $\Omega = A_1 \cup A_2 \cup \cdots \cup A_k$ and $A_1, \ldots, A_k$ are mutually exclusive (such $A_1, \ldots, A_k$ are called a **partition** of the sample space), then for any event B,

$$\mathbb{P}(B) = \mathbb{P}(A_1)\mathbb{P}(B|A_1) + \mathbb{P}(A_2)\mathbb{P}(B|A_2) + \ldots + \mathbb{P}(A_k)\mathbb{P}(B|A_k)$$



Proof: $B = \cup_{i=1}^{k}(A_k \cap B)$ where these $A_k \cap B$ are mutually exclusive (why?).Then $\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(A_k \cap B) = \sum_{i=1}^{k} \mathbb{P}(A_k)\mathbb{P}(B|A_k)$.

Remark: Sometimes the conditional probability $\mathbb{P}(B|A_k)$ is much straightforward to calculate than $\mathbb{P}(B)$. Then the rule of total probability can be useful.

Example

An individual has 3 different email accounts.

- 70% of her messages come into account #1
- 20% come into account #2
- the remaining 10% into account #3

Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively. What is the probability that a randomly selected message is spam?

Solution: First we translate the problem by math language. Denote $A_i = \{\text{message is from account } \#i\}$, $i = 1, 2, 3$, $B = \{\text{message is spam}\}$. The problem implies that

$$\mathbb{P}(A_1) = 0.7, \quad \mathbb{P}(A_2) = 0.2, \quad \mathbb{P}(A_3) = 0.1$$
$$\mathbb{P}(B|A_1) = 0.01, \quad \mathbb{P}(B|A_2) = 0.02, \quad \mathbb{P}(B|A_3) = 0.05.$$

Finally by rule of total probability,
$\mathbb{P}(B) = \mathbb{P}(A_1)\mathbb{P}(B|A_1) + \mathbb{P}(A_2)\mathbb{P}(B|A_2) + \mathbb{P}(A_3)\mathbb{P}(B|A_3) = 0.016.$

Bayes' Theorem

Bayes' Theorem: If $A_1, \ldots, A_k$ are a partition of sample space Ω, then for any event B,

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(A_j \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_j)\mathbb{P}(A_j)}{\sum_{i=1}^{k} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}, \quad j = 1, \ldots, k.$$

Remark:

- Proof: by rule of total probability
- In many cases, B is the "consequence" while A_1, ..., A_k are different "conditions" or "causes" of B. We may know each $\mathbb{P}(B|A_i)$ (given cause A_i, the probability that consequence B happens), and would like to infer "the probability that the consequence B is caused by A_j", i.e. the probability that A_j occurs given that B has occurred (which is $\mathbb{P}(A_j|B)$).

Example (revisited)

An individual has 3 different email accounts.

- 70% of her messages come into account #1
- 20% come into account #2
- the remaining 10% into account #3

Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively. We randomly select a *spam* message. What's the probability that it is from account #2?

Solution: Denote $A_i = \{\text{message is from account } \#i\}$, $i = 1, 2, 3$, $B = \{\text{message is spam}\}$. The problem implies that

$$\mathbb{P}(A_1) = 0.7, \quad \mathbb{P}(A_2) = 0.2, \quad \mathbb{P}(A_3) = 0.1$$
$$\mathbb{P}(B|A_1) = 0.01, \quad \mathbb{P}(B|A_2) = 0.02, \quad \mathbb{P}(B|A_3) = 0.05.$$

Then $\mathbb{P}(A_2|B) = \frac{\mathbb{P}(B|A_2)\mathbb{P}(A_2)}{\mathbb{P}(B)} = \frac{0.02\times 0.2}{0.016} = 0.25.$

Example (revisited)

Denote $A_i = \{\text{message is from account } \#i\}$, $i = 1, 2, 3$,
$B = \{\text{message is spam}\}$. The problem implies that

$$\mathbb{P}(A_1) = 0.7, \quad \mathbb{P}(A_2) = 0.2, \quad \mathbb{P}(A_3) = 0.1$$
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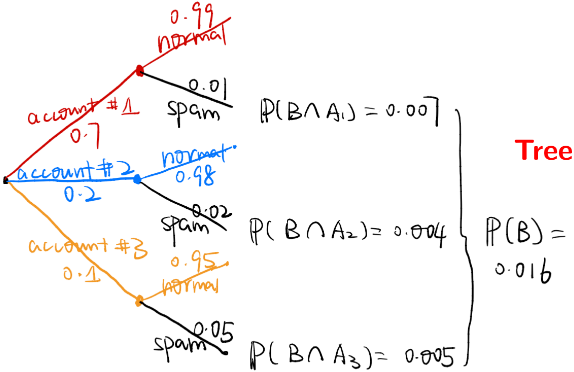
Then $\mathbb{P}(A_2|B) = \frac{\mathbb{P}(B|A_2)\mathbb{P}(A_2)}{\mathbb{P}(B)} = \frac{0.02\times 0.2}{0.016} = 0.25.$



Tree diagram may help!

More examples

Lab tests produce positive and negative results. Assume that for a lab test, *95% of patients with disease obtain positive results and only 2% of patients without disease obtain positive results*. Assume that 1% of the population has the disease.

- Pick one person at random. What is the chance that lab test will be positive?
- Pick one person at random. The lab test shows positive result. What is the chance that the person really has the disease?
- What is the probability that, given that the lab test shows a negative result, the person does not have the disease?

Solution: First we translate the problem by math language. Denote $A = \{\text{test positive}\}$, $D = \{\text{have the disease}\}$.Then

$$\mathbb{P}(A|D) = 0.95, \quad \mathbb{P}(A|D^c) = 0.02, \quad \mathbb{P}(D) = 0.01.$$

- $\mathbb{P}(A) = \mathbb{P}(D)\mathbb{P}(A|D) + \mathbb{P}(D^c)\mathbb{P}(A|D^c) = 0.01 \times 0.95 + 0.99 \times 0.02 = 2.93\%$

More examples

Lab tests produce positive and negative results. Assume that for a lab test, *95% of patients with disease obtain positive results and only 2% of patients without disease obtain positive results*. Assume that 1% of the population has the disease.

- Pick one person at random. What is the chance that lab test will be positive?
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Solution: First we translate the problem by math language. Denote $A = \{\text{test positive}\}$, $D = \{\text{have the disease}\}$.Then

$$\mathbb{P}(A|D) = 0.95, \quad \mathbb{P}(A|D^c) = 0.02, \quad \mathbb{P}(D) = 0.01.$$

- $\mathbb{P}(D|A) = \frac{\mathbb{P}(D\cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|D)\mathbb{P}(D)}{\mathbb{P}(A)} = \frac{0.95\times 0.01}{2.93\%} = 32.4\%$

More examples

Lab tests produce positive and negative results. Assume that for a lab test, *95% of patients with disease obtain positive results and only 2% of patients without disease obtain positive results*. Assume that 1% of the population has the disease.

- Pick one person at random. What is the chance that lab test will be positive?
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Solution: First we translate the problem by math language. Denote $A = \{\text{test positive}\}$, $D = \{\text{have the disease}\}$. Then

$$\mathbb{P}(A|D) = 0.95, \quad \mathbb{P}(A|D^c) = 0.02, \quad \mathbb{P}(D) = 0.01.$$

- $\mathbb{P}(D^c|A^c) = \frac{\mathbb{P}(D^c \cap A^c)}{\mathbb{P}(A^c)} = \frac{\mathbb{P}(A^c|D^c)\mathbb{P}(D^c)}{\mathbb{P}(A^c)} = \frac{[1-\mathbb{P}(A|D^c)]\mathbb{P}(D^c)}{1-\mathbb{P}(A)} = 99.95\%$

More examples: coffee purchase

The accompanying table gives information on the type of coffee selected by someone purchasing a single cup at a particular airport kiosk.

	Small	Medium	Large
Regular	14%	20%	26%
Decaf	20%	10%	10%

Consider randomly selecting such a coffee purchaser.

- What is the probability that the individual purchased a small cup? A cup of decaf coffee?

We can see that $S = (S \cap R) \cup (S \cap D)$ which are mutually exclusive, thus $\mathbb{P}(S) = \mathbb{P}(S \cap R) + \mathbb{P}(S \cap D) = 14\% + 20\% = 34\%$.
Similarly, $D = (D \cap S) \cup (D \cap M) \cup (D \cap L)$, thus
$\mathbb{P}(D) = \mathbb{P}(D \cap S) + \mathbb{P}(D \cap M) + \mathbb{P}(D \cap L) = 20\% + 10\% + 10\% = 40\%$

More examples: coffee purchase

The accompanying table gives information on the type of coffee selected by someone purchasing a single cup at a particular airport kiosk.

	Small	**Medium**	**Large**
Regular	14%	20%	26%
Decaf	20%	10%	10%

Consider randomly selecting such a coffee purchaser.

- If we learn that the selected individual purchased a small cup, what now is the probability that he/she chose decaf coffee?

$\mathbb{P}(D|S) = \frac{\mathbb{P}(D \cap S)}{\mathbb{P}(S)} = \frac{20\%}{34\%} = 58.8\%$

More examples: coffee purchase

The accompanying table gives information on the type of coffee selected by someone purchasing a single cup at a particular airport kiosk.

	Small	Medium	Large
Regular	14%	20%	26%
Decaf	20%	10%	10%

Consider randomly selecting such a coffee purchaser.

◦ If we learn that the selected individual purchased decaf, what now is the probability that a small size was selected?

$$\mathbb{P}(S|D) = \frac{\mathbb{P}(S \cap D)}{\mathbb{P}(D)} = \frac{20\%}{40\%} = 50\%$$

More examples: accident rate and seatbelt wearing

A recent Maryland highway safety study found that in 77% of all accidents the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers escaped serious injury (hospitalization or death), but only 63% of the nonbelted drivers were so fortunate.
What's the probability that a driver who was seriously injured wasn't wearing a seatbelt?

Denote $B = \{\text{the driver was wearing a seatbelt}\}$ and $I = \{\text{serious injury or death}\}$. Then

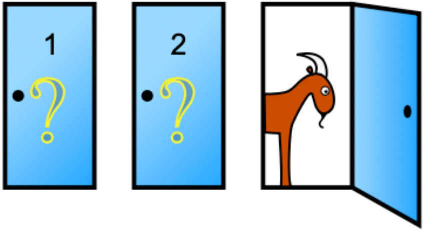
$$\mathbb{P}(B) = 0.77, \quad \mathbb{P}(I|B) = 1 - 0.92 = 0.08, \quad \mathbb{P}(I|B^c) = 1 - 0.63 = 0.37.$$

Thus
$$\mathbb{P}(B^c|I) = \frac{\mathbb{P}(I \cap B^c)}{\mathbb{P}(I)} = \frac{\mathbb{P}(I|B^c)\mathbb{P}(B^c)}{\mathbb{P}(I|B^c)\mathbb{P}(B^c) + \mathbb{P}(I|B)\mathbb{P}(B)} = \frac{0.37 \times 0.23}{0.37 \times 0.23 + 0.08 \times 0.77} = 0.580$$

The last example: Monty Hall problem

You're given the choice of three doors: Behind one door is an iPhone 13; behind the others, goats.
You pick a door, say No.1, and the host, **who knows what's behind the doors**, opens another door, say No.3, which has a goat. He then says to you, "Do you want to pick door No. 2?"

Question: Would you switch your choice?



We have to make an **assumption** about the host: If the iPhone is behind the door we pick, then the host randomly picks one door from the remaining two to open.

Reading list (optional)

- "Probability and Statistics for Engineering and the Sciences" (9th edition):
  - Chapter 2.4
- "OpenIntro statistics" (4th edition, free online, download [here]):
  - Chapter 3.2 (skip Chapter 3.2.6 for now)

Many thanks to

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