# Lecture 6: Independence and Probability Calculation

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Calculus-based Introduction to Statistics (S1201)

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# Recap: conditional probability

**Definition:** **Conditional probability** of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

**Multiplication rule:**

- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$
- $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A \cap B)$

**Rule of total probability:** If $\Omega = A_1 \cup A_2 \cup \cdots \cup A_k$ and $A_1, \ldots, A_k$ are mutually exclusive (such $A_1, \ldots, A_k$ are called a **partition** of the sample space), then for any event $B$,

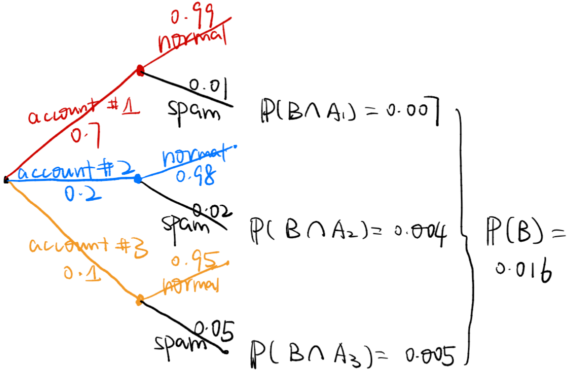
$$\mathbb{P}(B) = \mathbb{P}(A_1)\mathbb{P}(B|A_1) + \mathbb{P}(A_2)\mathbb{P}(B|A_2) + \ldots + \mathbb{P}(A_k)\mathbb{P}(B|A_k)$$

# Recap: Bayes' Theorem

**Bayes' Theorem:** If $A_1, \ldots, A_k$ are a partition of sample space $\Omega$, then for any event $B$,

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(A_j \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_j)\mathbb{P}(A_j)}{\sum_{i=1}^{k} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}, \quad j = 1, \ldots, k.$$

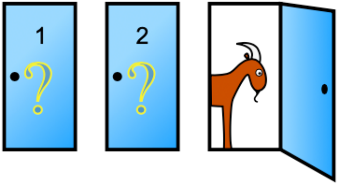
Tree diagram:



# Last lecture: Monty Hall problem

You're given the choice of three doors: Behind one door is an iPhone 13; behind the others, goats.
You **randomly** pick a door, say No.1, and the host, **who knows what's behind the doors**, opens another door, say No.3, which has a goat. He then says to you, "Do you want to pick door No.2?"

**Question:** Would you switch your choice?




**Underlying assumptions:**

- If the iPhone is behind the door we pick, then the host **randomly** picks one door from the remaining two to open.
- Otherwise, the host opens the door of goat.

# Last lecture: Monty Hall problem

**Solution (by Bayes Theorem):**

◦ $\mathbb{P}$(iPhone is behind door $i$) $= 1/3$, $i = 1, 2, 3$

Then

$$
\begin{aligned}
&\mathbb{P}(\text{win iPhone by staying with } \#1)\\
&= \mathbb{P}(\text{iPhone behind } \#1 | \text{host opens } \#3 \text{ and see a goat})\\
&= \frac{\mathbb{P}(\text{host opens } \#3 \text{ and see a goat} | \text{iPhone behind } \#1)\mathbb{P}(\text{iPhone behind } \#1)}{\sum_{i=1}^{3} \mathbb{P}(\text{host opens } \#3 \text{ and see a goat} | \text{iPhone behind } \#i)\mathbb{P}(\text{iPhone behind } \#i)}\\
&= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3}}\\
&= \frac{1}{3}.
\end{aligned}
$$

# Last lecture: Monty Hall problem

**Solution (by Bayes Theorem):**

∘ $\mathbb{P}(\text{iPhone is behind door } i) = 1/3$, $i = 1, 2, 3$

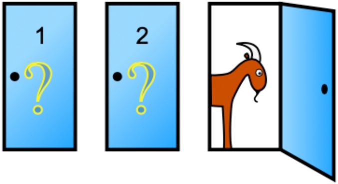
Then

$$\begin{aligned}
&\mathbb{P}(\text{win iPhone after switching to \#2}) \\
&= \mathbb{P}(\text{iPhone behind \#2} | \text{host opens \#3 and see a goat}) \\
&= \frac{\mathbb{P}(\text{host opens \#3 and see a goat} | \text{iPhone behind \#2})\mathbb{P}(\text{iPhone behind \#2})}{\sum_{i=1}^{3} \mathbb{P}(\text{host opens \#3 and see a goat} | \text{iPhone behind \#}i)\mathbb{P}(\text{iPhone behind \#}i)} \\
&= \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3}} \\
&= \frac{2}{3} > \frac{1}{3} \quad \Longrightarrow \text{We should switch!}
\end{aligned}$$

# Last lecture: Monty Hall problem (modified)

You **randomly** pick a door, say No.1, and the host, **who doesn't know what's behind the doors**, **randomly** opens another door among the remaining two, say No.3, which has a goat. He then says to you, "Do you want to pick door No.2?"

**Question:** Would you switch your choice?




<u>**Underlying assumptions:**</u>

- No matter which one you originally pick, the host always **randomly** picks one door from the remaining two to open.

# Last lecture: Monty Hall problem (modified)

**Solution (by Bayes Theorem):**

∘ $\mathbb{P}(\text{iPhone is behind door } i) = 1/3$, $i = 1, 2, 3$

Then

$$\begin{aligned}
&\mathbb{P}(\text{win iPhone by staying with } \#1) \\
&= \mathbb{P}(\text{iPhone behind } \#1 | \text{host opens } \#3 \text{ and see a goat}) \\
&= \frac{\mathbb{P}(\text{host opens } \#3 \text{ and see a goat} | \text{iPhone behind } \#1)\mathbb{P}(\text{iPhone behind } \#1)}{\sum_{i=1}^{3} \mathbb{P}(\text{host opens } \#3 \text{ and see a goat} | \text{iPhone behind } \#i)\mathbb{P}(\text{iPhone behind } \#i)} \\
&= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3}} \\
&= \frac{1}{2}.
\end{aligned}$$

# Last lecture: Monty Hall problem (modified)

**Solution (by Bayes Theorem):**

∘ $\mathbb{P}(\text{iPhone is behind door } i) = 1/3$, $i = 1, 2, 3$

Then

$$\begin{aligned}
&\mathbb{P}(\text{win iPhone after switching to \#2}) \\
&= \mathbb{P}(\text{iPhone behind \#2 and see a goat}|\text{host opens \#3}) \\
&= \frac{\mathbb{P}(\text{host opens \#3 and see a goat}|\text{iPhone behind \#2})\mathbb{P}(\text{iPhone behind \#2})}{\sum_{i=1}^{3} \mathbb{P}(\text{host opens \#3 and see a goat}|\text{iPhone behind \#}i)\mathbb{P}(\text{iPhone behind \#}i)} \\
&= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3}} \\
&= \frac{1}{2} \qquad \Longrightarrow \text{It doesn't matter!!}
\end{aligned}$$

# Reflection on Monty Hall problem

- Why is the answer so different?
- A more intuitive explanation of the **original** Monty Hall problem:

| Behind door 1 | Behind door 2 | Behind door 3 | Result if staying at door #1 | Result if switching to the door offered |
|---|---|---|---|---|
| Goat | Goat | **iPhone** | Wins goat | **Wins iPhone** |
| Goat | **iPhone** | Goat | Wins goat | **Wins iPhone** |
| **iPhone** | Goat | Goat | **Wins iPhone** | Wins goat |

  The host always helps us kick out a wrong choice in the remaining two!
- But in the modified version, the host selects the door **randomly**, which doesn't benefit switching.

# Today's goal

- Understand the definition of independence of events
- Know how to decide whether events are independent or not
- Comprehensively use all properties of probability we mentioned before to calculate relative problems

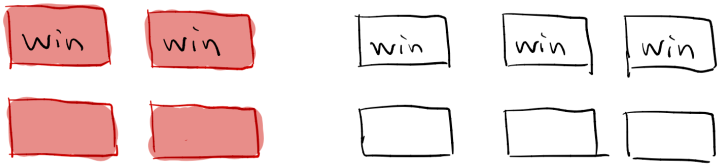
# Independence

# Intuition

In previous examples, often $\mathbb{P}(A|B) \neq \mathbb{P}(A)$, which means the information "B has occurred" resulted in a change in the likelihood of $A$ occurring.

But sometimes, $\mathbb{P}(A|B) = \mathbb{P}(A)$, which means the chance that A will occur or has occurred is not affected by knowledge that B has occurred.

**Example:** A box contains 4 red tickets and 6 white tickets. Pick a ticket at random from the box.




- What is the chance of getting a winning ticket?$(5/10)$
- If you see that it is a white one, what is the chance of getting a winning ticket?$(3/6)$

# Independence between two events

**Definition:** Two **events** $A$ and $B$ are **independent** if $\mathbb{P}(A|B) = \mathbb{P}(A)$, denoted as $A \perp\!\!\!\perp B$. They are **dependent** otherwise, denoted as $A \not\perp\!\!\!\perp B$.

Question: What can we say about $\emptyset$ and $\Omega$?

**Equivalent definitions for $A \perp\!\!\!\perp B$:**

- $\mathbb{P}(B|A) = \mathbb{P}(B)$
- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ (very useful!)

**Properties:** $A$ and $B$ are independent iff (if and only if) any of the following holds:

- $A$ and $B^c$ are independent
- $A^c$ and $B$ are independent
- $A^c$ and $B^c$ are independent

You will prove these in HW2!

# Example

Consider a gas station with six pumps numbered 1, 2,..., 6. Suppose that

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{6\}) = 0.1,\ \mathbb{P}(\{2\}) = \mathbb{P}(\{5\}) = 0.15,\ \mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = 0.25.$$

Define events $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$, $C = \{2, 3, 4, 5\}$.

Therefore:

- $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.5$, $\mathbb{P}(C) = 0.8$.
- $\mathbb{P}(A \cap B) = \mathbb{P}(\{2\}) = 0.15$
- $\mathbb{P}(B \cap C) = \mathbb{P}(\{2, 3\}) = 0.4$
- $\mathbb{P}(A \cap C) = \mathbb{P}(\{2, 4\}) = 0.4$

Hence:

- $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B) \Longrightarrow A \not\perp\!\!\!\perp B$
- $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C) \Longrightarrow B \perp\!\!\!\perp C$
- $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C) \Longrightarrow A \perp\!\!\!\perp C$

# Independence between multiple events

**Definition:** **Events** $A_1, \ldots, A_n$ are **(mutually) independent** if for every $k = 2, 3, \ldots, n$ and every subset of indices $i_1, i_2, \ldots, i_k$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k}).$$

- When $n = 2$: it reduces to $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$
- When $n = 3$: it reduces to (all the following conditions have to hold at the same time):
  - $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$
  - $\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\mathbb{P}(A_3)$
  - $\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)$
  - $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

Therefore, independence is often an idealistic assumption in practice!

# Independence between multiple events

**Theorem:** If $A_1, \ldots, A_n$ are mutually independent, then for any index sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_r\}$ where $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_r\} = \emptyset$,

$$\text{intersections/unions between } A_{i_1}, \ldots, A_{i_k} \perp\!\!\!\perp$$
$$\text{intersections/unions between } A_{j_1}, \ldots, A_{j_r}..$$

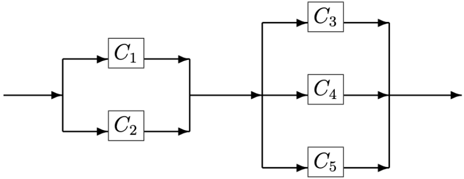
The events $A_{i_1}, \ldots, A_{i_k}$ and $A_{j_1}, \ldots, A_{j_r}$ can be replaced by their complements.

For example:

- $(A_1 \cup A_2) \perp\!\!\!\perp (A_3 \cap A_4)$
- $(A_1 \cap A_2^c \cup A_5) \perp\!\!\!\perp (A_3 \cup A_4)$
- $A_2 \perp\!\!\!\perp (A_1^c \cap A_4 \cap A_3)$
- ...

# Example: parallel system

A system consists of 5 **independent** components $C_1$, ..., $C_5$.
$\mathbb{P}(C_i$ works properly$) = 0.9$, $i = 1, \ldots, 5$. The chance that the system will work properly $=$?

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$\{$The whole system works$\} = (C_1 \cup C_2) \cap (C_3 \cup C_4 \cup C_5)$

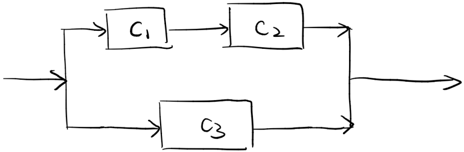
$$\begin{aligned}\mathbb{P}(\text{The whole system works}) &= \mathbb{P}((C_1 \cup C_2) \cap (C_3 \cup C_4 \cup C_5)) \\ &= \mathbb{P}(C_1 \cup C_2) \times \mathbb{P}(C_3 \cup C_4 \cup C_5).\end{aligned}$$

- $\mathbb{P}(C_1 \cup C_2) = 1 - \mathbb{P}(C_1^c \cap C_2^c) = 1 - \mathbb{P}(C_1^c)\mathbb{P}(C_2^c) = 0.99;$
- $\mathbb{P}(C_3 \cup C_4 \cup C_5) = 1 - \mathbb{P}(C_3^c)\mathbb{P}(C_4^c)\mathbb{P}(C_5^c) = 1 - 0.1^3 = 0.999$

$\Longrightarrow \mathbb{P}(\text{The whole system works}) = 0.99 \times 0.999 = 0.989$

# Example: parallel system 2

A system consists of 3 **independent** components $C_1$, $C_2$, $C_3$.
$\mathbb{P}(C_i \text{ works properly}) = 0.7$, $i = 1, \dots, 3$. The chance that the system will work properly $=$?

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$\{\text{The whole system works}\} = (C_1 \cap C_2) \cup C_3$

$$\begin{aligned}
\mathbb{P}(\text{The whole system works}) &= \mathbb{P}((C_1 \cap C_2) \cup C_3) \\
&= \mathbb{P}(C_1 \cap C_2) + \mathbb{P}(C_3) - \mathbb{P}(C_1 \cap C_2 \cap C_3) \\
&= \mathbb{P}(C_1)\mathbb{P}(C_2) + \mathbb{P}(C_3) - \mathbb{P}(C_1)\mathbb{P}(C_2)\mathbb{P}(C_3) \\
\\
&= 0.7^2 + 0.7 - 0.7^3 \\
&= 0.847
\end{aligned}$$

# Example: coffee purchase

The accompanying table gives information on the type of coffee selected by someone purchasing a single cup.

| | **Small** | **Medium** | **Large** |
|---|---|---|---|
| Regular | 14% | 20% | 26% |
| Decaf | 20% | 10% | 10% |

Consider randomly selecting such a coffee purchaser.

- Is "buying decaf coffee" independent with "buying medium-size coffee"?
- Are "buying regular coffee", "buying small-size coffee" and "buying medium-size coffee" mutually independent?

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$\mathbb{P}(\text{decaf} \cap \text{medium}) = 10\% \neq 40\% \times 30\% = \mathbb{P}(\text{decaf}) \times \mathbb{P}(\text{medium})$
$\Rightarrow$ decaf $\not\perp$ medium
Can we claim that $\{$regular$\}$, $\{$small$\}$ and $\{$medium$\}$ are NOT mutually independent?

# Review: Probability Calculation

# Review: Probability Calculation

$\mathbb{P}(\cdot) : A \in$ a set of all events $\to$ a number $\mathbb{P}(A)$ (a mapping/function which maps an event to a number)

**Three axioms:**

- $0 \leq \mathbb{P}(A) \leq 1$ for any event $A \subseteq \Omega$
- $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$
- If $A_1$, $A_2$, $A_3$, ..., $A_n$ is a collection of disjoint (mutually exclusive) events, then

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

**Theorem**: In an experiment consisting of $N$ outcomes **with equal probability**, for any event $A$,

$$\mathbb{P}(A) = \frac{N(A)}{N},$$

where we use counting techniques to calculate $N(A)$ and $N$.

# Review: Probability Calculation

<u>**General idea to calculate probability:**</u>

(1) Translate the conditions and the event of interest by probability language

(2) Calculate.

- For equally likely outcomes, consider $\mathbb{P}(A) = \frac{N(A)}{N}$ combined with counting techniques
- If conditioning probability is easier to calculate or it is given, consider conditional multiplication rule, rule of total probability, and Bayes' Theorem
- See whether there are independence or not, which may help us to do the calculation

# Example 1

Given $A$ and $B$, such that $\mathbb{P}(A) = \mathbb{P}(A^c \cap B) = 0.4$ and $\mathbb{P}(A \cap B) = 0.1$.

- Find $\mathbb{P}(A \cup B)$
- Are $A$ and $B$ independent?

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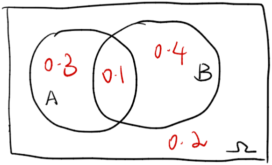
- Venn diagram:




Or:

  - ▷ $\mathbb{P}(B) = \mathbb{P}(A^c \cap B) + \mathbb{P}(A \cap B) = 0.4 + 0.1 = 0.5$
  - ▷ Then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
    $= 0.4 + 0.5 - 0.1 = 0.8$

- No,because $\mathbb{P}(A \cap B) = 0.1 \neq 0.4 \times 0.5 = \mathbb{P}(A) \times \mathbb{P}(B)$.

# Example 2

5 faculty members (3 pro-A and 2 pro-B) vote for the department chair from candidate A and B one by one in a random order.

The probability that "A remains ahead of B throughout the vote count" = ? (e.g., "AABAB" is OK, but not "ABBAA")

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Some observations:

- Permutation problem
- The last one has to vote for $B$. (Why?)
- The first one must vote for $A$. (Why?)
- So it has to be "ABAAB", "AABAB" or "AAABB"

$N(\Omega) = P_{5,5} = 5! = 120$
$N(\text{A remains ahead of B}) = 3 \times P_{3,3} \times P_{2,2} = 36$
$\mathbb{P}(\text{A remains ahead of B}) = \frac{N(\text{A remains ahead of B})}{N(\Omega)} = 0.3$

# Example 3

Randomly select a student at Columbia:

- $A = \{$the selected student has a American Express card$\}$
  $B = \{$the selected student has a Barclays card$\}$
  $C = \{$the selected student has a Chase card$\}$
- $\mathbb{P}(A) = 0.6, \mathbb{P}(B) = 0.4, \mathbb{P}(C) = 0.2.$
  $\mathbb{P}(A \cap B) = 0.3, \mathbb{P}(A \cap C) = 0.15, \mathbb{P}(B \cap C) = 0.1$
  $\mathbb{P}(A \cap B \cap C) = 0.08$

(1) The probability that the selected student has $\geq 1$ of 3 types of cards=?
(2) $\mathbb{P}(B|A) =?$, $\mathbb{P}(A|B) =?$

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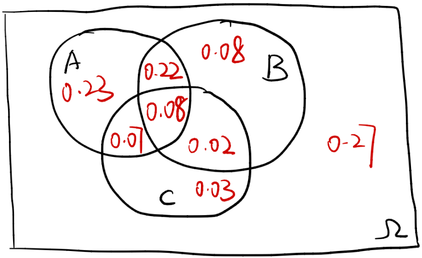


- $\mathbb{P}(A \cup B \cup C) = 0.73$
- $\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = 0.5,$
  $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 0.75$

# Example 3

Randomly select a student at Columbia:

- $A = \{$the selected student has a American Express card$\}$
  $B = \{$the selected student has a Barclays card$\}$
  $C = \{$the selected student has a Chase card$\}$
- $\mathbb{P}(A) = 0.6, \mathbb{P}(B) = 0.4, \mathbb{P}(C) = 0.2.$
  $\mathbb{P}(A \cap B) = 0.3, \mathbb{P}(A \cap C) = 0.15, \mathbb{P}(B \cap C) = 0.1$
  $\mathbb{P}(A \cap B \cap C) = 0.08$

(3) Given that the student has an American Express card, what is the probability that she/he has $\geq 1$ of the other two types of cards?

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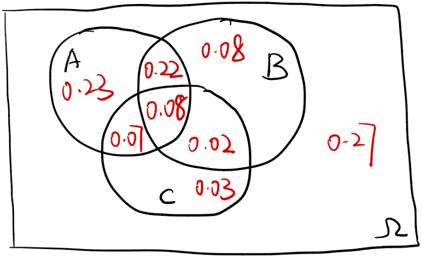
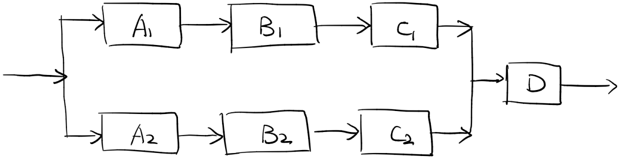



- $\mathbb{P}(B \cup C | A) = \frac{\mathbb{P}((B \cup C) \cap A)}{\mathbb{P}(A)} = \frac{0.22+0.08+0.07}{0.6} = 0.617$

# Example 4

A system consists of 7 **independent** components $A_1$, ..., $D$.
$\mathbb{P}(A_1 \text{ works}) = \mathbb{P}(A_2 \text{ works}) = 0.9$
$\mathbb{P}(B_1 \text{ works}) = \mathbb{P}(B_2 \text{ works}) = 0.8$
$\mathbb{P}(C_1 \text{ works}) = \mathbb{P}(C_2 \text{ works}) = 0.7$, $\mathbb{P}(D \text{ works}) = 0.95$
The chance that the system will work properly = ?

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$$
\begin{aligned}
&\mathbb{P}(\{\text{the system works}\}) = \mathbb{P}(((A_1 \cap B_1 \cap C_1) \cup (A_2 \cap B_2 \cap C_2)) \cap D)\\
&= \mathbb{P}((A_1 \cap B_1 \cap C_1) \cup (A_2 \cap B_2 \cap C_2))\mathbb{P}(D)\\
&= [2\mathbb{P}(A_1 \cap B_1 \cap C_1) - \mathbb{P}((A_1 \cap B_1 \cap C_1) \cap (A_2 \cap B_2 \cap C_2))]\mathbb{P}(D)\\
&= [2\mathbb{P}(A_1)\mathbb{P}(B_1)\mathbb{P}(C_1) - \mathbb{P}(A_1 \cap B_1 \cap C_1)\mathbb{P}(A_2 \cap B_2 \cap C_2)]\mathbb{P}(D)\\
&= [2 \times 0.9 \times 0.8 \times 0.7 - (0.9 \times 0.8 \times 0.7)^2] \times 0.95\\
&= 0.717
\end{aligned}
$$

# Conditional independence

**Independence:** Two **events** $A$ and $B$ are **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, denoted as $A \perp\!\!\!\perp B$. They are **dependent** otherwise, denoted as $A \not\perp\!\!\!\perp B$.

**Conditional independence:** Two **events** $A$ and $B$ are **independent** conditioning on event $C$ if $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$, denoted as $(A \perp\!\!\!\perp B)|C$. They are **dependent** otherwise, denoted as $(A \not\perp\!\!\!\perp B)|C$.

Can be extended to the case of multiple ($\geq 2$) events too.

# Example 5

- 5% of the population has a certain disease
- A diagnostic test:
  - ▹ correctly detects the presence of the disease 98% of the time
  - ▹ correctly detects the absence of the disease 99% of the time
- Randomly select an individual -> run tests twice -> Both positive
- The two test results are independent given the presence/absence of the disease on the selected individual

Question: The probability that this individual has the disease = ?

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- $P_i = \{i\text{-th test positive}\}$, $i = 1, 2$
- $D = \{\text{this individual has the disease}\}$
- Goal: $\mathbb{P}(D|P_1 \cap P_2) =?$
- What we know:
  - ▹ $\mathbb{P}(D) = 0.05$. $\mathbb{P}(P_i|D) = 0.98$, $\mathbb{P}(P_i^c|D^c) = 0.99$, $i = 1, 2$
  - ▹ $P_1 \perp\!\!\!\perp P_2$ given $D$, $P_1 \perp\!\!\!\perp P_2$ given $D^c$
- Bayes' Theorem!

# Example 5

- $P_i = \{i\text{-th test positive}\}$, $i = 1, 2$
- $D = \{\text{this individual has the disease}\}$
- Goal: $\mathbb{P}(D|P_1 \cap P_2) =?$
- What we know:
  - $\mathbb{P}(D) = 0.05$. $\mathbb{P}(P_i|D) = 0.98$, $\mathbb{P}(P_i^c|D^c) = 0.99$, $i = 1, 2$
  - $P_1 \perp\!\!\!\perp P_2$ given $D$, $P_1 \perp\!\!\!\perp P_2$ given $D^c$

---

By Bayes' Theorem,

$$\mathbb{P}(D|P_1 \cap P_2) = \frac{\mathbb{P}(P_1 \cap P_2|D)\mathbb{P}(D)}{\mathbb{P}(P_1 \cap P_2|D)\mathbb{P}(D) + \mathbb{P}(P_1 \cap P_2|D^c)\mathbb{P}(D^c)}$$

$$= \frac{\mathbb{P}(P_1|D)\mathbb{P}(P_2|D)\mathbb{P}(D)}{\mathbb{P}(P_1|D)\mathbb{P}(P_2|D)\mathbb{P}(D) + \mathbb{P}(P_1|D^c)\mathbb{P}(P_2|D^c)\mathbb{P}(D^c)}$$

$$= \frac{0.98 \times 0.98 \times 0.05}{0.98 \times 0.98 \times 0.05 + 0.01 \times 0.01 \times 0.95} = 0.998.$$

# Reading list (optional)

- "Probability and Statistics for Engineering and the Sciences" (9th edition):
  - Chapter 2.5
- "OpenIntro statistics" (4th edition, free online, download [here]):
  - Chapter 3.1.7 and 3.2.6

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