IEOR 8100: Lecture 3 Outline

1 Overview

Recall that for the loss network defined at the end of last lecture, the equilibrium distribution is given by

\[ \pi(n) = G(C)^{-1} \prod_{r \in R} \frac{\lambda_r^{n_r}}{n_r!}, \]

where

\[ G(C) = \sum_{n \in S(C)} \prod_{r \in R} \frac{\lambda_r^{n_r} n_r!}{n_r} \]

is a normalizing constant. The loss probability \( L_r \) on route \( r \) satisfies

\[ 1 - L_r = \frac{G(C - Ae_r)}{G(C)}. \]

Thus, we have closed-form expressions for the equilibrium distribution and the loss probabilities. However, in practice these quantities are very hard to compute. The difficulty lies with the normalizing constant \( G(C) \), the computation of which has been shown to be \#P-complete. Given the difficulty, one might also try to apply simulation techniques to the calculation of loss probabilities and eqm distr. One obvious approach is to sample the state \( n \) according to \( \exp\left(-\sum_r \lambda_r \prod_{i \in r - \{j\}} (1 - E_i)\right) \), i.e., sample coordinates of \( n \) as independent Poisson r.v. For each sampled \( n \), we then check whether \( n \in S(C) = \{ n \in \mathbb{Z}_+^{\|R\|} : An \leq C \} \); if \( n \in S(C) \), we “keep” it, otherwise we discard it. The resulting samples that we kept is then distributed as \( \pi \). However, to calculate loss probabilities, for example, this technique can be time-consuming as well, since we need to sum up \( O(\|R\|) \) terms.

A popular heuristic used in the industry for the calculation of the loss probabilities \( L_r \) is so-called Erlang’s fixed point. Assuming \( A_{jr} \in \{0, 1\} \) (and write \( j \in r \) iff \( A_{jr} = 1 \)), it is characterized by the set of fixed-point equations below:

\[ E_j = E \left( \sum_{r : j \in r} \lambda_r \prod_{i \in r - \{j\}} (1 - E_i), C_j \right), \]

for all \( j \), where recall that

\[ E(\nu, C) = \frac{\nu^C/C!}{\sum_{m=0}^C \nu^m/m!} \]

is the loss probability for a link with capacity \( C \) and a single stream of arrivals with rate \( \nu \). The interpretation is as follows. Let

\[ \rho_j = \sum_{r : j \in r} \lambda_r \prod_{i \in r - \{j\}} (1 - E_i). \]

Then \( (E_j) \) satisfies \( E_j = E(\rho_j, C_j) \). \( \rho_j \) can be thought of as the effective load on link \( j \) in the following sense. Suppose that for the flow of calls on route \( r \), passing through link \( i \) of route \( r \) reduces the load by a factor of \( (1 - E_i) \) – this is because of the interpretation of \( E_i \) as loss probabilities; hence in the long run, \( E_i \) fraction of the calls are lost. Then when this flow of route \( r \) reaches link \( j \), the effective load contributed by \( r \) is reduced by a factor of \( \prod_{i \in r - \{j\}} (1 - E_i) \) – assuming independence among the links, which is clearly not true. Combining flows from all the routes, the total effective load on link \( j \) is then \( \rho_j = \sum_{r : j \in r} \lambda_r \prod_{i \in r - \{j\}} (1 - E_i) \).

Later we will see that Erlang’s fixed point always exists (and is unique). For now, assuming existence and uniqueness, let us concentrate on its validity. This heuristic is surprisingly useful in practice, which makes one wonder whether there is some formal justification of its usefulness. This is the content of this and the next lecture. The justification is fairly involved, and this lecture is only part of it.
Outline of the lecture. To justify Erlang’s fixed point approximation, we will focus on a certain largesystem limiting regime, and then consider approximations to the equilibrium distributions. This approximation has nice structural properties, is reminiscent of the central limit theorem, and can be used to characterize the loss probabilities as well. In the next lecture, we modify the approach to justify the Erlang’s fixed point.

2 Scaling regime

Without further ado let us consider the scaling regime. Consider a sequence of loss networks indexed by $N$, where $N$ is to be thought of as a discrete sequence tending to $\infty$. These networks have fixed topology, i.e., they have the same matrix $A = (A_{jr})$. We assume that the arrival rates to the $N$th system are given by $\lambda(N)$, and the capacities of the links of the $N$th system are given by $C(N)$. We assume that there exists finite $\lambda$ and $C$ such that $\frac{1}{N}\lambda(N) \to \lambda$, and $\frac{1}{N}C(N) \to C$ as $N \to \infty$. For simplicity, we can think of $\lambda(N) = N\lambda$ and $C(N) = NC$. Now for any $N$, the state space $S^N(C) = \{n \in \mathbb{Z}^{|R|}_+: An \leq C(N)\}$ is discrete and large. However, if we consider the normalized state space $\frac{1}{N}S^N(C) = \{\frac{n}{N}: n \in \mathbb{Z}^{|R|}_+, A\frac{n}{N} \leq \frac{C(N)}{N} = C\}$, then it is intuitively clear that as $N \to \infty$, $\frac{1}{N}S^N(C)$ becomes denser and denser in the continuous set $\{x \geq 0: Ax \leq C\}$. Recall that one of the difficulties of computing the normalizing constant $G(\cdot)$ is that the underlying state space is too large, and hence too many terms to sum. With the approximation to a continuous set, we may be able to simplify the problem.

3 An optimization problem

Now the question is: what are we trying to approximate? Recall that the equilibrium distribution of a loss network is characterized by a number of independent Poisson r.v. conditioned on the state space $S(C)$. As we consider the large-system scaling, it is likely that the limit that arises out of the limiting procedure looks like a multivariate normal conditioned on some set. Without going into the formal details, we need to think about what the mean and variance may be, as well as the conditioning set, just to guess the limiting distribution.

Let us try to guess the means first. Computing them in a direct manner is clearly infeasible (think about why!), so instead of computing the means, we look for the mode of the equilibrium distributions. It is not obvious that the mode is a good proxy to the mean, but it all worked out in the end. This is a much easier task, since $\pi(n)$ is proportional to $\prod_{r \in R} \lambda_r^n / n!$, and we can simply try to maximize $\prod_{r \in R} \lambda_r^n / n!$ subject to the state space constraints. By the reasoning in the previous section, we will relax the discrete state space into a continuous one. We will also approximate the term $\prod_{r \in R} \lambda_r^n / n!$ using Stirling’s approximation, and discard lower-order terms. Without going into the detailed calculations (covered in lecture), the optimization problem of interest is the following PRIMAL problem:

$$\begin{align*}
\text{maximize} & \quad \sum_{r \in R} x_r \log \lambda_r - x_r \log x_r + x_r \\
\text{subject to} & \quad Ax \leq C, \quad x \geq 0.
\end{align*}$$

The DUAL problem is:

$$\begin{align*}
\text{minimize} & \quad \sum_{r} \lambda_r \exp \left( -\sum_{j} y_j A_{jr} \right) + \sum_{j} y_j C_j \\
\text{subject to} & \quad y \geq 0.
\end{align*}$$
The main theorem established in class concerns structural properties of the optimal solutions of PRIMAL and DUAL.

**Theorem 1.** There exists a unique optimal solution to PRIMAL and it can be expressed in product form as

$$\bar{x}_r = \lambda_r \prod_j (1 - B_j)^{A_{jr}}, \quad r \in \mathcal{R},$$

with $B = (B_j) \in [0,1]^J$ satisfying

$$\sum_r A_{jr} \lambda_r \prod_j (1 - B_j)^{A_{jr}} \left\{ \begin{array}{ll} = C_j & \text{if } B_j > 0 \\ \leq C_j & \text{if } B_j = 0. \end{array} \right. \quad (3)$$

Furthermore, solutions to (3) exist, and is unique if rank$(A) = J$.

### 4 Convergence theorems

For each system indexed by $N$, we have the corresponding $\bar{x}(N)$ and $B(N)$. Recall the limits $\lambda$ and $C$; we can also associate $\bar{x}$ and $B$ with these parameters.

For simplicity, assume rank$(A) = J$. The following lemma can be proved.

**Lemma 2.** As $N \to \infty$, $B(N) \to B$ and $\frac{1}{N}\bar{x}(N) \to \bar{x}$.

The following is the main theorem. First some notation. Let $u(N) = \frac{1}{\sqrt{N}} (n(N) - \bar{x}(N))$. Here $n(N)$ is a random state of the $N$th system, under the respective equilibrium distribution. Let $\mathcal{B} = \{ j : B_j > 0 \}$, and let $\mathcal{A}_\mathcal{B} = \{ A_{jr} : j \in \mathcal{B}, r \in \mathcal{R} \}$.

**Theorem 3.** Under technical conditions, $u(N) \Rightarrow u$, where notation $\Rightarrow$ means convergence in distribution, and $u$ is a multivariate normal $N(0, \bar{x})$ conditioned on the set $\mathcal{A}_\mathcal{B}u = 0$. Moments converge as well, and so

$$\frac{1}{N} \mathbb{E}[n_r(N)] \to \bar{x}_r, \quad r \in \mathcal{R}.$$

The proof is fairly involved, and omitted. A nice corollary of the theorem is the following.

**Corollary 4.** For $r \in \mathcal{R}$ and as $N \to \infty$,

$$L_r(N) \to L_r = 1 - \prod_j (1 - B_j)^{A_{jr}}.$$

Here $L_r(N)$ is the loss probability of route-$r$ calls in the $N$th system.

Theorem 3 is basically a characterization of the equilibrium distributions $\pi(N)$. Corollary 4 is a characterization of the loss probabilities $L_r(N)$. A remarkable feature of these statements is that performance quantities of interest can be expressed compactly using very low-dimensional information; in this case, only $J$ dimensions (as opposed to $|\mathcal{R}|$ dimensions).

### 5 Prelude to next lecture

What have we done so far? We have considered a limiting regime and “derived” a limiting theorem. After some hard work we now know that in the limit, the equilibrium distribution has a fairly simple form, and
so are the loss probabilities. The $B_j$ are fairly manageable since they are solutions to the DUAL problem, which can be easier to solve.

In the next lecture, we will see how the approach considered in this lecture can be modified to explore the Erlang’s fixed point approximation. In particular, we will consider a modification to the objective of the DUAL problem, and relate it to the Erlang’s fixed point.