Multistage Stochastic Mixed-Integer Programs for Optimizing Gas Contract and Scheduling Maintenance

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Abstract

In this paper, we present multistage stochastic mixed-integer programs for optimizing gas contract and scheduling maintenance for a gas-fueled thermal plant in a hydro dominated power system. We consider the specifications of the power generation obligation, the gas supply contract with take-or-pay and make-up clauses, the trading in the spot electricity market, and the maintenance scheduling issue. Since any decision made in one stage impact the future stages, and the future spot electricity price is the major unknown parameter, the problem is multistage and stochastic. We use a Two-reservoir Model to model the gas contract specifications, and the maintenance scheduling problem is then included to formulate the multistage SMIP computational model. We introduce solution methods including the two-stage simulation method and the multistage exact method to solve the problem. The two-stage simulation method concerns stochasticity in the second stage and uses Monte Carlo method to sample future realizations, which provides an estimation of the objective value. The L-shaped decomposition method, which is an important kind of multistage exact methods that apply Benders decomposition, can give optimal policies in each stage for the LP relaxation of the original model. We construct scenario trees of the stochastic parameter and conduct computational tests on our problem specifications for the various solution methods. The computational results and some comparisons and findings are presented.

Key words: gas contract optimization, maintenance scheduling, stochastic optimization, mixed-integer programming

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1 Introduction

In countries with abundant water resource such as Brazil, Chile, Columbia and Iceland, hydropower is the main source of power generation and hydro based power systems are adopted (Hreinsson, 2006; Wheeler). Hydro dominated systems have the characteristics that the spot electricity price is mainly determined by hydro plants and the spot price is variable mainly due to the volatility of hydrological conditions. During wet seasons the system has excess of hydro energy, which allows the power demand to be met using little thermal resource and results in very low spot prices. However, when dry seasons come, hydro plants alone are unable to supply the energy demand, thus the spot prices increase abruptly. The thermal plants will make supplements for the power system in such conditions (Chabar, 2005; Chabar et al., 2006).

A Regional Transmission Organization (RTO), or an Independent System Operator (ISO), is responsible for coordinating, controlling and monitoring the operation of the system. In order to meet the demand of the hydro based system with lowest possible operating cost, the RTO or ISO carries out a least-cost hydro-thermal power generation portfolio based on the operating data, such as the operational cost, the capacity and the must-run generation, of all participants of the system (Chabar et al., 2006). The thermal plants are obliged to supply certain amounts of electricity in each period according to the RTO's or ISO's arrangement.

For a gas-fueled thermal plant in the hydro based power system to meet its contract obligation, it is cost effective to generate more electricity when spot prices are high so that it may sell excess power to the spot market; while the thermal plant tends to generate less electricity and buy electricity from the spot market when spot prices are low. Such operational flexibility is an attractive feature for thermal plants in the hydro based systems, but is contradictory to the needs of gas suppliers. As a consequence, gas contracts, including *take-or-pay* (ToP) clauses and *make-up* clauses, between gas suppliers and gas-fueled thermal plants are signed which restrict the plants' purchase flexibility and thus reduce the volatility of the gas suppliers' revenues. The ToP clause requires that gas has to be paid whether taken or not, which specifies an obligation for the supplier to make available defined volumes of gas and for the buyer to purchase a minimum amount of gas independently of its consumption (Creti and Villeneuve, 2004). The gas purchased but not consumed in a period is virtually "stored" for a certain length of time during which at any time the gas can be recovered by the plant. This is the make-up clause (Chabar et al., 2006). The maintenance scheduling is another important issue for the plant to consider. This is not only because any maintenance has an execution cost, but also due to the impact on normal operation, thus the capacity in that period, of the maintenance. The scheduling of maintenance is made regarding both the maximal length of time without maintenance and the average maintenance duration. Therefore the thermal plant should develop an operation strategy taking into account jointly the uncertain spot market electricity price, the contract obligation to the RTO or ISO, the gas supply contract with the gas suppliers, and the maintenance scheduling, which becomes a complex problem.

The objective of this paper is to develop a model, which captures the aforementioned aspects (the power generation obligation, the gas contract with take-or-pay and make-up clauses, the opportunity of trading in the spot market, and the maintenance scheduling issue), and the solution methods, for a gas-fueled thermal plant to formulate operational strategies that minimize the overall (expected) cost. Our work relates most closely to that of Chabar et al. (2006), who make a comprehensive investigation of the Brazilian power system and present a stochastic multistage model applied on hydro-thermal dispatch problems, and the Stochastic Dual Dynamic Programming (SDDP) (Pereira and Pinto, 1991) is used to find optimal solutions of the problem. As Chiralaksanakul (2003) stated, the SDDP is more appropriated for solving multistage problems with interstage independent stochastic process. However, the spot market price in reality is interstage dependent regarding the natural evolution of hydrological conditions. Hence it is more appropriate to apply methods for interstage dependent problems introduced by Chiralaksanakul (2003). In this paper we model the problem as a multistage stochastic mixed-integer programming (SMIP) problem, and explore several solution methods. The two-stage simulation method concerns stochasticity in the second stage and uses Monte Carlo method to sample future realizations, it provides an estimation of the objective value. We also solve the LP relaxation with *Multistage L-shaped Method*, one of the multistage exact methods, which is computationally more demanding but gives optimal policies in each stage and each scenario. We use Benders decomposition in both the two-stage simulation method and the exact solution method and compare the results with those of the deterministic equivalent problems. The results provide an overview of the performance of solution methods including the direct solution of the original multistage SMIP problem, the two-stage simulation with and without Benders decomposition to the LP relaxation, and the multistage L-shaped method to the LP relaxation.

The remainder of this paper is organized as follows. Section 2 describes the general aspects of the problem including the power supply obligation, the gas supply contract, the maintenance scheduling, and the characteristics of decision under uncertainty. Section 3 presents the computational model and detailed formulation of the problem. Solution methods are introduced in Section 4. Section 5 presents the computational results of different solution methods and makes comparisons considering several evaluation factors, such as the CPU runtime, optimality gap, and the objective value. Section 6 concludes the paper.

2 Problem Description

In this section we present the general aspects of the problem and introduce the notation for the following explanation.

2.1 Power Supply Obligation and Gas Contract

In each month t, a gas-fueled thermal plant is obliged to supply a certain amount of electricity to the market coordinator (may be RTO or ISO), which is measured by the volume P of gas used to generate that amount of electricity. The electricity selling price is h_t per equivalent unit gas. There are two ways for the plant to fulfill this contract. In the first or a normal way, the plant purchases gas from gas suppliers and then generates electricity using the gas. In this way, there is a gas supply agreement, i.e., the gas contract, restricting the plant's purchase. Specifically, the plant can purchase as much as Q amount of gas monthly, with a price of c_t per unit gas. Considering the *take-or-pay* (ToP) clause, the plant is obliged to purchase (but not necessarily to consume) $X\%$ of the contract amount Q. At the end of the year, the plant should have purchased at least $Y\%$ ($Y \geq X$) of the annual contract amount, $12Q$. In each month, the gas already purchased but not consumed is "virtually" stored and can be recovered by the plant later at any time in the year without additional payment, which is called the *make-up* clause. However, any "virtually" stored gas that has not been recovered by the end of the year will be lost.

In the second way, the plant directly purchases electricity from the spot market at a price of q_t per equivalent unit gas. Then the amount of generated electricity and the purchased spot electricity should sum up to the contract amount, P. On the other hand, it is also permissible for the plant to generate more electricity than the contract amount. If this is the case, we assume that the plant can sell all excess electricity to the spot market at the spot price q_t, earning an additional profit. Then the difference between the amount of generated electricity and the sold electricity is equal to the contract amount, P.

2.2 Maintenance Scheduling

Maintenance scheduling is another important issue for the plant to consider. According to Balevic et al. (2004), maintenance inspection types may be broadly classified as standby, running and disassembly inspections. The standby inspection is performed during off-peak periods when the unit is not operating, and the running inspections is performed by observing key operating parameters while the turbine is still running. Both of them do not interfere with the normal operation. However, the disassembly inspection requires opening the turbine of internal components, resulting in the shutdown of certain parts of the machine, has much higher execution cost and substantial impact on the operation of the plant. There are three types of disassembly maintenance inspections: combustion inspection, hot gas path inspection, and major inspection. Each maintenance inspection has its own required maintenance frequency, average maintenance duration and maintenance cost. Table 1 demonstrates a set of maintenance parameters from Chabar et al. (2006).

Maintenance	Frequency (hours)	Avg. Duration (days)	Cost(MMR$)
Combustion	8,000	7	3.5
Hot gas path	24,000	14	10
Major	48,000	21	20

Table 1: Parameters of maintenance inspections

2.3 Decision under Uncertainty

As mentioned in the introduction part, the plant makes decision mainly depending on the spot electricity price evolution which is stochastic. When spot price is low, the plant tends to take less gas from the gas supplier and leave the paid but not consumed gas for future use. It will instead purchase electricity from the spot market to meet the contract obligation, which is more cost effective than generating itself. In addition, the plant prefers to schedule maintenance in such conditions since the operation load is not heavy. When spot price is high, the plant prefers to take more gas from the gas supplier, maybe by partly recovering the previously stored gas, to generate greater amount of electricity than the contract obligation and sell the excess in the spot market. The plant will tend not to schedule maintenance in such periods due to the heavy operation load. Because the spot price is mainly affected by the hydro plants as a consequence of the hydrological conditions, it is known only with conditional probability from the past records. Therefore the problem is stochastic and the thermal plant has to make decisions under uncertainty.

The objective of the plant is to minimize its net cost over a finite number of months $t = 1, \dots, T$. In each month the plant has to determine the amount of gas to purchase from the gas supplier (which, as mentioned before, should be between $X\%Q$ and Q), the amount of gas actually used for generating electricity, and whether to schedule maintenance for each kind of the maintenance inspections. The decision should result in a minimal expected cost over the future months in order to achieve the objective.

3 Computational Model

In this section we present the computational model and detailed formulation of the problem. We first develop a multistage deterministic mixed-integer programming (MIP) problem in which all parameters are known at the first stage. The purchase and storage clauses are modeled with a *Two-reservoir Model* which captures the characteristics of the gas contract and the trading in the spot market. Then we include the maintenance scheduling constraints to formulate the multistage MIP problem. Finally we include stochasticity and present the uncertainty using a rooted *scenario tree* over the decision horizon and formulate the multistage SMIP problem and its deterministic equivalent.

3.1 A Two-reservoir Model

As Figure 1 shows, we use reservoir A to store all monthly purchased gas that is not consumed and reservoir B to store the difference between the annual ToP amount and the sum of all monthly ToPs of a year. At the beginning of January in each year, $(Y\% - X\%)(12Q)$ amount of gas is pushed into reservoir B, while at the end of December in each year, all remaining gas in reservoir B is pushed into reservoir A and all gas in reservoir A is discarded. Variables a_t and b_t denote gas levels in reservoir A and reservoir B at the beginning of month t, respectively.

In month t, m_t is the amount of gas directly purchased from the gas supplier and f_t is the amount of gas transferred from reservoir B to reservoir A, both of which are at the contract gas price c_t. Then g_t amount of gas is used by the plant for generating electricity. We temporarily ignore the capacity constraint now, which will be considered in the maintenance scheduling problem presented later. Since the plant is assumed to operate in every month, the fixed operational cost V is incurred in every stage. The variable operational cost is v for each unit of gas used. Hence the total operational cost in month t is $V + vg_t$.

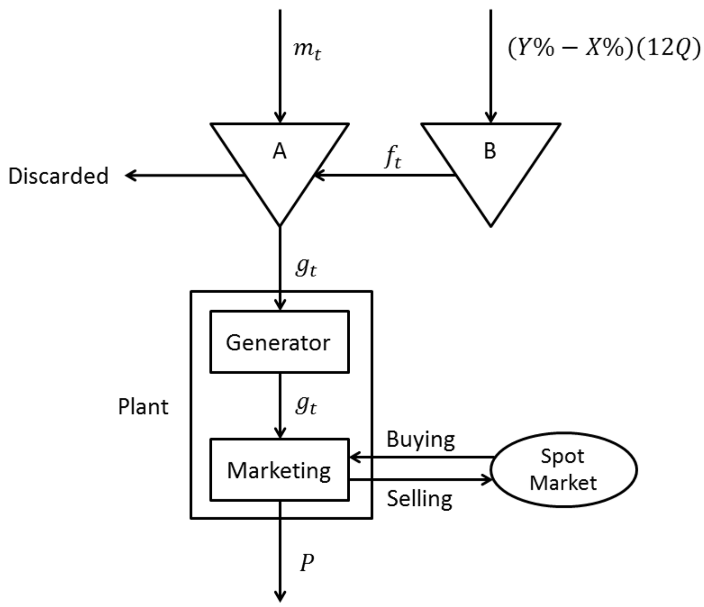

Figure 1: A Two-reservoir Model

The marketing department then buys electricity from or sells electricity to spot market at the price of q_t per unit of equivalent gas, based on the amount of contract electricity and the amount of electricity generated from purchased gas. The contract electricity is sold at the price of h_t per unit of equivalent gas. The aforementioned notation is restated below.

Parameters

Q : monthly contract gas volume

X : monthly ToP percentage, i.e. $ToP = X\%Q$

Y : annual ToP percentage, i.e. Annual $ToP = Y\%(12Q)$

P : monthly contract electricity supply, in equivalent gas volume

V : monthly fixed operational cost

c_t : contract gas price in month t

h_t : electricity price worth per equivalent unit gas in month t

q_t : spot electricity price worth per equivalent unit gas in month t

v : variable operational cost per unit of gas used

Variables

m_t : amount of gas directly purchased from gas producer for reservoir A in month t

f_t : amount of gas transfered from reservoir B to A in month t

g_t : amount of gas used by the power plant for generating electricity in month t

a_t : gas level in reservoir A at the beginning of month t

b_t : gas level in reservoir B at the beginning of month t

The gas portfolio management over months $t = 1, \ldots, T$ is given as

$$\min \quad \sum_{t=1}^{T} c_t(m_t + f_t) + vg_t + V - h_tP - q_t(g_t - P) \tag{1a}$$

$$\text{s.t.} \quad a_t = \begin{cases} a_{t-1} + m_{t-1} + f_{t-1} - g_{t-1} & t \neq 1 + 12k \\ 0 & t = 1 + 12k \end{cases} \quad k \in \mathbb{N},\ t \in \{1, \ldots, T\} \tag{1b}$$

$$b_t = \begin{cases} (Y\% - X\%)(12Q) & t = 1 + 12k \\ b_{t-1} - f_{t-1} & t \neq 1 + 12k \\ f_t & t = 12 + 12k \end{cases} \quad k \in \mathbb{N},\ t \in \{1, \ldots, T\} \tag{1c}$$

$$X\%Q \leq m_t + f_t \leq Q \quad t = 1, \ldots, T \tag{1d}$$

$$a_t, b_t, g_t, m_t, f_t \geq 0 \quad t = 1, \ldots, T, \tag{1e}$$

where (1a) is to minimize net cost over horizon T. Specifically, $c_t(m_t+f_t)$ is the cost of all purchased gas, vg_t+V is the operational cost, h_tP is the revenue obtained from the contract electricity supply, and $q_t(g_t - P)$ is the revenue from selling electricity to spot market when $g_t \geq P$ or the cost of buying electricity from spot market when $g_t \leq P$. Equation (1b) and (1c) represent the balance of gas in reservoir A and B during any month, notice that we discard all stored gas in A at the end of the year. Constraint (1d) restricts the monthly purchased gas amount to be at least $X\%Q$ and at most M in any month. Finally (1e) is the sign constraint.

3.2 Maintenance Scheduling Problem

We now consider the maintenance scheduling problem, which involves three types of maintenance inspections. For each maintenance i, the plant can work no longer than Δ^i before it must undergo the corresponding maintenance, and each maintenance takes an average duration of σ^i (which is shorter than a month) and cost u^i. Define r_t^i as the remaining time before the next required maintenance at the beginning of month t for maintenance i, and δ^i is the length of usable time in each month for maintenance i. When the plant is operating normally, it consumes C amount of gas per unit time, and it then has to determine when to schedule a maintenance for each maintenance inspection. We use a binary variable z_t^i to denote the decision of performing maintenance, such that $z_t^i = 1$ if the plant schedule maintenance during month t for maintenance i, and $z_t^i = 0$ otherwise. The aforementioned notation is restated below.

Parameters

Δ^i : maximal length of time without maintenance, i.e., the frequency, for maintenance i

δ^i : monthly length of usable time for maintenance i

σ^i : average maintenance duration for maintenance i

u^i : maintenance cost for maintenance i

C : gas consumption per unit of operating time

Variables

r_t^i : remaining hours before next required maintenance at the beginning of month t for maintenance i

z_t^i : binary variable denoting a decision of performing maintenance in month t for maintenance i

We have two preliminary specifications for modeling the maintenance constraints. First we assume that for any type of maintenance, the length of its monthly usable time is longer than its average maintenance duration and shorter than its maximal length of time without maintenance, i.e., $\sigma^i \leq \delta^i \leq \Delta^i$. Second, to simplify our modeling without violating the general condition, we pay no attention to the exact scheduling of a maintenance *within* a month, which means that if a maintenance is performed in a month, the remaining time before next required maintenance at the beginning of the next month is always set to the maximal length of non-maintenance time – whenever this maintenance is actually performed within this month.

The following constraints describe the feasible region of the maintenance scheduling problem.

$$r_{t+1}^i \leq r_t^i - \delta^i + (\Delta^i + \delta^i) z_t^i \quad \forall i = 1,2,3,\ t = 1,\dots,T \tag{2a}$$

$$0 \leq r_t^i \leq \Delta^i \quad \forall i = 1,2,3,\ t = 1,\dots,T \tag{2b}$$

$$z_t^i \geq \frac{\delta^i - r_t^i}{\Delta^i} \quad \forall i = 1,2,3,\ t = 1,\dots,T \tag{2c}$$

$$g_t \leq C(\delta^i - \sigma^i z_t^i) \quad \forall i = 1,2,3,\ t = 1,\dots,T \tag{2d}$$

$$z_t^i \in \{0,1\} \quad \forall i = 1,2,3,\ t = 1,\dots,T, \tag{2e}$$

where constraints (2a) and (2b) derive the remaining non-maintenance time. If maintenance is scheduled in month t for maintenance i, i.e., $z_t^i = 1$, the remaining non-maintenance time for the next month is bounded by the maximal length of non-maintenance time Δ^i; while if there is no maintenance performed in month t for maintenance i, the remaining non-maintenance time for the next month will be that for this month less the usable time in this month. Inequality (2c) forces z_t^i to be positive, i.e., to schedule a maintenance in month t for maintenance i, when the usable time for maintenance i in one month is less than its remaining non-maintenance time at the beginning of month t, which is obvious. Constraint (2d) illustrates the effect of a scheduled maintenance on the electricity generation. For each cycle, if no maintenance is scheduled in month t, the capacity of the plant, which is reflected by the amount of gas that can be used for generating electricity, is bounded by $C\delta^i$. But if maintenance is performed, the capacity for directly generating gas is reduced considering the maintenance period. Finally constraint (2e) makes z_t^i a binary decision variable.

Hence, the overall multistage deterministic MIP problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{t=1}^{T} \{c_t(m_t + f_t) + v g_t + V - h_t P - q_t(g_t - P) + \sum_{i=1,2,3} u^i z_t^i\} \\ \text{s.t.} \quad & (1\text{b}) - (1\text{e}), (2\text{a}) - (2\text{e}). \end{aligned} \tag{3}$$

3.3 The Multistage SMIP Model

In the previous subsections we assume all parameters to be known at the beginning of the decision horizon and formulate the deterministic model. In reality, the contract gas price c_t, the contract electricity price h_t and the spot electricity price q_t are unknown until the beginning of month t. For a T-stage stochastic problem, the stochastic parameters are given by a discrete time stochastic process $\{\tilde{\xi}_t\}_{t=1}^T$ defined on a finite probability space $(\Xi, \mathcal{F}, \mathcal{P})$, and a state in stage t is a realization of the random parameters $(c_t(\xi_t), h_t(\xi_t), q_t(\xi_t))$ corresponding to an elementary atom $\xi_t \in \Xi$, $\forall t = 1, \ldots, T$. The sequence of decisions, $\{\mathbf{x}_t\}_{t=1}^T$, is made under uncertainty, where $\mathbf{x}_t = (m_t, f_t, g_t, a_t, b_t, r_t^i, z_t^i)$, $\forall i = 1, 2, 3$, $t = 1, \ldots, T$.

We use a superscript t on an entity to denote its history through stage t, such that $\xi^t = (\xi_1, \ldots, \xi_t)$ and $\mathbf{x}^t = (\mathbf{x}_1, \ldots, \mathbf{x}_t)$. Realizations of $\{\tilde{\xi}_t\}_{t=1}^T$ is described by a finite scenario tree, i.e., a tree arising from $\{\tilde{\xi}_t\}_{t=1}^T$ whose supports Ξ_t, $\forall t = 1, \ldots, T$ are finite, such that the realizations of $\tilde{\xi}^t$ can be enumerated as $\xi^{t,1}, \ldots, \xi^{t,n_t}$, where n_t is the number of nodes at stage t. An example three-stage scenario tree is shown in Figure 2, and the corresponding notation is given by Table 2. The scenario tree at each stage consists of a set of nodes. We use ξ_t^j to denote node j in stage t and use $\xi^{t,j}$ to denote the history through node ξ_t^j. The scenario tree has a total number of n_T leaf nodes, one for each scenario $\xi^{T,j}$, $j = 1, \ldots, n_T$. For a particular node j in stage $t < T$ with history $\xi^{t,j}$, we denote $n(t, j)$ as the number of descendant nodes of node j in stage $t+1$. The descendant node ξ_{t+1}^k corresponds to the realization $\xi^{t+1,k}$, and we use D_t^j to represent the descendant index set of k, thus $|D_t^j| = n(t, j)$. The ancestor of ξ_t^j is denoted by $\xi_{t-1}^{a(j)}$, where $a(j)$ is an integer between 1 and n_{t-1}, hence $a(k) = j$, $\forall k \in D_t^j$.

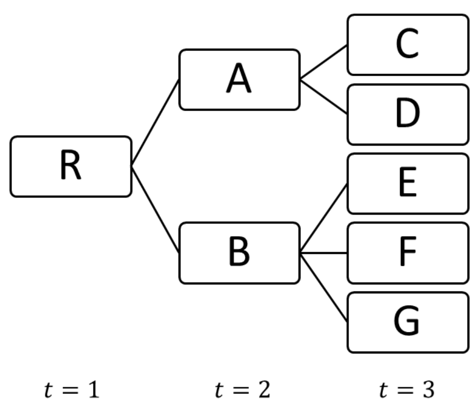

Figure 2: An example of a three-stage scenario tree

At the beginning of each month t, the information for that month, ξ_t, becomes available, and decision $\mathbf{x}_t$ is made with the knowledge of the past decisions $\mathbf{x}^{t-1}$ and of the realized random vectors ξ^t, such that the conditional expected cost, given the history ξ^t, is minimized. The multistage SMIP

	R	A	B	C	D	E	F	G
ξ_t^j	ξ_1^1	ξ_2^1	ξ_2^2	ξ_3^1	ξ_3^2	ξ_3^3	ξ_3^4	ξ_3^5
$n(t,j)$	2	2	3	0	0	0	0	0
D_t^j	{1,2}	{1,2}	{3,4,5}	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$a(j)$	–	1	1	1	1	2	2	2

Table 2: A notation for the scenario tree in Figure 2

model is given by the following recourse model:

$$\min_{\mathbf{x}_1} \quad \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \mathbb{E}_{\tilde{\xi}_2|\xi^1}[\alpha_1(\mathbf{x}_1, \tilde{\xi}_2)] \tag{4a}$$

$$\text{s.t.} \quad \mathbf{A}_1\mathbf{x}_1 \geq \mathbf{b}_1 \tag{4b}$$

$$\mathbf{x}_1 \in \mathbf{X}, \tag{4c}$$

where for $t = 2, \ldots, T$,

$$\alpha_{t-1}(\mathbf{x}_{t-1}, \xi_t) = \min_{\mathbf{x}_t} \quad \mathbf{c}_t(\xi_t)\mathbf{x}_t + \mathbb{E}_{\tilde{\xi}_{t+1}|\xi^t}[\alpha_t(\mathbf{x}_t, \tilde{\xi}_{t+1})] \tag{4d}$$

$$\text{s.t.} \quad \mathbf{A}_t\mathbf{x}_t \geq \mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1} \tag{4e}$$

$$\mathbf{x}_t \in \mathbf{X}, \tag{4f}$$

and $\alpha_T = 0$.

Here we assume that ξ_1 is known at stage $t = 1$ and only $\mathbf{c}_t$ is dependent of the uncertain data since the stochastic price parameters only exist in the coefficients of the objective functions. We denote $\mathbb{E}_{\tilde{\xi}_{t+1}|\xi_t}$ as the expectation with respect to the distribution of $\tilde{\xi}_{t+1}$ conditioned on the observation of ξ_t. For each stage t and realization of ξ_t, we suppose that $\mathbf{A}_t, \mathbf{B}_t, \mathbf{c}_t$, and $\mathbf{b}_t$ are rational matrices and vectors of conformable dimensions. The set $\mathbf{X}$ denotes restrictions that require some of the decision variables, i.e., the maintenance scheduling variables, to be binary.

The above model is formulated from the objective function (3) and constraints (1b)-(1e), (2a)-(2e). Specifically,

$$\mathbf{c}_t(\xi_t)\mathbf{x}_t = c_t(m_t + f_t) + vg_t + V - h_tP - q_t(g_t - P) + \sum_{i=1,2,3} u^i z_t^i, \tag{5}$$

and constraint (4b) or (4e) has the following set of constraints (note that the equality constraints are not presented as two corresponding "$\geq$" inequalities here, which does not affect our analysis):

$$m_t + f_t \geq X\%Q \quad \forall t = 1, \ldots, T \tag{6a}$$

$$-m_t - f_t \geq -Q \quad \forall t = 1, \ldots, T \tag{6b}$$

$$a_t + m_t + f_t - g_t \geq 0 \quad \forall t = 1, \ldots, T \tag{6c}$$

$$b_t - f_t \geq 0 \quad \forall t = 1, \ldots, T \tag{6d}$$

$$-r_t^i \geq -\Delta^i \quad \forall t = 1, \ldots, T, \ i = 1, 2, 3 \tag{6e}$$

$$\Delta^i z_t^i + r_t^i \geq \delta^i \quad \forall t = 1, \ldots, T, \ i = 1, 2, 3 \tag{6f}$$

$$-C\sigma^i z_t^i - g_t \geq -C\delta^i \quad \forall t = 1, \ldots, T, \ i = 1, 2, 3 \tag{6g}$$

$$-r_t^i \geq \delta^i - r_{t-1}^i - (\Delta^i + \delta^i) z_{t-1}^i \quad \forall t = 2, \ldots, T, \ i = 1, 2, 3 \tag{6h}$$

$$a_t = 0 \quad \forall t = 1, 13, 25, \ldots \tag{6i}$$

$$b_t = (Y\% - X\%)(12Q) \quad \forall t = 1, 13, 25, \ldots \tag{6j}$$

$$a_t = a_{t-1} + m_{t-1} + f_{t-1} - g_{t-1} \quad \forall t \neq 1, 13, 25, \ldots \tag{6k}$$

$$b_t = b_{t-1} - f_{t-1} \quad \forall t \neq 1, 13, 25, \ldots \tag{6l}$$

$$b_t - f_t = 0 \quad \forall t = 12, 24, 36, \ldots, \tag{6m}$$

and the domain of the decision variables (4c) and (4f) are

$$a_t, b_t, g_t, m_t, f_t, r_t^i \geq 0 \quad \forall t = 1, \ldots, T, \ i = 1, 2, 3 \tag{6n}$$

$$z_t^i \in \{0, 1\} \quad \forall t = 1, \ldots, T, \ i = 1, 2, 3. \tag{6o}$$

With the definition of the scenario tree, equations (4) can be restated by replacing the expectation operators with summations:

$$\min_{\mathbf{x}_1} \quad \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \sum_{j=1}^{n_2} p_2^{j|1} \alpha_1(\mathbf{x}_1, \xi_2^j) \tag{7a}$$

$$\text{s.t.} \quad \mathbf{A}_1 \mathbf{x}_1 \geq \mathbf{b}_1 \tag{7b}$$

$$\mathbf{x}_1 \in \mathbf{X}, \tag{7c}$$

where for all $k \in D_t^j, \ j = 1, \ldots, n_t, \ t = 2, \ldots, T,$

$$\alpha_{t-1}(\mathbf{x}_{t-1}, \xi_t^k) = \min_{\mathbf{x}_t} \quad \mathbf{c}_t(\xi_t^k)\mathbf{x}_t + \sum_{l \in D_t^k} p_{t+1}^{l|k} \alpha_t(\mathbf{x}_t, \xi_{t+1}^l) \tag{7d}$$

$$\text{s.t.} \quad \mathbf{A}_t \mathbf{x}_t \geq \mathbf{b}_t - \mathbf{B}_t \mathbf{x}_{t-1} \tag{7e}$$

$$\mathbf{x}_t \in \mathbf{X}, \tag{7f}$$

and $\alpha_T = 0$. The conditional probability mass function $p_t^{k|j}$ is defined as $p_t^{k|j} = P(\tilde{\xi}_t = \xi_t^k | \tilde{\xi}^{t-1} = \xi^{t-1,j}),\ k \in D_t^j,\ j = 1, \ldots, n_t,\ t = 2, \ldots, T$, and $p_{T+1}^{k|j} = 0,\ \forall j, k$. This shows that the problem has interstage dependency.

The multistage SMIP problem can also be stated as a large-scale MIP, i.e., its deterministic equivalent, of the following form:

$$\min \quad \sum_{t=1}^{T} \sum_{j=1}^{n_t} p_t^j \mathbf{c}_t(\xi_t^j) \mathbf{x}_t^j \tag{8a}$$

$$\text{s.t.} \quad \mathbf{A}_1 \mathbf{x}_1^1 \geq \mathbf{b}_1 \tag{8b}$$

$$\mathbf{A}_t \mathbf{x}_t^j \geq \mathbf{b}_t - \mathbf{B}_t \mathbf{x}_{t-1}^{a(j)} \quad \forall j = 1, \ldots, n_t,\ t = 2, \ldots, T \tag{8c}$$

$$\mathbf{x}_t^j \in \mathbf{X} \quad \forall j = 1, \ldots, n_t,\ t = 1, \ldots, T, \tag{8d}$$

where p_t^j is the probability of node ξ_t^j and $\mathbf{x}_t^j$ is the decision vector corresponding to node $\xi_t^j,\ \forall j = 1, \ldots, n_t,\ t = 1, \ldots, T$. Notice that the nonanticipativity constraints are implicitly included in the above deterministic equivalent model since every decision vector is associated with a single node in the scenario tree.

4 Solution Methods

In this section we introduce the solution methods for solving our multistage SMIP problem. We first present the two-stage simulation method which gives an estimation of the optimal objective value by using Monte Carlo sampling to generate sample scenario trees. Then we present the exact method which applies multistage L-shaped decomposition method to obtain optimal solutions to the LP relaxation of the original problem.

4.1 Two-stage Simulation Method

The multistage SMIP problems formulated in Section 3 can be solved directly via an optimization algorithm (with a possible numerical tolerance) which works on the recourse model or its deterministic equivalent, but the computational effort increases exponentially with the scenario tree size even if each stage has a manageable number of states. To reduce the computational difficulty while obtaining an estimation of the objective value, we can adopt Monte Carlo sampling-based method to solve the multistage stochastic programs. The method works by first generating a sample scenario tree using Monte Carlo sampling and then applying optimization algorithms to the sample scenario tree to get the optimal objective function value and the corresponding optimal solution, which is an estimation of the original problem based on the entire scenario tree.

To construct a *two-stage sample scenario tree*, we perform Monte Carlo sampling in the following fashion: we begin by associating the root node with the first stage realization ξ_1 which is known at the beginning of the decision horizon. Then we enumerate all $n(1,1) = n_2$ possible stage-2 realizations $\tilde{\xi}_2$ to form the nodes in the second stage. After this, for each node ξ_2^j, $\forall j = 1, \dots, n_2$ in stage 2, we form *one* descendant by drawing $n(2,j) = 1$ observation of $\tilde{\xi}_3$ from $F_3(\xi_3|\xi^{2,j})$. This process continues until we have sampled $n(T-1,j) = 1$ observation of $\tilde{\xi}_T$ from $F_T(\xi_T|\xi^{T-1,j})$ for each node ξ_{T-1}^j, $\forall j = 1, \dots, n_{T-1}$ in stage $T-1$. In this way, $n(t,j) = 1, D_t^j = \{j\}$, $\forall j = 1, \dots, n(1,1)$, $t = 2, \dots, T$, and $a(j) = j$, $\forall t = 3, \dots, T$, $a(j) = 1$, for $t = 2$. Figure 3 shows an example of the two-stage sample scenario tree construction.

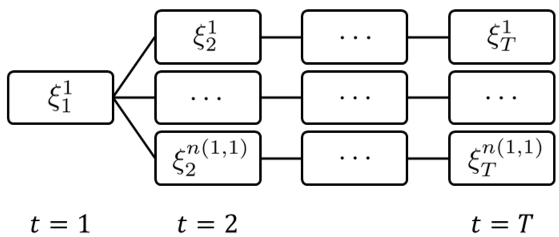

Figure 3: An example of a two-stage sample scenario tree

In the above sample scenario tree construction, stage-2 nodes consist of all possible descendants evolved from the first stage, and each of them is followed by an independently generated series of observations based on the conditional probability distributions. After constructing the sample scenario tree, each series of observations starting from stage 2, $\{\xi_t^j\}_{t=2}^T$, $\forall j = 1, \dots, n(1,1)$, will be regarded as a single deterministic problem, thus the original multistage stochastic problem can be solved as a two-stage stochastic problem. By solving the multistage SMIP based on this two-stage sample scenario tree we can obtain an approximated objective function value. We may improve the precision of the estimation by repeating the sampling and solution procedure and getting the mean of the objective values. If the number of states per stage is manageable, this method will give a good estimation result.

Based on the sample scenario tree, the two-stage SMIP problem is given by the following specific recourse model:

$$\begin{aligned}
\min_{\mathbf{x}_1} \quad & \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \mathbb{E}_{\tilde{\xi}_2|\xi^1}[\alpha(\mathbf{x}_1, \tilde{\xi}_2)] && \text{(9a)}\\
\text{s.t.} \quad & \mathbf{A}_1\mathbf{x}_1 \geq \mathbf{b}_1 && \text{(9b)}\\
& \mathbf{x}_1 \in \mathbf{X}, && \text{(9c)}
\end{aligned}$$

where for $j = 1, \ldots, n(1,1)$, i.e., all realizations of $\tilde{\xi}_2$ given ξ_1,

$$\begin{aligned}
\alpha(\mathbf{x}_1, \xi_2^j) = \min \quad & \sum_{t=2}^{T} \mathbf{c}_t(\xi_t^j)\mathbf{x}_t^j && \text{(9d)}\\
\text{s.t.} \quad & \mathbf{A}_t\mathbf{x}_t^j \geq \mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)} \quad \forall t = 2, \ldots, T && \text{(9e)}\\
& \mathbf{x}_t^j \in \mathbf{X} \quad \forall t = 2, \ldots, T. && \text{(9f)}
\end{aligned}$$

Considering the specific problem settings, $\mathbf{c}_t(\xi_t^j)\mathbf{x}_t^j,\ \forall j = 1, \ldots, n_t$ is similarly defined by (5) and constraints (9b) and (9e) have the set of constraints presented by (6) for all $j = 1, \ldots, n_t$ (note that the equality constraints are not presented as two corresponding "$\geq$" inequalities in (6), which does not affect the analysis).

Taking into account the two-stage sample scenario tree, the objective function (9a) can also be stated by replacing the expectation operator with summation:

$$\min_{\mathbf{x}_1} \quad \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \sum_{j=1}^{n(1,1)} p_2^{j|1}\alpha(\mathbf{x}_1, \xi_2^j), \tag{10}$$

where the conditional probability mass function $p_2^{j|1}$ is defined as $p_2^{j|1} = P(\tilde{\xi}_2 = \xi_2^j | \tilde{\xi}_1 = \xi_1),\ \forall j = 1, \ldots, n(1,1)$.

The deterministic equivalent of the above two-stage SMIP problem is given by

$$\begin{aligned}
\min \quad & \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \sum_{j=1}^{n(1,1)} p_2^{j|1} \sum_{t=2}^{T} \mathbf{c}_t(\xi_t^j)\mathbf{x}_t^j && \text{(11a)}\\
\text{s.t.} \quad & \mathbf{A}_1\mathbf{x}_1^1 \geq \mathbf{b}_1 && \text{(11b)}\\
& \mathbf{A}_t\mathbf{x}_t^j \geq \mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)} \quad \forall j = 1, \ldots, n(1,1),\ t = 2, \ldots, T && \text{(11c)}\\
& \mathbf{x}_t^j \in \mathbf{X} \quad \forall j = 1, \ldots, n(1,1),\ t = 1, \ldots, T. && \text{(11d)}
\end{aligned}$$

The aforementioned two-stage stochastic MIP problem can be solved directly by solving its deterministic equivalent (11), or by applying the decomposition method to its LP relaxation.

We first relax the binary restrictions (6o) as

$$0 \le z_t^i \le 1 \quad \forall t = 1, \ldots, T,\ i = 1, 2, 3, \tag{12}$$

to get the LP relaxation of the original MIP problem. We still use formulation (9) except that (12) replaces (6o) which makes it the LP relaxation. The LP relaxation of the two-stage stochastic MIP is restated as:

$$\min_{\mathbf{x}_1} \quad \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \sum_{j=1}^{n(1,1)} p_2^{j|1} \alpha(\mathbf{x}_1, \xi_2^j) \tag{13a}$$

$$\text{s.t.} \quad \mathbf{A}_1\mathbf{x}_1 \ge \mathbf{b}_1 \tag{13b}$$

$$\mathbf{x}_1 \ge 0, \tag{13c}$$

where for $j = 1, \ldots, n(1,1)$, i.e., all realizations of $\tilde{\xi}_2$ given ξ_1, the stage-2 subproblem sub(j) under realization ξ_2^j is

$$\alpha(\mathbf{x}_1, \xi_2^j) = \min \quad \sum_{t=2}^{T} \mathbf{c}_t(\xi_t^j)\mathbf{x}_t^j \tag{13d}$$

$$\text{s.t.} \quad \mathbf{A}_t\mathbf{x}_t^j \ge \mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)} \quad \forall t = 2, \ldots, T \tag{13e}$$

$$\mathbf{x}_t^j \ge 0 \quad \forall t = 2, \ldots, T, \tag{13f}$$

where (13b) and (13e) have included the second "$\le$" inequality of constraints (12) and (13c) and (13f) have included the first "$\le$" inequality of constraints (12).

Assigning dual variables $\pi_t,\ \forall t = 2, \ldots, T$ to constraints (13e), we can write the dual of sub(j), $\forall j = 1, \ldots, n(1,1)$ as

$$\max_{\pi_t} \quad \sum_{t=2}^{T} \pi_t(\mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)}) \tag{14a}$$

$$\text{s.t.} \quad \pi_t\mathbf{A}_t \le \mathbf{c}_t(\xi_t^j) \quad \forall t = 2, \ldots, T \tag{14b}$$

$$\pi_t \ge 0 \quad \forall t = 2, \ldots, T, \tag{14c}$$

and we denote π_t^j as the optimal dual solution for sub(j).

Since each optimal dual objective function provides a cut

$$\theta \ge \sum_{t=2}^{T} \pi_t^j(\mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)}) \tag{15}$$

for the stage-1 problem (the master problem), considering the conditional probability mass function of stage-2 subproblems, we should combine the $n(1,1)$ cuts (15) from stage-2 subproblems as one cut by their conditional probability as

$$\theta \geq \sum_{j=1}^{n(1,1)} p_2^{j|1} \sum_{t=2}^{T} \pi_t^j (\mathbf{b}_t - \mathbf{B}_t \mathbf{x}_{t-1}^{a(j)}). \tag{16}$$

Then each iteration of the decomposition method adds a cut (16) to the master problem until the optimality gap is within the given tolerance. We now restate the master problem as

$$\min_{\mathbf{x}_1, \theta} \quad \mathbf{c}_1(\xi_1)\mathbf{x}_1 + \theta \tag{17a}$$

$$\text{s.t.} \quad \mathbf{A}_1 \mathbf{x}_1 \geq \mathbf{b}_1 \tag{17b}$$

$$\mathbf{e}\theta \geq \vec{\mathbf{G}}\mathbf{x}_1 + \vec{\mathbf{g}} \tag{17c}$$

$$\mathbf{x}_1 \geq 0,\ \theta \text{ urs}, \tag{17d}$$

where constraint (17c) is the combined *cuts* from all iterations, and $\mathbf{e}$ is a vector of all 1's for combining all cuts. The cut-gradient matrix and the cut-intercept vector are denoted by $\vec{\mathbf{G}}$ and $\vec{\mathbf{g}}$, respectively. Observing the cut constraint (16), we form each row of the cut-gradient matrix and each component of the cut-intercept vector as

$$\mathbf{G} = -\sum_{j=1}^{n(1,1)} p_2^{j|1} \pi_2^j \mathbf{B}_2, \tag{18}$$

and

$$\mathbf{g} = \sum_{j=1}^{n(1,1)} p_2^{j|1} \sum_{t=2}^{T} \pi_t^j \mathbf{b}_t - \sum_{j=1}^{n(1,1)} p_2^{j|1} \sum_{t=3}^{T} \pi_t^j \mathbf{B}_t \mathbf{x}_{t-1}^{a(j)}, \tag{19}$$

respectively.

The decomposition algorithm for solving the LP relaxation of the two-stage SMIP problem is shown in Figure 4.

4.2 Multistage Exact Method

Exact solution methods for stochastic programs give an exact optimal solution to a stochastic program (Chiralaksanakul, 2003). For a stochastic problem whose random vector is finite with a relatively small number of realizations, its deterministic equivalent problem can be solved directly by applying the simplex method if it is a linear problem or the branch-and-bound algorithm if it

Step 0: Define $toler \geq 0$ and let $\overline{obj} = \infty$ (upper bound).

Initialize the cut for the master problem with $\theta \geq -M$.

Step 1: Solve the master problem and store its solution $(\boldsymbol{x}_1, \theta)$.

Let $\underline{obj} = \boldsymbol{c}_1 \boldsymbol{x}_1 + \theta$(lower bound).

Step 2: For j = 1 to n(1,1): (Forward traverse)

Form the right-hand side of sub(j), $\boldsymbol{b}_t - \boldsymbol{B}_t \boldsymbol{x}_{t-1}^{a(j)}$, with the optimal solution $\boldsymbol{x}_1$.

Solve sub(j), store its optimal primal and dual solutions $(\boldsymbol{x}_t^j, \pi_t^j), \forall t = 2, \ldots, T$.

Compute $\boldsymbol{G}$ and $\boldsymbol{g}$.

Step 3: Let $\widehat{obj} = \boldsymbol{c}_1 \boldsymbol{x}_1 + \sum_{j=1}^{n(1,1)} p_2^{j|1} \sum_{t=2}^{T} \boldsymbol{c}_t^j \boldsymbol{x}_t^j$.

If $\widehat{obj} < \overline{obj}$, let $\overline{obj} = \widehat{obj}$, and store the current solution of all nodes, $\boldsymbol{x}_t^j, \forall j, t$, as the optimal solution.

If $\overline{obj} - \underline{obj} \leq \min\left(|\overline{obj}|, |\underline{obj}|\right) * toler$, stop. The optimal solution is obtained within 100**toler*% of the optimal value.

Step 4: Augment the master problem's set of cuts with $\theta - \boldsymbol{G}\boldsymbol{x}_1 \geq \boldsymbol{g}$.

Goto Step 1.

Figure 4: The two-stage decomposition algorithm for the LP relaxation

is a mixed-integer problem. However, if the stochastic problem has random information with large number of realizations, its deterministic equivalent problem may be hard or even impossible to solve directly. In such conditions decomposition methods may be used for solving linear programming problems or the LP relaxation of MIP problems within a numerical tolerance (Lulli and Sen, 2004; Singh et al., 2009). Among decomposition methods, the L-shaped method (Van Slyke and Wets, 1969) is a good example which decomposes the original multistage problem by stage such that each stage has a collection of subproblems corresponding to the nodes in that stage in the scenario tree.

To apply the L-shaped method we should relax the binary restrictions (6o) as (12) to get the LP relaxation of the original MIP problem. We still use formulation (7) except that (12) replaces (6o) which makes it the LP relaxation.

Applying the single-cut L-shaped decomposition method to stage t subproblem under realization $\xi^{t,j}$, which is presented in (7d) to (7f), we can express this subproblem, denoted by sub(t, j), as

$$\min_{\mathbf{x}_t, \theta_t} \quad \mathbf{c}_t(\xi_t^j)\mathbf{x}_t + \theta_t \tag{20a}$$

$$\text{s.t.} \quad \mathbf{A}_t \mathbf{x}_t \geq \mathbf{b}_t - \mathbf{B}_t \mathbf{x}_{t-1}^{a(j)} \tag{20b}$$

$$\mathbf{e}\theta_t \geq \vec{\mathbf{G}}_t^j \mathbf{x}_t + \vec{\mathbf{g}}_t^j \tag{20c}$$

$$\mathbf{x}_t \geq 0, \ \theta_t \text{ urs}, \tag{20d}$$

for $j = 1, \ldots, n_t$, $t = 1, \ldots, T-1$. And for $t = T$, $\theta_T = 0$ and there is no constraint (20c).

Constraints (20c) are the *cuts*, which combine all cut constraints in matrix form. Specifically, $\mathbf{e}$ is a vector of all 1's combining all cuts. The cut-gradient matrix $\vec{\mathbf{G}}_t^j$ consisting of $\mathbf{G}_t^j$ and the cut-intercept vector $\vec{\mathbf{g}}_t^j$ consisting of $\mathbf{g}_t^j$ are computed from $\mathrm{sub}(t+1,k),\ k \in D_t^j$.

Assign dual variables π_t and μ_t to constraints (20b) and (20c), respectively, we can write the dual of $\mathrm{sub}(t,j)$ as

$$\max_{\pi_t,\mu_t} \quad \pi_t(\mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)}) + \mu_t\vec{\mathbf{g}}_t^j \tag{21a}$$

$$\text{s.t.} \quad \pi_t\mathbf{A}_t - \mu_t\vec{\mathbf{G}}_t^j \leq \mathbf{c}_t(\xi_t^j) \tag{21b}$$

$$\mu_t\mathbf{e} = 1 \tag{21c}$$

$$\pi_t,\ \mu_t \geq 0, \tag{21d}$$

for $j = 1,\ldots,n_t,\ t = 1,\ldots,T-1$, and

$$\max_{\pi_t} \quad \pi_t(\mathbf{b}_t - \mathbf{B}_t\mathbf{x}_{t-1}^{a(j)}) \tag{21e}$$

$$\text{s.t.} \quad \pi_t\mathbf{A}_t \leq \mathbf{c}_t(\xi_t^j) \tag{21f}$$

$$\pi_t \geq 0, \tag{21g}$$

for $j = 1,\ldots,n_T,\ t = T$. We denote (π_t^j, μ_t^j) as the optimal dual solution.

Since each optimal dual objective function provides a cut for the previous stage, by comparing (20c) with (21e) and noticing the conditional probability mass function, we form each row of the cut-gradient matrix and each component of the cut-intercept vector of $\mathrm{sub}(t,j)$ as

$$\mathbf{G}_t^j = -\sum_{k\in D_t^j} p_{t+1}^{k|j}\pi_{t+1}^k\mathbf{B}_{t+1}, \tag{22}$$

and

$$\mathbf{g}_t^j = \sum_{k\in D_t^j} p_{t+1}^{k|j}\pi_{t+1}^k\mathbf{b}_{t+1} + \sum_{k\in D_t^j} p_{t+1}^{k|j}\mu_{t+1}^k\vec{\mathbf{g}}_{t+1}^k, \tag{23}$$

respectively.

Since $\mathrm{sub}(T,j),\ j = 1,\ldots,n_T$, do not contain cut constraints and the variable θ_T, as presented in (21), the components of the cut-intercept vectors, $\mathbf{g}_{T-1}^j,\ j = 1,\ldots,n_{T-1}$, do not contain the last term in (23).

The multistage L-shaped decomposition algorithm is then presented in Figure 5.

Step 0: Define $toler \geq 0$ and let $\overline{obj} = \infty$ (upper bound).

Initialize the set of cuts for sub(t, j) with $\theta \geq -M$, for j = 1,...,n_t, t = 1,...,T-1.

Step 1: Solve sub(1,1) and store its solution $(\boldsymbol{x}_1, \theta_1)$.

Let $\underline{obj} = \boldsymbol{c}_1 \boldsymbol{x}_1 + \theta_1$(lower bound).

Step 2: For t = 2 to T: (Forward traverse)

For j = 1 to n_t:

Form the right-hand side of sub(t, j): $\boldsymbol{b}_t^j - \boldsymbol{B}_t^j \boldsymbol{x}_{t-1}^{a(j)}$.

Solve sub(t, j), store its solution $\boldsymbol{x}_t^j$.

If t = T, also store its optimal dual solution π_T^j.

Step 3: Let $\widehat{obj} = \boldsymbol{c}_1 \boldsymbol{x}_1 + \sum_{t=2}^{T} \sum_{j=1}^{n_t} p_t^j \boldsymbol{c}_t^j \boldsymbol{x}_t^j$.

If $\widehat{obj} < \overline{obj}$, let $\overline{obj} = \widehat{obj}$, and store the current solution of all nodes, $\boldsymbol{x}_t^j, \forall j, t$, as the optimal solution.

If $\overline{obj} - \underline{obj} \leq \min\left(|\overline{obj}|, |\underline{obj}|\right) * toler$, stop. The optimal solution is obtained within 100**toler*% of the optimal value.

Step 4: For t = T-1 to 2: (Backward traverse)

For j = 1 to n_t:

Augment sub(t, j)'s set of cuts with $\theta_t - \boldsymbol{G}_t^j \boldsymbol{x}_t \geq \boldsymbol{g}_t^j$.

Form the right-hand side of sub(t, j): $\boldsymbol{b}_t^j - \boldsymbol{B}_t^j \boldsymbol{x}_{t-1}^{a(j)}$.

Solve sub(t, j), store the optimal dual solution $\left(\pi_t^j, \mu_t^j\right)$.

Augment sub(1, 1)'s set of cuts with $\theta_1 - \boldsymbol{G}_1^1 \boldsymbol{x}_1 \geq \boldsymbol{g}_1^1$.

Goto Step 1.

Figure 5: The multistage L-shaped algorithm for the LP relaxation

5 Computational Results

In this section we conduct computational tests and present the computational results of different solution methods based on artificially generated data. We then make some comparisons concerning several evaluation factors such as the CPU runtime, the optimality gap, the objective value, the number of iterations, etc.

In our computational settings, the gas-fueled thermal plant has a monthly energy supply obligation of 650 units of equivalent gas at $170 per unit. The monthly contract gas volume is 1000 units at $100 per unit. In this contract, the percentages of monthly ToP and annual ToP are 50% and 60%, respectively. The plant has a fixed monthly operational cost of $6500 and a variable operational cost of 10 units of equivalent gas. The monthly length of usable time for each maintenance inspection is 30 days and the gas consumption per operating day is 20 units. The maintenance specifications are listed in Table 3.

Maintenance	Frequency (days)	Avg. Duration (days)	Cost(K$)
Combustion	70	5	10
Hot gas path	100	10	20
Major	160	15	25

Table 3: Maintenance specifications

The spot electricity price is the only stochastic parameter in our problem, which has S possible scenarios in each stage. In our computational tests we consider $S = 2$ and $S = 3$, which result in a binary scenario tree and a ternary scenario tree, respectively. For the binary scenario tree, the possible realizations and the transition probabilities of the spot electricity price are given by Table 4. The initial stage takes the low value 90 of the spot price.

q	90	130
90	0.8	0.2
130	0.2	0.8

Table 4: Possible spot electricity price and its transition probabilities for $S = 2$

For the ternary scenario tree, the possible realizations and the transition probabilities are given by Table 5. The initial stage takes the medium value 110 of the spot price.

q	90	110	130
90	0.7	0.2	0.1
110	0.15	0.7	0.15
130	0.1	0.2	0.7

Table 5: Possible spot electricity price and its transition probabilities for $S = 3$

We test the solution methods for fifteen problem instances that differ by the number of stages in the binary scenario tree (eight problems from 2 stages to 9 stages) and the number of stages in the ternary scenario tree (seven problems from 2 stages to 8 stages). For both the binary scenario tree and the ternary scenario tree, we generate the realizations (nodes) of the spot electricity price, $q_t^j,\ \forall j = 1, \ldots, n_t,\ t = 1, \ldots, T$, via the following procedure. Figure 6 illustrates the binary scenario tree of the three-stage problem instance.

Initialize q_1 with the given value, let its probability $p_1^1 = 1$;
for each stage $t = 1$ to $T - 1$ **do**
    for each node j in stage t **do**
        Generate S descendants of q_t^j by enumerating possible q values;
        for each descendant node $s = 1$ to S **do**
            Compute its probability as p_t^j times the transition probability;
        end for
    end for
end for

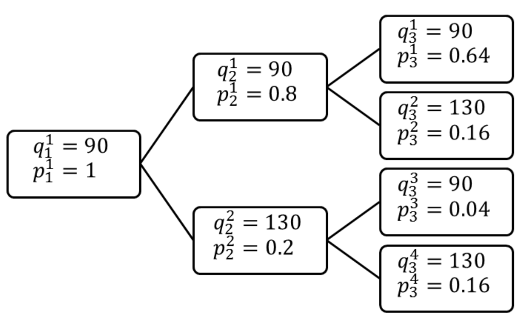

Figure 6: The problem instance with a three-stage binary scenario tree

The abbreviations we use to denote the various solution methods tested are listed in Table 6.

We now present detailed formulation of these solution methods.

For MIP-DE, the formulation given by (8) has the following detailed form in our computational tests:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T}\sum_{j=1}^{n_t} p_t^j\{c_t(m_t^j + f_t^j) + v g_t^j + V - h_t P - q_t^j(g_t^j - P) \\
& \quad + \sum_{i=1,2,3} u^i z_t^{j,i}\} && \text{(24a)} \\
\text{s.t.} \quad & m_t^j + f_t^j \geq X\%Q \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T && \text{(24b)} \\
& -m_t^j - f_t^j \geq -Q \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T && \text{(24c)} \\
& a_t^j + m_t^j + f_t^j - g_t^j \geq 0 \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T && \text{(24d)} \\
& b_t^j - f_t^j \geq 0 \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T && \text{(24e)} \\
& -r_t^{j,i} \geq -\Delta^i \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T,\ i = 1,2,3 && \text{(24f)} \\
& \Delta^i z_t^{j,i} + r_t^{j,i} \geq \delta^i \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T,\ i = 1,2,3 && \text{(24g)} \\
& -C\sigma^i z_t^{j,i} - g_t^j \geq -C\delta^i \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T,\ i = 1,2,3 && \text{(24h)} \\
& -r_t^{j,i} \geq \delta^i - r_{t-1}^{a(j),i} - (\Delta^i + \delta^i) z_{t-1}^{a(j),i} \\
& \qquad \forall j = 1,\ldots,n_t,\ t = 2,\ldots,T,\ i = 1,2,3 && \text{(24i)} \\
& a_t^j = 0 \quad \forall j = 1,\ldots,n_t,\ t = 1,13,25,\ldots && \text{(24j)} \\
& b_t^j = (Y\% - X\%)(12Q) \quad \forall j = 1,\ldots,n_t,\ t = 1,13,25,\ldots && \text{(24k)} \\
& a_t^j = a_{t-1}^{a(j)} + m_{t-1}^{a(j)} + f_{t-1}^{a(j)} - g_{t-1}^{a(j)} \\
& \qquad \forall j = 1,\ldots,n_t,\ t \neq 1,13,25,\ldots && \text{(24l)} \\
& b_t^j = b_{t-1}^{a(j)} - f_{t-1}^{a(j)} \quad \forall j = 1,\ldots,n_t,\ t \neq 1,13,25,\ldots && \text{(24m)} \\
& b_t^j - f_t^j = 0 \quad \forall j = 1,\ldots,n_t,\ t = 12,24,36,\ldots && \text{(24n)} \\
& a_t^j, b_t^j, g_t^j, m_t^j, f_t^j, r_t^{j,i} \geq 0 \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T,\ i = 1,2,3 && \text{(24o)} \\
& z_t^{j,i} \in \{0,1\} \quad \forall j = 1,\ldots,n_t,\ t = 1,\ldots,T,\ i = 1,2,3. && \text{(24p)}
\end{aligned}$$

For LP-DE, binary constraints (24p) are substituted by

$$\begin{aligned}
z_1^{1,i} \in \{0,1\} \quad \forall i = 2,2,3 && \text{(25a)} \\
-z_t^{j,i} \geq -1 \quad \forall j = 1,\ldots,n_t,\ t = 2,\ldots,T,\ i = 1,2,3 && \text{(25b)} \\
z_t^{j,i} \geq 0 \quad \forall j = 1,\ldots,n_t,\ t = 2,\ldots,T,\ i = 1,2,3, && \text{(25c)}
\end{aligned}$$

where we leave the binary variables $z_1^{1,i},\ \forall i = 1,2,3$ unchanged and relax the binary variables for all other nodes.

For LP-BD, subproblem sub(t, j), $\forall t = 1,\dots,T,\ j = 1,\dots,n_t$ given by (20) along with the dual variables has the following formulation.

$$
\begin{array}{lllr}
\min & c_t(m_t + f_t) + vg_t + V - h_tP - q_t(g_t - P) & & \\
 & \quad + \sum_{i=1,2,3} u^i z_t^i + \theta_t & \text{Duals} & \text{(26a)} \\
\text{s.t.} & m_t + f_t \geq X\%Q & \beta_t & \text{(26b)} \\
 & -m_t - f_t \geq -Q & \gamma_t & \text{(26c)} \\
 & a_t + m_t + f_t - g_t \geq 0 & \epsilon_t & \text{(26d)} \\
 & b_t - f_t \geq 0 & \zeta_t & \text{(26e)} \\
 & -r_t^i \geq -\Delta^i \quad \forall i = 1,2,3 & \eta_t^i & \text{(26f)} \\
 & \Delta^i z_t^i + r_t^i \geq \delta^i \quad \forall i = 1,2,3 & \theta_t^i & \text{(26g)} \\
 & -C\sigma^i z_t^i - g_t \geq -C\delta^i \quad \forall i = 1,2,3 & \kappa_t^i & \text{(26h)} \\
 & -r_t^i \geq \delta^i - r_{t-1}^{a(j),i} - (\Delta^i + \delta^i) z_{t-1}^{a(j),i} \quad \forall i = 1,2,3 & \iota_t^i & \text{(26i)} \\
 & a_t = 0 \quad \forall t = 13, 25, 37, \dots & \lambda_t & \text{(26j)} \\
 & b_t = (Y\% - X\%)(12Q) \quad \forall t = 13, 25, 37, \dots & \phi_t & \text{(26k)} \\
 & a_t = a_{t-1}^{a(j)} + m_{t-1}^{a(j)} + f_{t-1}^{a(j)} - g_{t-1}^{a(j)} \quad \forall t \neq 13, 25, 37, \dots & \nu_t & \text{(26l)} \\
 & b_t = b_{t-1}^{a(j)} - f_{t-1}^{a(j)} \quad \forall t \neq 13, 25, 37, \dots & \rho_t & \text{(26m)} \\
 & b_t - f_t = 0 \quad \forall t = 12, 24, 36, \dots & \tau_t & \text{(26n)} \\
 & -z_t^i \geq -1 \quad \forall i = 1,2,3 & \upsilon_t & \text{(26o)} \\
 & \mathbf{e}\theta_t \geq \vec{\mathbf{G}}_t^j \mathbf{x}_t + \vec{\mathbf{g}}_t^j & \vec{\mu}_t & \text{(26p)} \\
 & a_t, b_t, g_t, m_t, f_t, r_t^i, z_t^i \geq 0 \quad \forall i = 1,2,3 & & \text{(26q)} \\
 & \theta_t \text{ urs}, & & \text{(26r)}
\end{array}
$$

for $j = 1,\dots,n_t, t = 1,\dots,T-1$, except the constraints with explicitly stated t values. For $t = 1$, we do not assign dual variables, and constraints (26o) and the sign restriction on $z_1^i,\ \forall i = 1,2,3$ in (26q) are substituted by

$$z_1^1 \in \{0,1\}, \tag{27}$$

i.e., we do not relax the binary variables in stage 1. For $t = T$, $\theta_T = 0$ and there is no constraint (26p). Also notice that $\mu_t,\ \forall t=2,\ldots,T-1$ are dual vectors for all cuts. Comparing the above formulation with (22) and (23) we have

$$\begin{aligned}\mathbf{G}_t^j\mathbf{x}_t = -\sum_{k\in D_t^j} p_{t+1}^{k|j}\{&\sum_{i=1,2,3}\iota_{t+1}^{k,i}(r_{t+1}^{k,i}+(\Delta^i+\delta^i)z_{t+1}^{k,i})-\rho_{t+1}^k(b_{t+1}^k-f_{t+1}^k)\\&-\nu_{t+1}^k(a_{t+1}^k+m_{t+1}^k+f_{t+1}^k-g_{t+1}^k)\},\end{aligned}\tag{28}$$

and

$$\begin{aligned}\mathbf{g}_t^j = \sum_{k\in D_t^j} p_{t+1}^{k|j}\{&\beta_{t+1}^k(X\%Q)-\gamma_{t+1}^kQ-\sum_{i=1,2,3}\eta_{t+1}^{k,i}\Delta^i+\sum_{i=1,2,3}\theta_{t+1}^{k,i}\delta^i\\&-C\sum_{i=1,2,3}\kappa_{t+1}^{k,i}\delta^i+\phi_{t+1}^k(Y\%-X\%)(12Q)\\&-\sum_{i=1,2,3}\upsilon_{t+1}^k+V-h_{t+1}P+q_{t+1}^kP\}\\+\sum_{k\in D_t^j}&p_{t+1}^{k|j}\vec{\mu}_{t+1}\vec{\mathbf{g}}_{t+1}^k,\end{aligned}\tag{29}$$

where $\mathbf{g}_{T-1}^j,\ \forall j=1,\ldots,n_{T-1}$ do not contain the last term in (29).

After constructing the binary and ternary scenario trees, LP-DE and LP-BD are tested and the corresponding computational results are obtained.

For TS-DE and TS-BD, we first construct the two-stage sample scenario tree in the following fashion (for both the binary tree and the ternary tree). Figure 7 illustrates a two-stage ternary sample scenario tree of the four-stage problem instance.

Initialize q_1 with the given value, let its probability $p_1^1=1$;
Generate S stage-2 descendant nodes by enumerating possible q values, store the conditional probabilities as their probabilities;
if Decision horizon $T=2$ **then**
    stop;
else
    for each stage-2 node $s=1$ to S **do**
        for each stage $t=3$ to T **do**
            Sample node q_t^s based on the transition probability distribution;
        end for
    end for

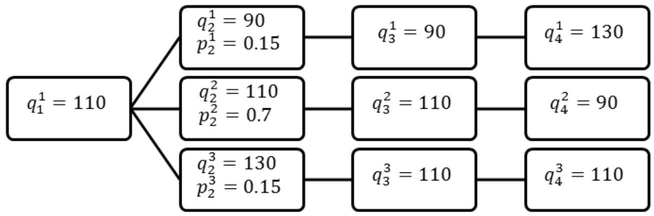

Figure 7: The four-stage problem instance with a two-stage ternary sample scenario tree

end if

After constructing the sample scenario tree, we test TS-DE and TS-BD. For TS-DE, the deterministic equivalent of the LP relaxation of (17) is as follows.

$$\begin{aligned} \min \quad & c_1(m_1^1 + f_1^1) + vg_1^1 + V - h_1P - q_1^j(g_1^1 - P) + \sum_{i=1,2,3} u^i z_1^{1,i} \\ & + \sum_{j=1}^{S} p_2^j \sum_{t=2}^{T} \{c_t(m_t^j + f_t^j) + vg_t^j + V - h_tP - q_t^j(g_t^j - P) \\ & + \sum_{i=1,2,3} u^i z_t^{j,i}\} \qquad (30a) \\ \text{s.t.} \quad & (24b) - (24o), (25a) - (25c). \end{aligned}$$

For TS-BD, the stage-2 subproblem sub(j), $\forall j = 1, \ldots, S$ given by (13d)-(13f), along with the dual variables, has the following formulation.

$$
\begin{aligned}
\min \quad & \sum_{t=2}^{T}\{c_t(m_t^j + f_t^j) + vg_t^j + V - h_tP - q_t^j(g_t^j - P) \\
& \quad + \sum_{i=1,2,3} u^i z_t^{j,i}\} && \text{Duals} && (31a)\\
\text{s.t.} \quad & m_t^j + f_t^j \geq X\%Q \quad \forall t = 2,\ldots,T && \beta_t && (31b)\\
& -m_t^j - f_t^j \geq -Q \quad \forall t = 2,\ldots,T && \gamma_t && (31c)\\
& a_t^j + m_t^j + f_t^j - g_t^j \geq 0 \quad \forall t = 2,\ldots,T && \epsilon_t && (31d)\\
& b_t^j - f_t^j \geq 0 \quad \forall t = 2,\ldots,T && \zeta_t && (31e)\\
& -r_t^{j,i} \geq -\Delta^i \quad \forall t = 2,\ldots,T,\ i = 1,2,3 && \eta_t^i && (31f)\\
& \Delta^i z_t^{j,i} + r_t^{j,i} \geq \delta^i \quad \forall t = 2,\ldots,T,\ i = 1,2,3 && \theta_t^i && (31g)\\
& -C\sigma^i z_t^{j,i} - g_t^j \geq -C\delta^i \quad \forall t = 2,\ldots,T,\ i = 1,2,3 && \kappa_t^i && (31h)\\
& -r_t^{j,i} \geq \delta^i - r_{t-1}^{a(j),i} - (\Delta^i + \delta^i)z_{t-1}^{a(j),i} \\
& \quad \forall t = 2,\ldots,T,\ i = 1,2,3 && \iota_t^i && (31i)\\
& a_t^j = 0 \quad \forall t = 13,25,37,\ldots && \lambda_t && (31j)\\
& b_t^j = (Y\% - X\%)(12Q) \quad \forall t = 13,25,37,\ldots && \phi_t && (31k)\\
& a_t^j = a_{t-1}^{a(j)} + m_{t-1}^{a(j)} + f_{t-1}^{a(j)} - g_{t-1}^{a(j)} \quad \forall t \neq 13,25,37,\ldots && \nu_t && (31l)\\
& b_t^j = b_{t-1}^{a(j)} - f_{t-1}^{a(j)} \quad \forall t \neq 13,25,37,\ldots && \rho_t && (31m)\\
& b_t^j - f_t^j = 0 \quad \forall t = 12,24,36,\ldots && \tau_t && (31n)\\
& -z_t^{j,i} \geq -1 \quad \forall t = 2,\ldots,T,\ i = 1,2,3 && \upsilon_t && (31o)\\
& a_t^j, b_t^j, g_t^j, m_t^j, f_t^j, r_t^{j,i}, z_t^{j,i} \geq 0 \quad \forall t = 2,\ldots,T,\ i = 1,2,3. && && (31p)
\end{aligned}
$$

Then we can write the cut given by (16) as

$$
\begin{aligned}
\theta \geq \sum_{j=1}^{S} p_2^j \sum_{t=2}^{T} \{ & \beta_t^j(X\%Q) - \gamma_t^j Q - \sum_{i=1,2,3}\eta_t^{j,i}\Delta^i + \sum_{i=1,2,3}\theta_t^{j,i}\delta^i \\
& - C\sum_{i=1,2,3}\kappa_t^{j,i}\delta^i + \sum_{i=1,2,3}\iota_t^{j,i}(\delta^i - r_{t-1}^{a(j),i} - (\Delta^i + \delta^i)z_{t-1}^{a(j),i}) \\
& + \phi_t^j(Y\% - X\%)(12Q) + \nu_t^j(a_{t-1}^{a(j)} + m_{t-1}^{a(j)} + f_{t-1}^{a(j)} - g_{t-1}^{a(j)}) \\
& + \rho_t^j(b_{t-1}^{a(j)} - f_{t-1}^{a(j)}) - \sum_{i=1,2,3}\upsilon_t^j + V - h_tP + q_t^jP\}. \qquad (32)
\end{aligned}
$$

Notice that some of the dual variables, including $\lambda_t^j, \phi_t^j, \nu_t^j, \rho_t^j,\ \forall j = 1, \ldots, S,\ t = 2, \ldots, T$, only exist for certain values of t, but we include them in the summation from $t = 2$ to T in (32) for the sake of uniformity.

Comparing (16) with (18) and (19) and noticing that $a(j) = 1,\ \forall j = 1, \ldots, S,\ t = 2$ and $a(j) = j,\ \forall j = 1, \ldots, S,\ t = 3, \ldots, T$, we can write $\mathbf{Gx_1}$ and $\mathbf{g}$ as

$$\mathbf{G}\mathbf{x}_1 = -\sum_{j=1}^{S} p_2^j \{ \sum_{i=1,2,3} \iota_2^{j,i}(r_1^{1,i} + (\Delta^i + \delta^i) z_1^{1,i}) - \nu_2^j (a_1^1 + m_1^1 + f_1^1 - g_1^1) - \rho_2^j (b_1^1 - f_1^1) \}, \tag{33}$$

and

$$\begin{aligned}
\mathbf{g} =& \sum_{j=1}^{S} p_2^j \sum_{t=2}^{T} \{ \beta_t^j (X\%Q) - \gamma_t^j Q - \sum_{i=1,2,3} \eta_t^{j,i} \Delta^i + \sum_{i=1,2,3} \theta_t^{j,i} \delta^i \\
&- C \sum_{i=1,2,3} \kappa_t^{j,i} \delta^i + \phi_t^j (Y\% - X\%)(12Q) - \sum_{i=1,2,3} \upsilon_t^j + V - h_t P + q_t^j P \} \\
&+ \sum_{j=1}^{S} p_2^j \sum_{t=3}^{T} \{ \sum_{i=1,2,3} \iota_t^{j,i} (\delta^i - r_{t-1}^{j,i} - (\Delta^i + \delta^i) z_{t-1}^{j,i}) + \rho_t^j (b_{t-1}^j - f_{t-1}^j) \\
&+ \nu_t^j (a_{t-1}^j + m_{t-1}^j + f_{t-1}^j - g_{t-1}^j) \}.
\end{aligned} \tag{34}$$

Finally the master problem given by (17) is as follows.

$$\min \quad c_1(m_1^1 + f_1^1) + vg_1^1 + V - h_1 P - q_1^1(g_1^1 - P) + \sum_{i=1,2,3} u^i z_1^{1,i} + \theta \tag{35a}$$

$$\text{s.t.} \quad m_1^1 + f_1^1 \geq X\%Q \tag{35b}$$

$$- m_1^1 - f_1^1 \geq -Q \tag{35c}$$

$$a_1^1 + m_1^1 + f_1^1 - g_1^1 \geq 0 \tag{35d}$$

$$b_1^1 - f_1^1 \geq 0 \tag{35e}$$

$$- r_1^{1,i} \geq -\Delta^i \quad \forall i = 1, 2, 3 \tag{35f}$$

$$\Delta^i z_1^{1,i} + r_1^{1,i} \geq \delta^i \quad \forall i = 1, 2, 3 \tag{35g}$$

$$- C\sigma^i z_1^{1,i} - g_1^1 \geq -C\delta^i \quad \forall i = 1, 2, 3 \tag{35h}$$

$$a_1^1 = 0 \tag{35i}$$

$$b_1^1 = (Y\% - X\%)(12Q) \tag{35j}$$

$$\mathbf{e}\theta \geq \vec{\mathbf{G}}\mathbf{x}_1 + \vec{\mathbf{g}} \tag{35k}$$

$$a_1^1, b_1^1, g_1^1, m_1^1, f_1^1, r_1^{1,i} \geq 0 \quad \forall i = 1, 2, 3 \tag{35l}$$

$$z_1^{1,i} \in \{0, 1\} \quad \forall i = 1, 2, 3 \tag{35m}$$

$$\theta \text{ urs.} \tag{35n}$$

In our computational tests, we test TS-DE and TS-BD on ten independently generated sample scenario trees for each problem instance (we construct only one sample scenario tree for the two-stage problem instance since the Monte Carlo sampling starts from the third stage), and we compute the mean value of each evaluation factor for comparisons. The overall TS-DE and TS-BD computational tests are displayed as follows.

for each problem instance (T, S) **do**
    if $T = 2$ **then**
        Construct one sample scenario tree;
    else
        Construct ten sample scenario trees independently;
    end if
    for each sample scenario tree **do**
        Perform TS-DE and TS-BD, get computational results;
    end for

Compute mean value of each evaluation factor;

end for

We implement and test our solution methods on a desktop computer with an Intel Core 2 Duo 2.8 GHz processor and 3 GB of RAM. The programs are coded in Visual Studio 2008 developing environment and the problems are solved with CPLEX, version 12.1, from IBM ILOG.

Abbreviation	Formulation and Solution Method
MIP-DE	Original SMIP problem, solved as a deterministic equivalent.
LP-DE	LP relaxation of the original SMIP problem, solved as a deterministic equivalent.
TS-DE	Two-stage simulation model of the LP relaxation, solved as a deterministic equivalent.
LP-BD	LP relaxation of the original SMIP problem, solved with Benders decomposition method.
TS-BD	Two-stage simulation model of the LP relaxation, solved with Benders decomposition method.

Table 6: Abbreviation for various solution methods

Table 7 displays the scenario tree statistics for the fifteen problem instances, along with their objective values as the original SMIP problem, the LP relaxation, and the two-stage simulation method. For MIP, the largest problem instance with a ternary scenario tree it can solve within 60 seconds is the seven-stage problem. The objective values obtained from LP are smaller than those from MIP for most of the problem instances, which is natural since we are solving minimization problems. The objective values of TS are more or less (very close) around those of LP, which is the consequence of the sample scenario tree construction using Monte Carlo sampling.

In Table 8 we compare the solution times for each method. The solution times increase exponentially with the tree size for all methods tested. The solution times for MIP-DE increase most rapidly due to the increasing complexity of large scale MIP problems. The value in parentheses for MIP-DE gives the optimality gap at 60 seconds. LP-BD's solution times increase faster than LP-DE's, which is the consequence of using the multistage L-shaped algorithm in such computational settings. In each iteration the L-shaped decomposition method makes a forward traversal (solving

Scenario tree statistics (num.)				Objective value ($)		
Stages	State/stage	Scenarios	Nodes	MIP	LP	TS
2	2	2	3	-70,600	-70,600	-70,600
2	3	3	4	-65,300	-65,300	-65,300
3	2	4	7	-95,460	-103,300	-103,140
3	3	9	13	-88,340	-96,280	-96,660
4	2	8	15	-109,536	-131,779	-131,763
4	3	27	40	-100,918	-123,396	-123,713
5	2	16	31	-142,681	-158,565	-158,328
5	3	81	121	-133,825	-149,104	-147,508
6	2	32	63	-141,464	-182,551	-182,987
6	3	243	364	-131,224	-172,266	-171,851
7	2	64	127	-172,566	-205,166	-205,867
7	3	729	1,093	-163,374	-194,200	-192,917.6
8	2	128	255	-186,431	-227,716	-225,611.8
8	3	2187	3,280	–	-216,190	-214,419.2
9	2	256	511	-209,457	-250,193	-249,922.8

Table 7: Objective values for each method

each subproblem) and a backward traversal (generating cuts), and in each traversal all subproblems, at each node of the scenario tree, are solved once, which significantly lengthens the solution times. Two dashes in the column of LP-BD indicate that these problem instances run out of the RAM when tested with our desktop computer, since the consumption of memory using Benders decomposition methods increases exponentially with the tree size. We find that TS-DE and TS-BD have the lowest mean solution times for most of the problem instances, and their solution times have no obvious trend of increasing, this is because the sample scenario tree size increases linearly when the number of stages or states per stage grows. Therefore TS can provide good estimations of the objective value within shortest solution times.

Table 9 presents the solution times for LP-BD to reach different relative optimality gaps. We may find that performing LP-BD to reach a 5% relative optimality gap can save solution times significantly when the scenario tree size increases, and a 5% relative optimality gap gives good estimation of the objective values in our problem instances.

Table 10 displays the number of iterations when LP-BD and TS-BD terminates (The number of iterations for TS-BD is the mean value of the ten sample scenario trees). It shows similar results as the tables for solution times. We may also find that it is better to run LP-BD to reach a 5% relative optimality gap, since it significantly saves the number of iterations to reach the gap and 5% is good enough for our problem instances.

6 Conclusions

We have developed a multistage SMIP model for optimizing the gas contract and scheduling the maintenance inspections for a gas-fueled thermal plant in a hydro dominated power system. The model jointly takes into consideration the specifications of the power supply obligation, the gas supply contract with take-or-pay and make-up clauses, the potential profit of trading in the spot electricity market, and the maintenance scheduling problem. The problem involves decision making under uncertainty because any decision made in one stage has impact in the future stages considering the evolution of the stochastic parameters. We explore several solution methods for the multistage SMIP problem and conduct computational tests. From the computational results we find that the two-stage simulation method using Monte Carlo sampling can provide a good estimation of the objective value in linear times. However, we should solve the LP relaxation, either by solving its deterministic equivalent directly or by applying decomposition methods such as the L-shaped method, to obtain the optimal policy (optimal solutions for each stage and each sce-

Scenario tree statistics (num.)				Deterministic equivalent (sec.)			Benders decomposition (sec.)	
Stages	State/stage	Scenarios	Nodes	MIP-DE	LP-DE	TS-DE	LP-BD	TS-BD
2	2	2	3	0.187	0.203	0.203	0.265	0.359
2	3	3	4	0.172	0.234	0.203	0.203	0.359
3	2	4	7	0.141	0.234	0.1545	0.421	0.3749
3	3	9	13	0.156	0.14	0.1612	0.485	0.2344
4	2	8	15	0.157	0.25	0.1503	0.984	0.3783
4	3	27	40	0.219	0.172	0.1561	1.344	0.2283
5	2	16	31	0.188	0.25	0.1564	2.047	0.3892
5	3	81	121	0.391	0.25	0.1625	4.437	0.3046
6	2	32	63	0.484	0.218	0.1688	4.859	0.3812
6	3	243	364	5.156	0.563	0.161	24.765	0.2528
7	2	64	127	0.875	0.297	0.1546	13.516	0.4092
7	3	729	1093	20.468	1.594	0.1641	–	0.2921
8	2	128	255	5.047	0.469	0.1576	27.578	0.3969
8	3	2187	3280	(13.65%)	6.687	0.1642	–	0.3327
9	2	256	511	25.031	0.812	0.1561	75.578	0.3095

Table 8: Solution times, in CPU seconds, for each method

Scenario tree statistics (num.)				LP-BD (sec.)			
Stages	State/stage	Scenarios	Nodes	10%	5%	1%	0%
2	2	2	3	0.172	0.266	0.297	0.265
2	3	3	4	0.188	0.188	0.297	0.203
3	2	4	7	0.281	0.516	0.422	0.421
3	3	9	13	0.297	0.359	0.625	0.485
4	2	8	15	0.438	0.547	0.625	0.984
4	3	27	40	0.609	0.719	1.156	1.344
5	2	16	31	0.812	0.875	1.375	2.047
5	3	81	121	1.719	1.688	2.781	4.437
6	2	32	63	1.235	1.828	2.532	4.859
6	3	243	364	5.469	7.844	11.234	24.765
7	2	64	127	2.75	3.422	5.344	13.516
7	3	729	1093	–	–	–	–
8	2	128	255	6.375	7.734	10.547	27.578
8	3	2187	3280	–	–	–	–
9	2	256	511	15.172	20.812	25	75.578

Table 9: Solution times for LP-BD to reach relative optimality gaps of 10%, 5%, 1%, and 0%

				Benders decomposition				
Scenario tree statistics (num.)				LP-BD				TS-BD
Stages	State/stage	Scenarios	Nodes	10%	5%	1%	0%	0%
2	2	2	3	1	1	3	3	3
2	3	3	4	1	1	1	1	3
3	2	4	7	3	4	6	6	5
3	3	9	13	2	3	5	6	2
4	2	8	15	4	5	7	12	5
4	3	27	40	3	4	8	10	2
5	2	16	31	6	6	11	17	5
5	3	81	121	4	4	8	14	2
6	2	32	63	5	8	12	23	4
6	3	243	364	4	7	11	26	2
7	2	64	127	7	9	15	33	5
7	3	729	1093	–	–	–	–	2
8	2	128	255	7	9	13	33	5
8	3	2187	3280	–	–	–	–	2
9	2	256	511	9	13	16	42	3

Table 10: Number of iterations for LP-BD (to reach relative optimality gaps of 10%, 5%, 1%, and 0%) and TS-BD (to reach 0% relative optimality gap)

nario). By testing the Benders decomposition methods to reach different relative optimality gaps, we find 5% is good since it most effectively saves the solution time and the number of iterations while resulting in small deviation from optimality. Although our research is based on a specific background, i.e., for a gas-fueled thermal plant in a hydro dominated power system, the model we formulated and the solution methods we developed can be applied to other problem specifications with similar characteristics.

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