Adaptive Capacity Allocation with Censored Demand Data: Application of Concave Umbrella Functions

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Abstract

One of the classical problems in revenue management is the capacity allocation problem, where the manager must allocate a fixed capacity among several demand classes that arrive sequentially in the order of increasing fares. The objective is to maximize expected revenue. For this classical problem, it has been known that one can compute the optimal protection levels in terms of the fares and the demand distributions. Contrary to conventional approaches in the literature, we consider the capacity allocation problem when the demand distributions are unknown and we only have access to historical sales, which represent censored demand data. We develop an adaptive algorithm for setting protection levels based on historical sales, show that the average expected revenue of our algorithm converges to the optimal revenue, and establish the rate of convergence. Our algorithm converges faster than any previously known algorithm for this problem. Our analysis relies on a novel concept of a concave umbrella function, which provides a lower bound for the revenue function while achieves the same maximizer and the same maximum value. Extensive numerical results show that our adaptive algorithm performs well.

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1 Introduction

Revenue management has been applied successfully to many industries, such as airlines, hotels, and restaurants (see Smith et al. (1992) for early examples). A number of overviews and surveys of the revenue management literature and practices have appeared: Etschmaier and Rothstein (1974), Belobaba (1987), Kimes (1989), Weatherford and Bodily (1992), McGill and Van Ryzin (1999), Talluri and Van Ryzin (2004), and Phillips (2005).

One of the classical problems in revenue management is the capacity allocation of a single resource among several classes of demand. Examples of such a resource include seats on a single-leg flight being sold to customers in multiple fare classes, and hotel rooms for a given night being sold to customers with different rates. In this problem, there is a fixed capacity \( C \) that must be allocated among \( N \) demand classes, indexed by \( 1, 2, \ldots, N \). Each demand class has a corresponding fare, which is exogenous and constant. Demands are stochastic and arrive sequentially, starting with class \( N \) demand and ending with class 1 demand. When the fares of demand classes are increasing with respect to the time of arrival, fare classes are said to be monotonic. This case has been analyzed in the literature by Belobaba (1987, 1989), Curry (1989), Wollmer (1992) and Li and Oum (2002). Brumelle and McGill (1993) have given an optimality condition that recursively defines a sequence of optimal protection levels in terms of “fill event” probabilities. This optimality condition is generalized by Robinson (1995) to the case of non-monotonic fare classes.

In much of the existing literature, the structure, analysis, and computation of the optimal policy hinge on the assumption that the demand distributions are known a priori. In particular, the optimality condition presented by Brumelle and McGill (1993) and Robinson (1995) requires explicit knowledge of the demand distributions. In many applications, however, the revenue manager does not know the exact demand distributions and must make allocation decisions based on historical data. Furthermore, the available data is often limited to past sales, representing censored demand observations. This occurs, for instance, when a customer does not make a booking request for sold-out fares, or when a customer’s denied booking request is not recorded in the database (as it does not involve any financial transaction).

We study the capacity allocation problem with monotonic fare classes, which is repeated over multiple selling seasons. A capacity of \( C \) units becomes available at the beginning of each season, and then the demand for each class arrives sequentially. Any demand that is not satisfied immedi-
ately upon arrival is lost. At the end of a season, any remaining capacity perishes. The manager, who does not know the demand distributions, observes only the historical sales (censored demand) data as well as the protection levels used in the past. In this setting, we develop an adaptive policy \( \phi = \{ y_t \in \mathbb{R}^N_+: t \geq 1 \} \), where \( y_t \in \mathbb{R}^N_+ \) represents the vector of protection levels in the \( t^{th} \) selling season. We require that the policy \( \phi \) is *non-anticipatory*, i.e., the decision \( y_t \) in season \( t \) may depend only on *historical sales and protection levels during the preceding \( t - 1 \) seasons*.

To our knowledge, the only other adaptive algorithm in the literature for the capacity allocation problem is due to Van Ryzin and McGill (2000), who also consider monotonic fare classes. Their algorithm, which we refer to as the VM ALGORITHM, iteratively updates the protection levels of the next selling season from the current protection levels. The updates make use of the optimality condition of Brumelle and McGill (1993). Using the results from the stochastic approximation theory (see, for example, Robbins and Monro (1951)), they show that the sequence of protection levels \( \{ y_t^{VM} : t \geq 1 \} \) generated by the VM ALGORITHM converges to the optimal protection levels at the rate of \( O \left( t^{-\beta/2^{N-1}} \right) \) for some \( \beta \in (0, 1) \). Note that the asymptotic convergence rate depends on \( N \), the number of demand classes.

In contrast, our algorithm – which we call the **Adaptive Revenue Management (ARM) Algorithm** – does *not* rely on the optimality condition of Brumelle and McGill (1993). Instead, the ARM ALGORITHM is based directly on the expected revenue \( R(\cdot) \), which is a function of the protection levels. While \( R(\cdot) \) is not a concave function, we introduce a novel concept of a *concave umbrella*, which is a lower bound of the revenue function but retains the same maximizer and the same maximum value. Maximizing the revenue function is thus equivalent to maximizing its concave umbrella. (We briefly contrast the definition of a concave umbrella with the notion of a “concave envelope” used in the mathematical programming literature, e.g., Falk and Soland (1969). Loosely speaking, a concave envelope is the “lowest” concave function that fits above the original function. Thus, while the concave envelope attains the same maximizer and the same maximum value as the original function, it provides an *upper bound*, not a lower bound.)

Our ARM ALGORITHM is based on a stochastic ascent method, where the protection levels in the current season are updated from previous protection levels. In the update, the adjustment vector corresponds to an estimate of the gradient for a *concave umbrella* of the revenue function. The step sizes are deterministic and pre-determined. While the VM ALGORITHM only allows step
sizes of the form $\Theta(1/t)$ in the analysis, our ARM ALGORITHM allows a family of step sizes in the order of $\Theta(1/t^{\alpha})$ for any $0 < \alpha \leq 1$.

To evaluate the performance of an adaptive policy $\phi$, we compare its expected $T$-season average revenue to the optimal expected revenue $R^*$ that the manager could have earned had she known the true demand distributions a priori. Let $R^\phi_t$ be a random variable representing the revenue in period $t$ under the policy $\phi$. We define the $T$-season average expected regret of $\phi$ as

$$\Delta^\phi_T \equiv R^* - E\left[\frac{1}{T}\sum_{t=1}^{T} R^\phi_t\right].$$

As the first contribution of the paper, we prove that the sequence of protection levels produced by our ARM ALGORITHM converges to the optimal protection levels, and the $T$-season average expected regret decreases at the rate of $O(1/\sqrt{T})$. This convergence rate can be improved to $O\left(1/T^{1-\varepsilon}\right)$ for any $\varepsilon > 0$ or even $O\left(\log T/T\right)$ under mild additional assumptions (see Theorem 1 and 13). Note that the asymptotic convergence rate does not deteriorate as the number of demand classes increases. We also carefully compare our algorithm to the VM ALGORITHM of Van Ryzin and McGill (2000), and report on the computational results. Table 1 summarizes the differences between our ARM ALGORITHM and the VM ALGORITHM. We provide a more detailed discussion of the comparison in Section 6.

As our second contribution, we introduce a new concept of the "concave umbrella" function, which is used in the ARM ALGORITHM (see Section 4). This idea enables us to tackle a potentially complex stochastic optimization problem by solving instead a collection of convex optimization problems. We believe that this idea is novel and can be applied to problems in many other settings in revenue and supply chain management.

The analysis in this paper is partly based on recent advances in stochastic online convex optimization. We extend the current research in this field by establishing the performance guarantee for the stochastic gradient method when the step sizes are $\Theta\left(1/t^{\alpha}\right)$ where $0 < \alpha \leq 1$ (see Theorem 2 and 3). To our knowledge, this result is new to the literature and represents our third contribution.

There has been a limited number of papers on non-parametric algorithms in the revenue management literature, until recently. Rusmevichientong et al. (2006) consider pricing decisions when

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\(^1\)The big-$\Theta$ notation refers to an asymptotically tight bound whereas the big-$O$ notation refers to an asymptotic upper bound. The general stochastic approximation approach works when the step sizes are $\Theta(1/t^\alpha)$ for any $0.5 < \alpha \leq 1$, but the proofs in Van Ryzin and McGill (2000) use $\Theta(1/t)$ step sizes.
Table 1: Comparison between the VM Algorithm and our ARM Algorithm.

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>VM Algorithm</th>
<th>ARM Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Allowable Step Sizes</strong></td>
<td>$\Theta \left( \frac{1}{t^\alpha} \right)$ for any $0.5 &lt; \alpha \leq 1$</td>
<td>$\Theta \left( \frac{1}{t^\alpha} \right)$ for any $0 &lt; \alpha \leq 1$</td>
</tr>
<tr>
<td>(see Theorems 1 and 13)</td>
<td></td>
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</tr>
<tr>
<td><strong>Adjustment Vector</strong></td>
<td>High Variability</td>
<td>Low Variability</td>
</tr>
<tr>
<td>(see Theorems 12 and Section 8.2 for more details)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>T-season Average Regret</strong></td>
<td>$O \left( \frac{1}{T^{\beta/2}N^{\beta-1}} \right)$</td>
<td>$O \left( \frac{1}{\sqrt{T}} \right)$</td>
</tr>
<tr>
<td>(see Theorems 1 and 13)</td>
<td>Depends on the number of demand classes $N$</td>
<td>Can be improved to $O \left( \frac{1}{T^{1-\epsilon}} \right)$ or $O \left( \log T/T \right)$</td>
</tr>
</tbody>
</table>

demands for multiple products depend on the prices. Ball and Queyranne (2006) and Karaesmen et al. (2006) take the competitive ratio approach against the worst possible demand realizations. Eren and Maglaras (2006) take the maximum entropy approach in the case of limited market information. Besbes and Zeevi (2006) and Kachani et al. (2006) consider demand learning when prices are decision variables. We note that all of these papers, except Van Ryzin and McGill (2000) and Rusmevichientong et al. (2006), are either concurrent with this paper or in preparation. For the stochastic inventory control problems, Levi et al. (2005) have developed nonparametric algorithms when historical demand is not censored.

Huh and Rusmevichientong (2006) have developed an adaptive algorithm that generates a sequence of order-up-to levels for inventory control problems with lost sales and censored demand. The algorithm also depends on stochastic online convex optimization, as in this paper; however, the application of the stochastic online convex optimization to the capacity allocation problem requires significantly more involved arguments. In the inventory control problem, replenishment may occur in every period, and lost sales necessarily implies stock out. For the capacity allocation problem, in contrast, no replenishment of inventory is allowed and denied sales may occur despite available inventory. Although both problems can be formulated via dynamic programming, the cost-to-go function in inventory control is convex, while the revenue-to-go function here in the capacity allocation problem is not concave. Furthermore, in this paper, a difficulty arises because the protection levels $y_t \in \mathbb{R}_+^N$ in each selling season $t$ must satisfy the monotonicity property, i.e.,
\(y_1^t \leq y_2^t \leq \cdots \leq y_N^t\). (In Section 5.2, we discuss this requirement in detail.) The monotonicity restriction, not present in the inventory control problem, and the non-concavity of the revenue function, motivate the introduction of new concepts such as the “extended revenue” and the “concave umbrella” functions. The adaptive updating of protection levels requires a collection of subtle yet important ideas to ensure both the observability of required information and provability of the convergence rate (see Section 5.2 for more details). In addition, our generalization of stochastic online convex optimization allows a family of step sizes and achieves a better performance bound than the results in Huh and Rusmevichientong (2006). This paper also includes a detailed comparison of our algorithm with the stochastic approximation method used in Van Ryzin and McGill (2000). We believe that the techniques introduced in this paper are quite general and will find other applications in various stochastic systems.

The remainder of this paper is organized as follows. Section 2 contains a detailed formulation of the capacity allocation problem and the main result of the paper. In Section 3, we prove new results on the stochastic online convex optimization. While this section can be read independently of the other sections in this paper, the results established here become useful in the analysis of the ARM Algorithm. In Section 4, we define a concave umbrella function of the expected revenue function, and establish its properties and relationships to the original revenue function. The concave umbrella function forms the basis for our adaptive algorithm, which appears in Section 5. In this section, we analyze the performance of the ARM Algorithm under the \(\Theta(1/\sqrt{T})\) step sizes, establishing \(O(1/\sqrt{T})\) convergence rate of the average expected regret. In Section 6, we carefully compare the ARM Algorithm to the VM Algorithm in terms of step sizes, adjustment directions, and performance guarantees. In Section 7, we improve the convergence rates to \(O\left(1/T^{1-\epsilon}\right)\) and \(O\left(\log T/T\right)\) under mild additional assumptions. A summary of the computational results is given in Section 8. Finally, we conclude in Section 9.

2 Problem Formulation and the Main Result

We consider a single-resource capacity allocation problem with multiple selling seasons. In each selling season, the manager must allocate a fixed capacity \(C\) among \(N\) demand classes indexed by \(\{1, 2, \ldots, N\}\). Demands in a selling season arrive sequentially, with class \(N\) demand arriving first followed by demand for class \(N - 1\), and the demand for class 1 arriving at the end of each selling
Each season is subdivided into $N$ periods, indexed backwards by $j = N, N-1, \ldots, 1$, where demand for class $j$ arrives in the corresponding period $j$.

For any $t \geq 1$ and $1 \leq j \leq N$, let the nonnegative random variable $D_j^t$ denote the class $j$ demand in the $t^{th}$ selling season. For any $j$, we assume that class $j$ demands in each selling season $D_j^1, D_j^2, \ldots$ are independent and identically distributed, and we will write $D_j$ to denote the common demand distribution for class $j$. In addition, we assume that the demand in each period of each selling season $\{D_j^t | t \geq 1, j = 1, \ldots, N\}$ are independent random variables.

Throughout the paper, we will make the following technical assumption regarding the demand distribution for each class. We remark that while Assumption 1 is not required in describing the ARM Algorithm, it is used in the analysis of its performance.

**Assumption 1.** There exists $M > 0$ such that for any $1 \leq j \leq N$, the nonnegative demand random variable $D_j$ for class $j$ demand has a continuous density function that is bounded above by $M$.

For any $1 \leq j \leq N$, the per-unit sale price (or fare) for class $j$ demand is exogenously fixed at $f_j > 0$. In each selling season, the manager must decide how much of the demand for class $j$ to satisfy. Any remaining capacity becomes available for class $j - 1$ demand, except for the last demand class ($j = 1$) where any excess capacity at the end of a selling season is scrapped. The fares associated with demand classes are monotonic with $f_N < f_{N-1} < \cdots < f_1$, and we say class $j$ is higher than $j'$ if $f_j < f_j'$, i.e., class $j$ demand arrives later than class $j'$ demand. We allow neither cancellation nor overbooking.

The decision in each selling season is denoted by a vector $(y_{N-1}, \ldots, y_1)$ where the protection level $y_j$ denotes the amount of capacity that will be reserved for demand classes $j$ and higher, i.e., classes $j, j-1, \ldots, 1$. Since there is no overbooking and the initial capacity is $C$, we require that $0 \leq y_j \leq C$ for any $1 \leq j < N$. (Note that neither $y_N$ nor $y_0$ is a decision variable and we set $y_N = C$ and $y_0 = 0$ for convenience.) The expected total revenue from all classes under the protection levels $(y_{N-1}, \ldots, y_1)$ is denoted by $\mathcal{R}(y_{N-1}, \ldots, y_1|C)$. Let $\mathcal{R}^*(C)$ and $(y_{N-1}^*, \ldots, y_1^*)$ denote the optimal expected revenue and the corresponding optimal protection levels, respectively, i.e.,

$$\mathcal{R}^*(C) \equiv \mathcal{R}(y_{N-1}^*, \ldots, y_1^*|C) = \sup_{(y_{N-1}, \ldots, y_1) \in [0,C]} \mathcal{R}(y_{N-1}, \ldots, y_1|C).$$

Since the fares are monotonic, it is a well known result (see Brumelle and McGill (1993) and Talluri and Van Ryzin (2004) for the proof) that protection-level policies are optimal and the optimal
protection levels are monotonic, i.e., \( y^*_N \geq y^*_{N-1} \geq \cdots \geq y^*_1 \). When the distributions of the demands \( D_1, \ldots, D_N \) are known, the optimal protection levels \((y^*_N, \ldots, y^*_1)\) can be computed via dynamic programming (see Brumelle and McGill (1993); Talluri and Van Ryzin (2004)).

Contrary to conventional approaches, we assume in this paper that the manager does not have any information about the underlying demand distributions \textit{a priori}, and only has access to historical sales (censored demand) and the protection levels of previous seasons. The sequence of events during period \( j \) (when class \( j \) demand arrives) in the \( t \)th season is given as follows:

1. The manager observes the remaining capacity \( x^t_j \), where \( x^t_j = C \) if \( j = N \).
2. The manager determines the protection level \( y^t_{j-1} \) for the remaining demand classes \( j-1 \) and higher, i.e., classes \( j-1, j-2, \ldots, 1 \). This decision can depend only on the historical sales and protection levels from the past.
3. Demand \( d^t_j \) for class \( j \) is realized. However, we only observe the sales quantity \( u^t_j \) given by \( u^t_j = \min\{d^t_j, (x^t_j - y^t_{j-1})^+\} \).
4. The revenue \( f_j \cdot u^t_j \) is collected and recorded. If \( j > 1 \), the remaining capacity of \( x^t_{j-1} = x^t_j - u^t_j \) is available for class \( j-1 \) demand. If \( j = 1 \), any unused capacity is lost.

Note that if protection levels are monotonic (i.e., \( y^t_{N-1} \geq y^t_{N-2} \geq \cdots \geq y^t_1 \)), then for any realization of demands, the remaining capacity level \( x^t_j \) at the beginning of period \( j \) is at least \( y^t_j \); otherwise, it is possible that \( y^t_j \) exceeds \( x^t_j \), in which case no unit will be sold in period \( j \) (i.e., \( u^t_j = 0 \)).

We aim to develop an adaptive algorithm that generates a sequence of protection levels whose average expected revenue converges to the optimal expected revenue. The main result of this section is stated in the following theorem. The proof of this result appears in Section 5.3.

**Theorem 1.** Under Assumption 1, there exists a sequence of protection levels \( \{(y^t_{N-1}, y^t_{N-2}, \ldots, y^t_1) : t \geq 1\} \) such that for any \( t \), \((y^t_{N-1}, \ldots, y^t_1)\) depends only on historical sales and protection levels in the previous \( t-1 \) seasons, and the \( T \)-season average expected regret, for any \( T \geq 1 \), satisfies

\[
\mathcal{R}^* (C) - E \left[ \frac{1}{T} \sum_{t=1}^{T} \mathcal{R} \left( y^t_{N-1}, \ldots, y^t_1 | C \right) \right] \leq \left\{ \frac{2C \sqrt{T} (2 + MC)^{N-1}}{1 + MC} \right\} \cdot \frac{1}{\sqrt{T}}.
\]

The above theorem shows that we can adaptively compute a sequence of protection levels based on historical data whose expected average revenue converges to the optimal expected revenue at the rate of \( O \left( 1/\sqrt{T} \right) \). Although our algorithm relies on historical sales data to make decisions in each
season, as the benchmark for evaluating performance, we use the optimal expected revenue $\mathcal{R}^*(C)$ that we would have earned had we known the underlying demand distributions. The adaptive algorithm for generating such a sequence of protection levels – which we will refer to as the ARM Algorithm – is given in Section 5. The algorithm adjusts the protection levels in each selling season based on the gradient ascent algorithm applied to the umbrella of the extended revenue function $\tilde{\mathcal{R}}$ that will be defined in Section 4.

3 Stochastic Online Convex Optimization: Preliminaries

In this section, we present and extend the current research on stochastic online convex optimization. In online convex optimization, the optimizer does not know any information about the objective function a priori except its convexity. At each iteration, the optimizer chooses a feasible solution and obtains some information about the objective function at that feasible solution. If this information is the exact gradient of the function at the feasible solution, Zinkevich (2003) has presented an adaptive algorithm whose expected average regret over $T$ periods – the difference between the current cost and the optimal cost – diminishes at the rate of $O\left(\frac{1}{\sqrt{T}}\right)$. In stochastic online convex optimization, the information available at each iteration may be a random variable. Flaxman et al. (2004) have extended Zinkevich (2003)’s results to the setting where we only have access to an unbiased estimate of the gradient in each iteration, and have shown that the expected average regret continues to diminish at the same asymptotic rate. By imposing additional requirements on the objective function, Hazan et al. (2006) have presented an algorithm with an improved asymptotic convergence rate of $O(\log T/T)$ using step sizes of the form $\Theta(1/t)$.

In this section, we extend the results of Hazan et al. (2006) to the case where we may have biased estimates of the gradients. In addition, we establish explicit error bounds for a family of step sizes of the form $O(1/t^\alpha)$ where the parameter $\alpha$ ranges from 0 to 1. These results are important in the development and analysis of our algorithm; moreover, we believe that these results are interesting in their own right and can be used in other applications.

Let $\mathcal{S}$ be a compact convex set in $\mathbb{R}^n$. We denote by $P_{\mathcal{S}}$ the standard projection operator onto $\mathcal{S}$, i.e., for any $x \in \mathcal{S}$,

$$P_{\mathcal{S}}(x) = \arg \min \{ z \in \mathcal{S} : \|z - x\| \},$$

where $\arg \min \{ z \in \mathcal{S} : \|z - x\| \}$ denotes the set of points that minimize the distance to $x$ in $\mathcal{S}$.
where \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^n \). Also, we denote the diameter of \( S \) by \( \text{diam}(S) \), i.e., \( \text{diam}(S) = \sup_{z_1, z_2 \in S} |z_1 - z_2| \).

The main results of this section are stated in Theorems 2 and 3. The proofs of these theorems appear in Appendix A. For \( \alpha \in (0, 1) \), \( \gamma > 0 \), and \( T \geq 1 \), define

\[
\xi^I(T, \alpha, \gamma) = \frac{\text{diam}(S)}{2} \cdot \left[ \frac{1}{T^{1-\alpha}} + \frac{1}{1-\alpha} \cdot \frac{\gamma}{T^\alpha} \right],
\]

where \( \overline{B} \) is defined in Theorem 2.

**Theorem 2.** Let \( \Phi : S \rightarrow \mathbb{R} \) be a convex function and let \( z^* = \text{arg min}_{z \in S} \Phi(z) \) be its maximizer. For any \( z \in S \), let \( H(z) \) be an \( n \)-dimensional random vector defined on \( S \), and define \( \delta(z) = E[H(z) \mid z] - \nabla \Phi(z) \). Suppose that there exist \( \overline{B} \) such that for all \( z \in S \), \( E \|H(x)\|^2 \leq \overline{B}^2 \).

Let the sequence \( (z_t : t \geq 1) \) be defined by

\[
z_{t+1} = P_S(z_t - \epsilon_t H(z_t)), \quad \text{where} \quad \epsilon_t = \frac{\gamma \text{diam}(S)}{\overline{B}} \cdot \frac{1}{t^{\alpha}}
\]

for some \( \gamma > 0 \) and \( \alpha \in (0, 1) \), where \( z_1 \) is any point in \( S \). Then, for all \( T \geq 1 \),

\[
E \left[ \frac{1}{T} \sum_{t=1}^{T} \Phi(z_t) \right] - \Phi(z^*) \leq \xi(T, \alpha, \gamma) + \frac{\text{diam}(S)}{T} \sum_{t=1}^{T} E[\|\delta(z_t)\|]
\]

holds where \( \xi(T, \alpha, \gamma) = \xi^I(T, \alpha, \gamma) \).

Note that if \( \alpha \in (0, 1) \), then \( \xi^I(T, \alpha, \gamma) = O(1/T^\alpha + 1/T^{(1-\alpha)}) = O(1/T^{\min\{\alpha, 1-\alpha\}}) \), which is minimized when \( \alpha = 1/2 \). Theorem 2 extends the previous result by Huh and Rusmevichientong (2006), which establishes a similar bound for the case when \( \alpha = 1/2 \). The result of Theorem 2 will be used in the analysis of our ARM ALGORITHM in Section 4.

When we have additional information regarding the objective function, we can obtain improved convergence rate as shown in Theorem 3. We will use the result of Theorem 3 in the analysis of the extended ARM ALGORITHM in Section 7. For \( \alpha \in (0, 1] \), \( \gamma > 0 \), and \( T \geq 1 \), define

\[
\xi^H(T, \alpha, \gamma) = \begin{cases} 
\frac{\text{diam}(S)}{2} \cdot \left( \frac{\alpha \text{diam}(S)}{\gamma \bar{\epsilon} \delta^2} \right)^{1/\alpha} \cdot \frac{1}{T^{1/\alpha}} + \frac{1}{1-\alpha} \cdot \frac{\gamma}{T^{\alpha}} & \text{if } 0 < \alpha < 1 \\
\frac{\text{diam}(S)}{2} \cdot \left( \frac{1}{T} + \frac{\text{diam}(S)}{2} \cdot \frac{2 \gamma \log T}{T} \right) & \text{if } \alpha = 1 
\end{cases}
\]

where \( \bar{\epsilon} > 0 \) is given in the statement of Theorem 3 below.
**Theorem 3.** Suppose the conditions of Theorem 2 hold except that $\alpha$ belongs to $(0, 1]$. In addition, suppose that $\Phi$ is twice differentiable with continuous first and second derivatives, and the Hessian of $\Phi$ is strictly positive definite in a compact neighborhood of $z^*$, i.e., there exist a pair of $\bar{\epsilon} > 0$ and $0 < \delta < \text{diam}(S)$ such that for any $z \in S$ satisfying $\|z - z^*\| \leq \delta$, $w^T [\nabla^2 \Phi(z)] w \geq \bar{\epsilon}$ holds for any unit-length vector $w \in \mathbb{R}^n$. In addition, if $\alpha = 1$, assume $\gamma \geq \text{diam}(S) \bar{B}/(\bar{\epsilon} \delta^2)$. Then, for any $T \geq 1$, the inequality in (1) holds with $\xi(T, \alpha, \gamma) = \xi^{II}(T, \alpha, \gamma)$.

Note that if $\Phi$ is twice differentiable with continuous first and second derivatives and $\nabla^2 \Phi$ is strictly positive definite at $z^*$, then we can find $\bar{\epsilon}$ and $\delta$ satisfying the requirements in Theorem 3. In case of $0 < \alpha < 1$ (and $\gamma > 0$), Theorem 3 improves the convergence guarantee of $O(1/T^{\min\{\alpha, 1-\alpha\}})$ of Theorem 2 to $O(1/T^\alpha)$. While $O(1/T^\alpha)$ remains sub-linear, we can choose $\alpha$ as close to 1 as possible to obtain $O(1/T^{1-\epsilon})$ for any sufficiently small $\epsilon > 0$. In case of $\alpha = 1$, we obtain the convergence rate of $O(\log T/T)$ provided that we choose the scaling factor $\gamma$ sufficiently large so that $\gamma$ exceeds $\text{diam}(S) \bar{B}/(\bar{\epsilon} \delta^2)$. To ensure that $\gamma$ is sufficiently large, the knowledge of the threshold value (or its upper bound) is required. In Theorem 3, the case of $\alpha = 1$ with unbiased gradient estimates has recently been shown by Hazan et al. (2006); the case of $\alpha < 1$ does not exist in the literature, to our knowledge.

### 4 Relaxation of the Revenue Function and Its Umbrella

In this section, we introduce an extension $\tilde{R}$ of the original revenue function $R$. The extended function $\tilde{R}$ coincides with $R$ if the protectional levels are monotonic; otherwise, it provides a lower bound on $R$. We also introduce an umbrella function $U$ which is a separable concave function that provides an upper bound on the difference between the expected revenue and the optimal revenue. The umbrella function forms a basis for our adaptive method defined in Section 5.

#### 4.1 The Extended Revenue Function

The main idea behind the extended revenue function is as follows. If the protections levels are monotonic, it is the same as the original revenue function. Otherwise, the manager will be required to purchase additional capacity to meet the required protection level for future demand classes.
For any protection levels \((y_{N-1}, \ldots, y_1) \in [0, C]^{N-1}\), we define the extended revenue function \(\tilde{R}\) recursively as follows. For any \(j\), if the remaining capacity \(x_j\) at the beginning of period \(j\) is at least as large as the protection level \(y_{j-1}\), we earn a revenue of \(f_j \cdot \min \{d_j, x_j - y_{j-1}\}\), where \(d_j\) denotes the realized demand for class \(j\). The remaining capacity \(\max \{x_j - d_j, y_{j-1}\}\) is then available for the arrival of demand for class \(j - 1\). This case is the same as the original revenue function. However, if the remaining capacity \(x_j\) is less than the protection levels \(y_{j-1}\), then under the definition of the extended revenue function \(\tilde{R}\), the manager must purchase an additional capacity of \(y_{j-1} - x_j\) units at a cost \(f_j\) per unit, bringing the remaining capacity to \(y_{j-1}\). In this case, the value under the extended revenue in this period will be \(f_j \cdot (x_j - y_{j-1})\), which is negative, representing the cost of purchasing additional capacity. (Under the original revenue function, in contrast, the revenue from class \(j\) demand will be zero because there is no available capacity, and moreover, the available capacity for period \(j - 1\) will remain at \(x_j\).)

For \(1 \leq j \leq N\), let \(\tilde{R}_j(x_j, y_{j-1}, \ldots, y_1)\) denote the expected extended revenue earned during periods \(j, j - 1, \ldots, 1\) given that \(x_j\) units of capacity are available at the beginning of period \(j\) and that we use the protection levels \(y_{j-1}, \ldots, y_1\). It follows from the above definition that

\[
\tilde{R}_j(x_j, y_{j-1}, \ldots, y_1) = E \left[ f_j \cdot \min \{D_j, x_j - y_{j-1}\} + \tilde{R}_{j-1} \left( \max \{x_j - D_j, y_{j-1}\}, y_{j-2}, \ldots, y_1 \right) \right],
\]

where \(\tilde{R}_0 \equiv 0\) and \(y_0 = 0\). We define \(\tilde{R}(y_{N-1}, \ldots, y_1|C) = \tilde{R}_N(C, y_{N-1}, \ldots, y_1)\).

The following lemma establishes a relationship between the extension \(\tilde{R}\) and the original revenue function. The first part shows that the extended revenue function and the original revenue functions coincide when the protection levels are monotonic. The second part shows that if the protection levels are not monotonic, the extended revenue function is bounded above by the original revenue function evaluated at suitably “monotonized” protection levels. The last part shows that both the extended revenue and the original revenue functions share the same maximizer and the same maximum value. The proof of Lemma 4 appears in Appendix B.

**Lemma 4.** For any protection levels \((y_{N-1}, \ldots, y_1) \in [0, C]^{N-1}\), the following statements hold.

(i) If \(y_{N-1} \geq \cdots \geq y_1\), then \(\tilde{R}(y_{N-1}, \ldots, y_1|C) = R(y_{N-1}, \ldots, y_1|C)\).

(ii) Let \((\bar{y}_{N-1}, \ldots, \bar{y}_1) \in [0, C]^{N-1}\) be defined by \(\bar{y}_j = \max \{y_j, y_{j-1}, \ldots, y_1\}\) for all \(j\). Then, \(\tilde{R}(y_{N-1}, \ldots, y_1|C) \leq R(\bar{y}_{N-1}, \ldots, \bar{y}_1|C)\).
(iii) The optimal protection level \( (y_{N-1}^*, \ldots, y_1^*) \) under the original revenue function also maximizes the extension \( \tilde{R} \), and the maximum value \( \tilde{R}^*(C) \) of \( \tilde{R} \) coincides with the maximum value \( R^*(C) \) of \( R \), i.e.,

\[
\tilde{R} (y_{N-1}, \ldots, y_1|C) \leq \tilde{R} (y_{N-1}^*, \ldots, y_1^*|C) = \tilde{R}^*(C) = R^*(C) .
\]

The following example shows the extension \( \tilde{R} \) when we have two demand classes.

**Example 1.** When there are two demand classes \( (N = 2) \), it is easy to verify that

\[
R (y_1|C) = \tilde{R} (y_1|C) = f_2 \cdot E[\min \{D_2, C - y_1\}] + f_1 E[\min \{D_1, \max \{y_1, C - D_2\}\}] .
\]

Since \( D_1 \) and \( D_2 \) are independent, we have that

\[
R' (y_1|C) = \tilde{R}' (y_1|C) = -f_2 \cdot P \{D_2 > C - y\} + f_1 P \{D_2 > C - y, D_1 > y\}
\]

At the optimal protection level \( y_1^* \), we have \(-f_2 + f_1 P \{D_1 > y^*_1\} = 0\). It follows that the derivative is nonnegative for any \( y \leq y_1^* \) and is nonpositive for \( y > y_1^* \). Therefore, both the original and the extended revenue functions are quasi-concave. The quasi-concavity of the extended revenue function, however, is not true in general.

### 4.2 The Umbrella Function

Since the extended revenue function \( \tilde{R} \) may not be quasi-concave, it motivates us to instead consider a concave umbrella function of the extended revenue function, whose properties we make use of in designing our adaptive algorithm. We have the following definition.

**Definition 5.** For any function \( f : S \subseteq \mathbb{R}^K \to \mathbb{R} \), we say \( g : S \to \mathbb{R} \) is an umbrella function of \( f \) if the followings hold:

(i) \( g \) is concave.

(ii) \( g \) has the same maximizer and the same maximum value as \( f \).

(iii) If \( g^* \) and \( f^* \) are the maximum values of \( g \) and \( f \) respectively, then \( f^* - f(x) \leq g^* - g(x) \) for any \( x \in S \).
(iv) $g$ is separable, i.e., there exist $g_1, g_2, \ldots, g_K$ such that $g_i : \mathbb{R} \to \mathbb{R}$ for all $1 \leq i \leq K$ and for any $x \in S$, $g(x) = \sum_{i=1}^{K} g_i(x_i)$.

If such an umbrella function $g$ exists, then maximizing $f$ is equivalent to maximizing a concave and separable function $g$, which provides an upper bound on the gap between the current and optimal objective values.

The following lemma provides an expression for the partial derivatives of $\tilde{R}_j$, motivating the definition of an umbrella function. The proof of this result follows directly from the recursive definition of $\tilde{R}_j$, and we omit the details.

**Lemma 6.** For any $x_j$ and $(y_{j-1}, \ldots, y_1)$, the following results hold

$$
\frac{\partial \tilde{R}_j}{\partial x_j} (x_j, y_{j-1}, \ldots, y_1) = f_j + E \left[ 1(D_j \leq x_j - y_{j-1}) \cdot \left\{ -f_j + \frac{\partial \tilde{R}_{j-1}}{\partial x_{j-1}} (x_j - D_j, y_{j-2}, \ldots, y_1) \right\} \right]
$$

$$
\frac{\partial \tilde{R}_j}{\partial y_{j-1}} (x_j, y_{j-1}, \ldots, y_1) = P \{ D_j \geq x_j - y_{j-1} \} \cdot \left\{ -f_j + \frac{\partial \tilde{R}_{j-1}}{\partial x_{j-1}} (y_{j-1}, y_{j-2}, \ldots, y_1) \right\}.
$$

As shown in the above lemma, the partial derivative of $\tilde{R}_j$ with respect to the protection level $y_{j-1}$ consists of the product of two factors. The second factor is decreasing in $y_{j-1}$ if $y_{j-2}, \ldots, y_1$ correspond to the optimal protection levels (Brumelle and McGill (1993)). However, the first factor $P \{ D_j \geq x_j - y_{j-1} \}$ is nonnegative and increasing in $y_{j-1}$, contributing to the non-concavity of the extended revenue function. To avoid this problem, we will define the umbrella $U_{j-1}$ of $\tilde{R}_j$ as follows. For any $(y_j, y_{j-1}, \ldots, y_1)$, let

$$
U_j (y_j | y_{j-1}, \ldots, y_1) = \int_{y_{j-1}}^{y_j} V_j(z | y_{j-1}, \ldots, y_1) dz,
$$

where

$$
V_j (z | y_{j-1}, \ldots, y_1) = -f_{j+1} + \frac{\partial \tilde{R}_j}{\partial x_j} (z, y_{j-1}, \ldots, y_1)
$$

for any $z \in \mathbb{R}$.

The following lemma establishes properties of the function $U_j(\cdot | y_j^*, \ldots, y_1^*)$. The proof of this result appears in Appendix C.

**Lemma 7.** For any $1 \leq j \leq N - 1$, the following statements hold:

(i) $V_j(\cdot | y_j^*, \ldots, y_1^*)$ is nonincreasing and $U_j(\cdot | y_j^*, \ldots, y_1^*)$ is concave and achieves its maximum at $y_j^*$.

(ii) For any $x_j$ and $y_{j-1}$,

$$
\tilde{R}_j (x_j, y_j^* - y_{j-1}, \ldots, y_1^*) - \tilde{R}_j (x_j, y_j, y_j^* - y_{j-2}, \ldots, y_1^*) \leq U_{j-1} (y_j^* - y_{j-2}, \ldots, y_1^*) - U_{j-1} (y_j - y_{j-2}, \ldots, y_1^*)
$$
(iii) For any \( x_j \) and \( (y_{j-1}, \ldots, y_1) \),

\[
\tilde{R}_j(x_j, y^*_j \ldots, y^*_1) - \tilde{R}_j(x_j, y_{j-1}, \ldots, y_1) \leq \sum_{i=1}^{j-1} U_i(y^*_i | y^*_{i-1}, \ldots, y^*_1) - U_i(y_i | y^*_{i-1}, \ldots, y^*_1).
\]

We are now ready to define the umbrella function \( U \) of the extended revenue function \( \tilde{R} \). For any \( (y_{N-1}, \ldots, y_1) \in [0, C]^{N-1} \), define

\[
U(y_{N-1}, \ldots, y_1) = \sum_{j=1}^{N-1} U_j(y_j | y^*_j \ldots, y^*_1),
\]

and let \( U^* \) denote the maximum value of \( U \). We verify that \( U \) is indeed an umbrella function of \( \tilde{R} \).

From construction, \( U \) is a separable function, and Lemma 7(i) implies that \( U \) is jointly concave and has the same maximizer as \( \tilde{R} \). Furthermore, by Lemma 7(iii), \( U \) provides an upper bound on the difference between the revenue function and the optimal revenue. The following lemma summarizes how the umbrella function of the extended revenue function relates to the original revenue function.

**Lemma 8.** For any \( C > 0 \) and \( (y_{N-1}, \ldots, y_1) \in [0, C]^{N-1} \), let \( \bar{y}_j = \max\{y_j, y_{j-1}, \ldots, y_1\} \) for all \( j \).

Then,

\[
U^* - U(y_{N-1}, \ldots, y_1) \geq \tilde{R}^*(C) - \tilde{R}(y_{N-1}, \ldots, y_1 | C) \geq R^*(C) - R(\bar{y}_{N-1}, \ldots, \bar{y}_1 | C).
\]

**Proof.** The first inequality follows from the properties of an umbrella function and the second inequality follows from Lemma 4.

The following example shows the umbrella function in the case of two demand classes.

**Example 2.** Consider Example 1 with two demand classes. Suppose that \( f_2 = 5 \), \( f_1 = 12 \), and \( C = 20 \). Figure 1 compares the original revenue function (which in this case coincides with the extended revenue function) and its umbrella. In this case, we assume that \( D_2 \) and \( D_1 \) follow Gaussian distributions with means of 12 and 10, respectively, and variances of 4 and 16, respectively. Figure 1(a) compares the values of the two functions while Figure 1(b) shows their derivatives.

Although the definition of the umbrella function \( U \) depends on the optimal protection levels \( y^*_N-1, \ldots, y^*_1 \), we assure the reader that our adaptive algorithm will not require any knowledge of the optimal protection levels. The algorithm does not compute the umbrella function explicitly; rather, it only requires an approximation to the gradient of the umbrella function. The algorithm generates
a sequence of protection levels whose average expected value of the umbrella function converges to its maximum \( U^* \). It follows from Lemma 8 that a sequence of protection levels generated by the algorithm will converge to the optimal expected revenue with respect to the original function \( R \).

5 An Adaptive Revenue Management (ARM) Algorithm

In this section, we describe and analyze the ARM ALGORITHM that will generate a sequence of protection levels with the performance bound given in Theorem 1. The definition of the algorithm is given in Section 5.1. The algorithm maintains two sequences of protection levels, an auxiliary sequence and an implemented sequence, and we motivate the use of the two sequences in Section 5.2. Then, in Section 5.3, we analyze the performance of the algorithm and prove Theorem 1.
5.1 Description of the Algorithm

In this section, we define the adaptive revenue management algorithm – which we call the ARM Algorithm – that will generate a sequence of protection levels whose average expected revenue converges to the optimal. The ARM Algorithm updates the protection levels in each selling season based on an estimate of the gradient of the umbrella function $U$. Recall that for any $(y_{N-1}, \ldots, y_1)$,

$$ U(y_{N-1}, \ldots, y_1) = \sum_{j=1}^{N-1} U_j \left( y_j \left| y_{j-1}^*, \ldots, y_1^* \right. \right) . $$

To compute the gradient of the umbrella function $U$, it suffices to determine the derivative of $U_j \left( y_j \left| y_{j-1}^*, \ldots, y_1^* \right. \right)$ for each $j$ separately. Since we do not know the optimal protection levels $y_{N-1}^*, \ldots, y_1^*$, the ARM Algorithm will instead compute the derivative of $U_j \left( y_j \left| y_{j-1}^*, \ldots, y_1^* \right. \right)$, using it as an approximation of the true derivative. Recall that for any $(y_j, \ldots, y_1)$, we have that

$$ V_j \left( y_j \left| y_{j-1}^*, \ldots, y_1^* \right. \right) = \frac{\partial U_j}{\partial y_j} \left( y_j \left| y_{j-1}^*, \ldots, y_1^* \right. \right) . $$

Lemma 9 below provides explicit formulas for the derivative $V_j$ of the umbrella function $U_j$, from which an estimator can be constructed.

We define, for any $1 \leq j \leq N - 1$,

$$ K_j(y_j, y_{j-1}, \ldots, y_1) = \max \{ i \mid 1 \leq i \leq j \text{ and } D_j + D_{j-1} + \ldots + D_{i+1} + D_i \geq y_j - y_{i-1} \} , $$

i.e., the random variable $K_j(y_j, y_{j-1}, \ldots, y_1)$ denotes the first demand class (starting from class $j$) that is sold out assuming the capacity available at the beginning of period $j$ is $y_j$ and we use protection levels $y_{j-1}, \ldots, y_1$ thereafter. Recall $y_0 = 0$. Note that if $y_j \leq y_{j-1}$, then $K_j(y_j, y_{j-1}, \ldots, y_1) = j$.

**Lemma 9.** Let $(y_{N-1}, \ldots, y_1) \in [0, C]^{N-1}$. For any $1 \leq j \leq N - 1$, we have

\[
V_j \left( y_j \left| y_{j-1}, \ldots, y_1 \right. \right) = \begin{cases} 
-f_{j+1} + f_j + E \left[ 1 (y_j - D_j > y_{j-1}) \cdot V_{j-1} \left( y_j - D_j \left| y_{j-2}, \ldots, y_1 \right. \right) \right] , & \text{if } j \geq 2, \\
-f_2 + \partial \tilde{R}_1 (y_1) / \partial x_1 = -f_2 + f_1 \cdot \mathbb{P} \{ D_1 > y_1 \} , & \text{if } j = 1.
\end{cases}
\]

and

\[
V_j \left( y_j \left| y_{j-1}, \ldots, y_1 \right. \right) = -f_{j+1} + \sum_{i=1}^{j} f_i \cdot \mathbb{P} \{ K_j(y_j, y_{j-1}, \ldots, y_1) = i \} .
\]
Since the results of Lemma 9 involves routine analysis, we omit the details. The first part of Lemma 9 follows immediately from the definition of $V_j$ and the recursion relating the partial derivative of $\tilde{R}_{j-1}$ given in Lemma 6. The intuition underlying the second part of Lemma 9 is as follows. If $K_j(y_j, y_{j-1}, \ldots, y_1) = j$, then the demand for class $j$ exceeds the available capacity $y_j - y_{j-1}$, implying that the benefit of an additional unit of $y_j$ is $f_j$. If $K_j(y_j, y_{j-1}, \ldots, y_1) = j - 1$, then the additional unit of $y_j$ becomes useful not in period $j$, but in period $j - 1$. In general, if $K_j(y_j, y_{j-1}, \ldots, y_1) = i$, then the arrival of class $i$ demand represents the first time that we have used up the allocated capacity, and thus, the benefit of the additional unit is $f_i$. We will use the second part of Lemma 9 in the description of our ARM Algorithm.

The ARM Algorithm maintains two sequences: the auxiliary protection levels $\{(\tilde{y}_t^j, \ldots, \tilde{y}_t^1) : t \geq 1\}$ and the actual implemented protection levels $\{(\bar{y}_t^j, \ldots, \bar{y}_t^1) : t \geq 1\}$. While the auxiliary protection levels may not be monotonic, the implemented protection levels under the ARM Algorithm will always be monotone, i.e., $\bar{y}_t^j \geq \cdots \geq \bar{y}_t^1$. The definition of these sequences are given as follows.

**ARM Algorithm**

**Initialization**: Let $\kappa$ be any positive number. Let $(\tilde{y}_N^1, \ldots, \tilde{y}_1^1)$ be any vector in $[0, C]^{N-1}$. Define a vector $(\tilde{y}_N^1, \ldots, \tilde{y}_1^1)$ as follows: for any $j$,

$$\tilde{y}_j^1 = \max \{\tilde{y}_j^1, \tilde{y}_{j-1}^1, \ldots, \tilde{y}_1^1\}.$$ 

Note that $\tilde{y}_{N-1}^1 \geq \tilde{y}_{N-2}^1 \geq \cdots \geq \tilde{y}_1^1$ and $\bar{y}_j^1 \geq \tilde{y}_j^1$ for all $j$, by our construction.

**Algorithm**: For each selling season $t = 1, 2, \ldots$.

1. Implement the protection levels $(\tilde{y}_N^1, \ldots, \tilde{y}_1^1)$.

2. Demand for each class is realized. Record the sales quantity associated with each demand class. For any $j$, let $u_j^t$ denote the unit sales associated with class $j$ under the protection levels $(\tilde{y}_N^1, \ldots, \tilde{y}_1^1)$, i.e., for any $j$,

$$u_j^t = \min \{d_j^t, x_j^t - \tilde{y}_{j-1}^t\},$$

where $d_j^t$ denotes the realized demand for class $j$ in the $t^{th}$ selling season and $x_j^t$ denotes the capacity remaining at the beginning of period $j$ (before the arrival of class $j$ demand), where $x_N^t = C$ and $x_j^t = x_j^{t+1} - u_j^{t+1}$. Note that $x_j^t - \tilde{y}_{j-1}^t$ is always nonnegative by the monotonicity of the implemented protection levels.
3. For any $1 \leq j \leq N - 1$, compute $K^t_j \left( \bar{y}^t_j, \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1 \right)$, which denotes the largest index $k \in \{1, 2, \ldots, j - 1, j\}$ such that

$$u^t_j + u^t_{j-1} + \cdots + u^t_k \geq \bar{y}^t_j - \bar{y}^t_{k-1},$$

where $\bar{y}^t_0 = 0$. If no such $k$ exists, set $K^t_j = 0$. Note that $K^t_j \left( \bar{y}^t_j, \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1 \right)$ denotes the first fare class (starting from class $j$) that is sold out if the seat capacity at the beginning of class $j$ is $\bar{y}^t_j$ and we use protection levels $\bar{y}^t_{j-1}, \ldots, \bar{y}^t_1$ thereafter. Also note that if $\bar{y}^t_j \leq \bar{y}^t_{j-1}$, then $K^t_j = j$ by definition. Although we implement the monotonic protection levels $(\bar{y}^t_{N-1}, \ldots, \bar{y}^t_1)$, it is possible to compute $K^t_j \left( \bar{y}^t_j, \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1 \right)$ using the observed sales data of each class (see Section 5.2 for details).

4. For $1 \leq j \leq N - 1$, compute $H^t_j \left( \bar{y}^t_j | \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1 \right)$ given by

$$H^t_j \left( \bar{y}^t_j | \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1 \right) = -f_{j+1} + f_{K^t_j(\bar{y}^t_j, \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1)}$$

where $f_0 = 0$.

5. Update protection levels for the next selling season. For $1 \leq j \leq N - 1$, let

$$\bar{y}^{t+1}_j = P_{[0, C]} \left( \bar{y}^t_j + \epsilon^t \cdot H^t_j \left( \bar{y}^t_j | \bar{y}^t_{j-1}, \ldots, \bar{y}^t_1 \right) \right)$$

$$\hat{y}^{t+1}_j = \max \left\{ \bar{y}^{t+1}_j, \hat{y}^{t+1}_{j-1}, \ldots, \hat{y}^{t+1}_1 \right\}$$

where $\epsilon^t = \kappa / \sqrt{t}$ and $P_{[0, C]}$ denotes the projection operator onto the interval $[0, C]$, i.e., for any $z \in \mathbb{R}$, $P_{[0, C]}(z) = \min\{z^+, C\}$.

Output: Sequences of protection levels $\{(\bar{y}^t_{N-1}, \ldots, \bar{y}^t_1) : t \geq 1\}$ and $\{(\hat{y}^t_{N-1}, \ldots, \hat{y}^t_1) : t \geq 1\}$.

The analysis of this algorithm is contained in Section 5.3. Before we proceed to the analysis, we briefly review the intuition underlying the ARM ALGORITHM in the next subsection.

### 5.2 Observability and the Need to Maintain Two Sequences

The ARM ALGORITHM maintains both the auxiliary sequence of protection levels $\{(\bar{y}^t_{N-1}, \ldots, \bar{y}^t_1) : t \geq 1\}$ and the actual implemented sequence of protection levels $\{(\hat{y}^t_{N-1}, \ldots, \hat{y}^t_1) : t \geq 1\}$. Since the algorithm aims to maximize the concave umbrella function $U(y_{N-1}, \ldots, y_1)$, it would be desirable to use its gradient to update the auxiliary protection levels for the next selling season. Yet, we
cannot obtain an unbiased estimator of $V_j \left( \hat{y}_j^t | y_{j-1}^*, \ldots, y_1^* \right)$ since the optimal protections levels \( \left( y_{j-1}^*, \ldots, y_1^* \right) \) are not known.

One possible solution is to estimate $V_j \left( \hat{y}_j^t | \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t \right)$ instead of $V_j \left( \hat{y}_j^t | y_{j-1}^*, \ldots, y_1^* \right)$. According to the second part of Lemma 9, such a computation requires an evaluation of the random variable $K_j \left( \hat{y}_j^t, \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t \right)$, representing the first fare class (starting from class $j$) that is sold out under the protection levels $\hat{y}_j^t, \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t$ for each sample path of realized demands. However, unless the protection levels are monotonic (i.e., $\hat{y}_N^t \geq \cdots \geq \hat{y}_1^t$), it is impossible to compute such an estimate of the derivative because we only observe sales quantities – corresponding to censored demand observations – in each season. For instance, consider three demand classes with non-monotonic protection levels. Suppose that $C = 10$, $y_2 = 2$ and $y_1 = 8$. Since $y_2 < y_1$, if the realized demand for class 3 is $d_3 = 7$, then the remaining capacity levels just before the arrival of class 2 and class 1 demands will both be $10 - 7 = 3$. Then, it is not possible to determine whether or not the realized demand for class 1 actually exceeds $y_1 = 8$ because we only have 3 units of capacity remaining and lost sales are assumed to be unobservable.

This motivates us to define monotonic protection levels, which correspond to the implemented protection levels \( \{(\hat{y}_{N-1}^t, \ldots, \hat{y}_1^t) : t \geq 1\} \). We then use the monotonic protection levels to evaluate the random variable $K_j^t \left( \hat{y}_j^t, \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t \right)$ – which is computed based on observed sales data. Note that if $\hat{y}_j^t \leq \hat{y}_{j-1}^t$, then $K_j^t \left( \hat{y}_j^t, \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t \right) = j$ by definition. Otherwise, if $\hat{y}_j^t > \hat{y}_{j-1}^t$, then we have, for any $k \leq j$,

$$\hat{y}_j^t - \hat{y}_{k-1}^t \leq \hat{y}_j^t - \hat{y}_{k-1}^t \leq x_j^t - x_{k-1}^t,$$

where the first inequality follows from the definition of $\hat{y}_j^t$. The second inequality follows from the fact that under the implemented monotonic protection levels \( \left( \hat{y}_{N-1}^t, \ldots, \hat{y}_1^t \right) \), the remaining capacity $x_j^t$ (just before the arrival of class $j$ demand) is always at least as large as $\hat{y}_j^t$. Since the remaining capacity $x_j^t - \hat{y}_{k-1}^t$ for class $k$ demand ($k \leq j$) is at least as large as $\hat{y}_j^t - \hat{y}_{k-1}^t$, we can determine whether $K_j^t \left( \hat{y}_j^t, \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t \right)$ equals $k$ by checking whether the sales of class $j$ through $k$ is at least $\hat{y}_j^t - \hat{y}_{k-1}^t$. We can thus compute the random variable $K_j^t \left( \hat{y}_j^t, \hat{y}_{j-1}^t, \ldots, \hat{y}_1^t \right)$ based on the observed sales data.
5.3 Analysis of the ARM Algorithm and Proof of Theorem 1

In this section, we establish the performance bound for the ARM algorithm, proving that as the number of seasons increases, the average expected value under the umbrella function $U$ converges to the maximum $U^*$ at the rate of $O(1/\sqrt{T})$. This result is stated in the following lemma.

**Lemma 10.** Let $\{(\hat{y}_{t}^j)_{t \geq 1} : t \geq 1\}$ denote the auxiliary sequence of protection levels generated by the ARM Algorithm. Then, for any $T \geq 1$,

$$U^* - E \left[ \frac{1}{T} \sum_{t=1}^{T} U(\hat{y}_{N-1}^j, \ldots, \hat{y}_1^j) \right] \leq \left\{ \frac{\kappa^2 f_1 + C^2}{\kappa} \cdot \frac{(2 + MC)^{N-1}}{1 + MC} \right\} \cdot \frac{1}{\sqrt{T}}.$$

Before we proceed to the proof of Lemma 10, let us show how the above result leads to the proof of Theorem 1.

**Proof of Theorem 1:** It follows from Lemma 8 and 10 that

$$R^*(C) - E \left[ \frac{1}{T} \sum_{t=1}^{T} R(\hat{y}_{N-1}^j, \ldots, \hat{y}_1^j) \right] \leq \left\{ \frac{\kappa^2 f_1 + C^2}{\kappa} \cdot \frac{(2 + MC)^{N-1}}{1 + MC} \right\} \cdot \frac{1}{\sqrt{T}},$$

and the desired result follows from setting $\kappa = C/\sqrt{f_1}$.

The proof of Lemma 10 relies on the following lemma, which establishes an upper bound on the difference in the derivative. In particular, the bound on the difference in $V_j$ is established using the differences in the umbrella function values of $U_k$’s for $k \in \{j - 1, j - 2, \ldots, 1\}$.

**Lemma 11.** For any $C$ and $(y_{N-1}, \ldots, y_1)$, suppose $0 \leq y_j \leq C$ for each $j = N - 1, \ldots, 1$. Let $ar{y}_j = \max\{y_j, y_{j-1}, \ldots, y_1\}$. Then,

$$0 \leq V_j(y_j, y_{j-1}, \ldots, y_1) - V_j(y_j^*, y_{j-1}^*, \ldots, y_1^*) \leq M \sum_{i=1}^{j-1} \max_{k \in \{1, 2, \ldots, i\}} \{U_k(y_k^* | y_{k-1}^*, \ldots, y_1^*) - U_k(y_k | y_{k-1}, \ldots, y_1)\}.$$

Recall that $M > 0$ is an upper bound on the density function of demand random variables defined in Assumption 1. The proof of Lemma 11 appears in Appendix E.

The proof of Lemma 10 makes use of the result from online stochastic convex optimization (Theorem 2) and is given below.

**Proof.** In this proof, we repeatedly use the single-dimensional version of Theorem 2 with $\alpha = 1/2$, $\text{diam}(S) = C$ and $\bar{B} = f_1$. Note that by Lemma 9, the absolute value of $V_j$ is bounded above
by $f_1$. Since the step size under the ARM algorithm is given by $\varepsilon_t = \kappa / \sqrt{t}$, we have that $\varepsilon_t = \gamma \cdot \text{diam}(S) / (B \sqrt{t})$, where $\gamma = \kappa f_1 / C$. Let

$$
\eta(T) = \left(\gamma + \frac{1}{\gamma}\right) \frac{\text{diam}(S)B}{\sqrt{T}}.
$$

For any $j = 1, 2, \ldots, N - 1$, let $\Delta S_j^T$ and $\Delta V_j^T$ be defined by

$$
\Delta U_j^T = \frac{1}{T} \cdot \sum_{t=1}^{T} E \left[ U_j(y_j^*|y_{j-1}^*, \ldots, y_1^*) - U_j(\hat{y}_j^*|y_{j-1}^*, \ldots, y_1^*) \right], \quad \text{and}
$$

$$
\Delta V_j^T = \frac{1}{T} \cdot \sum_{t=1}^{T} E \left[ V_j(\hat{y}_j^*|y_{j-1}^*, \ldots, y_1^*) - V_j(\hat{y}_j^*|y_{j-1}^*, \ldots, y_1^*) \right].
$$

For $j = 1$, we have $\Delta U_1^T = \sum_{t=1}^{T} E \left[ U_1(y_1^*) - U_1(\hat{y}_1^*) \right] / T$, and

$$
H_1^T(\hat{y}_1^*) = -f_2 + f_1 \cdot 1[D_1^T \geq \hat{y}_1^*].
$$

Thus, $H_1^T(\hat{y}_1^*)$ is an unbiased estimator of $V_1(\hat{y}_1^*)$. By Theorem 2, we obtain $\Delta U_1^T \leq \eta(T)$.

For $j \geq 2$, by Theorem 2,

$$
\Delta U_j^T \leq \eta(T) + \text{diam}(S) \cdot \Delta V_j^T
$$

$$
\leq \eta(T) + \text{diam}(S) \cdot M \sum_{i=1}^{j-1} \max_{1 \leq k \leq i} \Delta U_k^T
$$

$$
\leq \eta(T) + \text{diam}(S) \cdot M \sum_{i=1}^{j-1} \sum_{k=1}^{i} \Delta U_k^T
$$

where the second inequality follow from Lemma 11. We claim that the above recursive bound, along with $\Delta U_1^T \leq \eta(T)$, implies

$$
\Delta U_j^T \leq (2 + \text{diam}(S) \cdot M)^{j-1} \cdot \eta(T).
$$

(See Appendix D for the proof of this claim.) Substituting the value of $\text{diam}(S)$ and $\eta(T)$, we obtain for $j \geq 1$,

$$
\Delta U_j^T \leq (2 + C \cdot M)^{j-1} \left( \frac{\kappa \cdot f_1}{C} + \frac{C}{\kappa \cdot f_1} \right) \cdot C \cdot f_1 \cdot \frac{1}{\sqrt{T}}
$$

$$
= \frac{\kappa^2 f_1 + C^2}{\kappa} \cdot (2 + CM)^{j-1} \cdot \frac{1}{\sqrt{T}},
$$

and therefore,

$$
U^* - E \left[ \frac{1}{T} \sum_{t=1}^{T} U(\hat{y}_{N-1}^t, \ldots, \hat{y}_1^t) \right] = \sum_{j=1}^{N-1} \Delta U_j^T \leq \left\{ \frac{\kappa^2 f_1 + C^2}{\kappa} \cdot \sum_{j=1}^{N-1} (2 + MC)^{j-1} \right\} \cdot \frac{1}{\sqrt{T}}
$$

$$
\leq \left\{ \frac{\kappa^2 f_1 + C^2}{\kappa} \cdot \frac{(2 + MC)^{N-1}}{1 + MC} \right\} \cdot \frac{1}{\sqrt{T}},
$$

22
which is the desired result.

6 Comparison to Van Ryzin and McGill (2000)

In this section, we compare the ARM Algorithm and the adaptive algorithm proposed by Van Ryzin and McGill which we refer to as the VM Algorithm. Both algorithms maintain the auxiliary and implemented sequences of protection levels, where the implemented protection level for each demand class is obtained by taking the maximum among auxiliary protection levels of demand classes with higher fares. In both algorithms, the auxiliary protection levels in each iteration are updated from the previous auxiliary levels by adding adjustment vectors (scaled by an appropriate step size).

The step size and the adjustment direction distinguish the VM Algorithm and our ARM Algorithm. To facilitate our discussion, let us introduce the following notations. For any \( j \) and \( y_j \geq \ldots \geq y_1 \), define the fill event \( A_j(y_j, \ldots, y_1) \) as follows:

\[
A_j(y_j, \ldots, y_1) = [D_1 \geq y_1, D_1 + D_2 \geq y_2, \ldots, D_1 + \ldots + D_j \geq y_j].
\]

The random variable \( A_j(y_j, \ldots, y_1) \) correspond to the event that the entire capacity of \( y_j \) units are sold out between periods \( j \) and \( 1 \) when the protection levels \( y_j, \ldots, y_1 \) are used. Brumelle and McGill (1993) and Robinson (1995) have shown that the optimal protection levels \( (y^*_{N-1}, y^*_{N-2}, \ldots, y^*_1) \) satisfy

\[
P[A_j(y^*_j, \ldots, y^*_1)] = \frac{f_{j+1}}{f_1}
\]

for each \( j = N - 1, \ldots, 1 \). The above optimality condition provides the basis for the development of the VM Algorithm.

We briefly describe the VM Algorithm. Whereas the ARM Algorithm is motivated by recent developments in stochastic online convex optimization, the VM Algorithm is motivated by the classical stochastic approximation algorithm. We denote the auxiliary protection levels of the VM Algorithm by \( (y^*_N, \ldots, y^*_1) \) and denote the implemented protection levels by \( (\bar{y}^*_N, \ldots, \bar{y}^*_1) \), where for any \( 1 \leq j < N \), \( \bar{y}^*_j = \max \left\{ y^*_j, y^*_{j-1}, \ldots, y^*_1 \right\} \). The three components defining the VM Algorithm are given as follows.

- Adjustment direction:

\[
H_{j}^{V, t} = H_{j}^{V, t} \left( \frac{y^*_j}{\bar{y}^*_{j-1}}, \ldots, y^*_1 \right) = -f_{j+1} + f_{1} \cdot 1 \left[ A_j \left( y^*_j, y^*_{j-1}, \ldots, y^*_1 \right) \right],
\]
where \( 1[\cdot] \) denotes the indicator random variable that equals one if the event in the bracket is true and zero otherwise.

- Step size: \( \{ \varepsilon^{M,t} : t \geq 1 \} \) where \( \varepsilon^{M,t} = (1/f_1) \cdot \sigma_A/(t + \sigma_B) \) for some constants \( \sigma_A \geq 0 \) and \( \sigma_B \geq 0 \). (We note that in the original definition of the VM Algorithm, Van Ryzin and McGill (2000) used step sizes of the form \( \sigma_A/(t + \sigma_B) \), without the scaling factor of \( 1/f_1 \). The adjustment vector in their paper, however, was scaled by \( 1/f_1 \). Thus, the resulting updates are equivalent to the one given above. We choose this particular representation to facilitate a convenient comparison with our ARM Algorithm.)

- Auxiliary Protection Level Update: For any \( 1 \leq j < N \),
  \[
y_j^{V_{M,t+1}} = P_{[0,C]} \left( y_j^{V_{M,t}} + \varepsilon^{V_{M,t}} H_j^{V_{M,t}} \right).
  \]

### 6.1 Step Size Comparison

The step size in the ARM Algorithm decreases at the rate of \( O(1/\sqrt{t}) \), while the step-size of the VM Algorithm decreases at the rate of \( O(1/t) \). Thus, asymptotically, our step size is bigger than the step size of the VM Algorithm. In Section 7, we show how to incorporate a larger family of step sizes can into the ARM Algorithm.

While Van Ryzin and McGill (2000) have used the step sizes of the form \( \sigma_A/(t + \sigma_B) \), the VM Algorithm, allows, without the convergence proof, other step sizes provided that the following condition holds: \( \sum_t \varepsilon^{M,t} = +\infty \) and \( \sum_t \left( \varepsilon^{V_{M,t}} \right)^2 < +\infty \). This condition is quite common in the classical stochastic approximation literature and is satisfied if \( \varepsilon_t = O(1/t^\alpha) \) for \( \alpha \in (0.5, 1] \). Since the ARM Algorithm uses a square-root sequence of step sizes, it does not belong to the general framework of stochastic approximation algorithm.

In addition, the VM Algorithm is not fully specified in the sense that Van Ryzin and McGill (2000) do not offer any guideline regarding how a particular sequence of step sizes should be chosen for each problem instance. Finding the appropriate step sizes for each problem often requires trials and errors. Given that the manager does not have any information about the demand distributions a priori, it is not clear how she should select \( \sigma_A \) and \( \sigma_B \) in the VM Algorithm. In contrast, the selection of step sizes for our ARM Algorithm is guided by Theorem 1 (and also Theorem 13), which provides an explicit upper bound on the performance of the ARM Algorithm for the given choice of step sizes.
6.2 Adjustment Direction Comparison

We now compare the adjustment directions under the VM and ARM algorithms. For any \(1 \leq j < N\), recall that the adjustment vectors \(H_j^{VM}\) and \(H_j\) under the VM and ARM algorithms, respectively, are given by: for any \((y_j, y_{j-1}, \ldots, y_1)\),

\[
H_j^{VM} (y_j | y_{j-1}, \ldots, y_1) = -f_{j+1} + f_1 \cdot 1 \{A_{i-1} (y_{i-1}, \ldots, y_1)\}
\]

\[
H_j (y_j | y_{j-1}, \ldots, y_1) = -f_{j+1} + f_{K_j (y_j, y_{j-1}, \ldots, y_1)} .
\]

The main result of this section is stated in the following theorem.

**Theorem 12.** For any \(1 \leq j < N\), the following statements hold.

(i) For any \((y_j, y_{j-1}, \ldots, y_1)\),

\[
E \left[ H_j (y_j | y_{j-1}, \ldots, y_1) - H_j^{VM} (y_j | y_{j-1}, \ldots, y_1) \right] = f_1 \cdot \sum_{i=1}^{j} P \{ K_j (y_j, y_{j-1}, \ldots, y_1) = i \} \cdot (P \{ A_{i-1} (y_i^* - 1, \ldots, y_1^*) \} - P \{ A_{i-1} (y_i, y_{i-1}, \ldots, y_1) \}) .
\]

(ii) For any \(y_j\),

\[
E \left[ H_j^{VM} (y_j | y_{j-1}^*, \ldots, y_1^*) \left| H_j (y_j | y_{j-1}, \ldots, y_1) \right. \right] = H_j (y_j | y_{j-1}, \ldots, y_1) .
\]

(iii) \(H_j (y_j | y_{j-1}^*, \ldots, y_1^*)\) is smaller than \(H_j^{VM} (y_j | y_{j-1}^*, \ldots, y_1^*)\) in the convex order,\(^2\) i.e., for all convex functions \(\phi : \mathbb{R} \rightarrow \mathbb{R}\),

\[
E \left[ \phi (H_j (y_j | y_{j-1}^*, \ldots, y_1)) \right] \leq E \left[ \phi (H_j^{VM} (y_j | y_{j-1}^*, \ldots, y_1)) \right] .
\]

(iv) For any \(y_j\),

\[
Var \left( H_j (y_j | y_{j-1}^*, \ldots, y_1) \right) \leq Var \left( H_j^{VM} (y_j | y_{j-1}^*, \ldots, y_1) \right) .
\]

Before we proceed to the proof of the above theorem, let us discuss the implications of this result. The first part of Theorem 12 establishes an expression for the difference between the adjustment vectors under the VM and ARM algorithms for any protection levels \((y_j, \ldots, y_1)\). The second

\(^2\)For more details on convex ordering between two random variables, see Chapter 2 in Shaked and Shanthikumar (1994).
part of Theorem 12 shows that under the optimal protection levels, the conditional expectation of $H_j^{VM}$ given $H_j$ coincides with $H_j$; furthermore, $H_j^{VM}$ is a mean-preserving spread of $H_j$. Thus, the distribution of $H_j^{VM}$ second-order stochastically dominates the distribution of $H_j$ (see page 357 in Shaked and Shanthikumar (1994) for more details). This condition implies that $H_j$ is smaller than $H_j^{VM}$ in the convex order, implying that the variance of the adjustment vector under ARM ALGORITHM is smaller than the variance of the adjustment vector under the VM ALGORITHM.

Proof of Theorem 12. We will first prove part (i). It follows from the definition of the ARM ALGORITHM that the adjustment vector $H_j$ is given by (see Section 6.2):

$$H_j (y_j|y_{j-1}, \ldots, y_1) = -f_{j+1} + \sum_{i=0}^{j} f_i \cdot 1[K_j (y_j, \ldots, y_1) = i]$$

$$= -f_{j+1} + f_1 \cdot \sum_{i=1}^{j} 1[K_j (y_j, y_{j-1}, \ldots, y_1) = i] \cdot P[A_{i-1} (y_{i-1}, \ldots, y_1^*)],$$

where the second equality follows from the optimality conditions of Brunelle and McGill (1993) which shows that for any $1 \leq j \leq N - 1$, $P[A_j (y_j^*, \ldots, y_1^*)] = f_{j+1}/f_1$.

Consider the fill event $A_j(y_j, y_{j-1}, \ldots, y_1)$, which corresponds to the event that the capacity of $y_j$ is used up during period $j$ through 1, given that we use protection levels $(y_{j-1}, \ldots, y_1)$. We can partition the event $A_j (y_j, \ldots, y_1)$ based on the first period when the protection level becomes tight, and can obtain, for any $j$,

$$A_j(y_j, y_{j-1}, \ldots, y_1) = \bigcup_{i=1}^{j} [K_j (y_j, y_{j-1}, \ldots, y_1) = i, A_{i-1} (y_{i-1}, \ldots, y_1)].$$

(The proof of the above equality result follows from a straightforward induction on $j$ and we omit the details.) Since $H_j^{VM} (y_j|y_{j-1}, \ldots, y_1) = -f_{j+1} + f_1 \cdot 1[A_{i-1} (y_{i-1}, \ldots, y_1)]$ (see Section 6.2), it follows that

$$H_j^{VM} (y_j|y_{j-1}, \ldots, y_1) = -f_{j+1} + f_1 \cdot \sum_{i=1}^{j} 1[K_j (y_j, y_{j-1}, \ldots, y_1) = i] \cdot 1[A_{i-1} (y_{i-1}, \ldots, y_1)].$$

Taking the difference between the expressions of $H_j$ and $H_j^{VM}$ completes the proof of part (i).

We now proceed to part (ii) of Theorem 12. Observe that the possible values of the random variable $H_j (y_j|y_{j-1}^*, \ldots, y_1^*)$ are \{-f_{j+1} + f_i | i = 0, 1, \ldots, j\}. By definition, given that $H_j = f_{j+1} + f_i$ for some $i$, this is equivalent to conditioning on the value of $K_j (y_j, y_{j-1}^*, \ldots, y_1^*)$ being $i$,
which implies that the remaining capacity at the beginning of period \(i - 1\) is exactly \(y^*_{i-1}\). Thus, for any \(i\),

\[
E \left[ H^V_j (y_j | y^*_{j-1}, \ldots, y^*_1) \right] = -f_{j+1} + f_i
\]

\[
= E \left[ H^V_j (y_j | y^*_{j-1}, \ldots, y^*_1) \right] = i
\]

\[
= E \left[ -f_{j+1} + f_1 \cdot 1 \left[ A_{i-1} (y^*_{i-1}, y^*_{i-2}, \ldots, y^*_1) \right] \right]
\]

\[
= -f_{j+1} + f_1 \cdot P \left[ A_{i-1} (y^*_{i-1}, y^*_{i-2}, \ldots, y^*_1) \right]
\]

\[
= -f_{j+1} + f_i
\]

where the second equality follows from the expression of \(H^V_j\) and the last equality follows the fact that \(P[A_{i-1}(y^*_{i-1}, \ldots, y^*_1)] = f_i/f_1\) (Brumelle and McGill (1993)).

The last two parts of Theorem 12 directly follow from part (ii) and the results by Shaked and Shanthikumar (1994) (see Theorem 12.C.5 on page 357).

6.3 Performance Guarantee Comparison

Van Ryzin and McGill (2000) have proved the convergence of the VM Algorithm, showing that the auxiliary protection levels \((y^V_{N-1}, \ldots, y^V_1)\) satisfy

\[
E \left[ y^V_{j,t} - y^*_{j,t} \right]^2 = O \left( t^{-\beta/2j-1} \right)
\]

for some \(\beta \in (0,1)\). Thus, the convergence rate guarantee depends on the index of fare class, e.g., \(O \left( t^{-\beta} \right)\) for class 1, \(O \left( t^{-\beta/2} \right)\) for class 2, \(O \left( t^{-\beta/4} \right)\) for class 3, and so on. While the above performance bound is with respect to the distance between the current auxiliary protection levels and the optimal protection levels, we can establish a similar bound on the difference of the expected profits. From Lemma 8 and the boundedness of its partial derivatives \(V_j\) of the umbrella function \(U\), we obtain

\[
R^*(C) - \frac{1}{T} \sum_{t=1}^{T} R \left( y^V_{N-1}, \ldots, y^V_1 | C \right) = O \left( T^{-\beta/2^{N-1}} \right)
\]

We remark that the above performance bound is a provable bound, and the actual convergence may be faster.

In contrast, we show in Theorem 1 that the running average revenue under our ARM ALGORITHM converges to the optimal at the asymptotic rate of \(O \left( T^{-1/2} \right)\), regardless of the number of demand classes.
The convergence rates of both the VM Algorithm and the ARM Algorithm require a set of technical assumptions that are easily satisfied. While the VM Algorithm requires that the distributions of the partial sums $D_1 + D_2 + \cdots + D_j$ are Lipschitz continuous, the ARM Algorithm has a slightly stronger requirement that the density function of each $D_j$ is continuous and bounded above by $M$. While the ARM Algorithm requires the boundedness of the support of the demand distributions, but the VM Algorithm does not. The VM Algorithm, however, requires an additional technical condition regarding the expected adjustment vector, which is not required by the ARM Algorithm. When this technical condition (stated in Assumption 2) holds, however, we can improve the convergence rate of the ARM Algorithm (see Theorem 13 in Section 7).

7 Extensions

In this section, we take advantage of Theorem 3, another stochastic online convex optimization result, to obtain an improved convergence rate. We first find a sufficient condition for the application of Theorem 3 in the revenue management setting. It follows from Lemma 9 that we can express $V_j(y_j|y_{j-1}^*, \ldots, y_1^*)$ as follows:

$$V_j(y_j|y_{j-1}^*, \ldots, y_1^*) = -f_{j+1} + \sum_{i=1}^j P[K_j(y_j, y_{j-1}^*, \ldots, y_1^*) = i] \cdot f_i$$

By the convexity of $U(y_{N-1}, \ldots, y_1)$ (Lemma 7), $V_j(y_j|y_{j-1}^*, \ldots, y_1^*)$ is a nonincreasing function of $y_j$. Assumption 2 below ensures that $V_j(y_j|y_{j-1}^*, \ldots, y_1^*)$ is strictly decreasing, and its derivative is bounded away from 0. This is equivalent to Assumption (A2) of Van Ryzin and McGill (2000).

**Assumption 2.** For each $j = 1, \ldots, N - 1$, the demand distribution $D_j$ has a proper density. Furthermore, there exists $\varepsilon > 0$ such that, for all $y_j \in [0, C]$,

$$\left| V_j(y_j|y_{j-1}^*, \ldots, y_1^*) \right| \geq \varepsilon.$$

Recall that the definition of the umbrella function $U(y_{N-1}, \ldots, y_1)$ is a separable function, where its partial derivative with respect to $y_j$ is $V_j(y_j|y_{j-1}^*, \ldots, y_1^*)$. Thus, the Hessian matrix of $U(y_{N-1}, \ldots, y_1)$ is a diagonal matrix with diagonal entries given by the derivatives of $U_j(y_j|y_{j-1}^*, \ldots, y_1^*)$’s. Therefore, Assumption 2 implies $w^T [\nabla^2 U(y_{N-1}, \ldots, y_1)] w \geq \varepsilon$ for any $(y_{N-1}, \ldots, y_1) \in [0, C]^{N-1}$, satisfying the hypothesis of Theorem 3.
We define the modified ARM Algorithm by changing the step size of the ARM Algorithm; in particular, we set \( \epsilon_t = \kappa/t^\alpha \) for some \( \alpha \in (0,1] \). Thus, in case of \( \alpha = 1/2 \), the modified ARM Algorithm coincides with the ARM Algorithm.

**Theorem 13.** Consider the modified ARM Algorithm, and let its auxiliary sequence of protection levels be denoted by \( \{ (\bar{y}_{N-1}, \hat{y}_{N-2}, \ldots, \hat{y}_1) : t \geq 1 \} \). Suppose Assumptions 1 and 2 hold. In addition, if \( \alpha = 1/2 \), we set \( \kappa > 1/\varepsilon \) holds. Then,

\[
U^* - E \left[ \frac{1}{T} \sum_{t=1}^{T} U(\bar{y}_{N-1}, \ldots, \hat{y}_1) \right] \leq \begin{cases} 
\left( \frac{(2+MC)N-1}{1+MC} \right)^{N-1} \cdot \left( \frac{C^2}{2 \kappa} \cdot \beta \cdot \frac{1}{T^\alpha} + \frac{\kappa f_1^2}{2(1-\alpha)} \cdot \frac{1}{T^\alpha} \right), & \text{if } \alpha \in (0,1) \\
\left( \frac{(2+MC)N-1}{1+MC} \right)^{N-1} \cdot \left( \frac{C^2}{2 \kappa} \cdot \frac{1}{T} + \frac{\kappa f_1^2}{2} \cdot \frac{\log T}{T} \right), & \text{if } \alpha = 1.
\end{cases}
\]

**Proof.** Since most of the analysis for Theorem 1 remains valid with the new definition of \( \eta(T) \) (see the proof of Theorem 1 in Section 5.3) based on \( \xi_{II}(T, \alpha, \gamma) \) defined in Section 3. Using \( \text{diam}(S) = C, \overline{B} = f_1, \delta = C, \varepsilon = \xi, \text{ and } \gamma = \kappa f_1/C \), we let

\[
\eta(T) = \begin{cases} 
\frac{C^2}{2 \kappa} \cdot \left( \frac{\alpha}{\kappa} \right) \cdot \frac{1}{T^\alpha} \cdot \frac{1}{T^\alpha} + \frac{\kappa f_1^2}{2} \cdot \frac{1}{T^\alpha} \cdot \frac{1}{T^\alpha}, & \text{if } 0 < \alpha < 1; \\
\frac{C^2}{2 \kappa} \cdot \frac{1}{T} + \frac{\kappa f_1^2}{2} \cdot \frac{\log T}{T}, & \text{if } \alpha = 1.
\end{cases}
\]

In case of \( \alpha = 1, \kappa > 1/\xi \) implies

\[
\frac{\text{diam}(S) \cdot \overline{B}}{\varepsilon \cdot \delta^2} = \frac{C \cdot f_1}{\xi \cdot C^2} = \frac{1}{\xi} \cdot \frac{f_1}{C} \leq \frac{\kappa f_1}{C},
\]

satisfying the conditions of Theorem 3. Using the same technique as in the proof of Lemma 10, we can show that for any \( 1 \leq j < N \), \( \Delta U_j^T \leq (2 + \text{diam}(S) \cdot M)^{j-1} \cdot \eta(T) \), yielding the desired results. \( \square \)

Theorem 13 implies that with Assumption 2, we obtain the convergence rate of \( O \left( (1/T)^{1-a} \right) \) for any sufficiently small positive number \( a \). By selecting the step size with \( \alpha = 1 \) and \( \kappa > 1/\xi \), it is possible to achieve \( O \left( \log T/T \right) \) convergence rate as well.

### 8 Experiments

We evaluate the performance of our adaptive algorithm under several demand distributions and parameter settings. In Section 8.1, we consider the setting involving four demand classes, replicating the setup considered in the paper by Van Ryzin and McGill (2000). We show that the revenues generated by our ARM algorithm are comparable to those generated in Van Ryzin and McGill.
In Section 8.2, we show that the adjustment vector at each iteration under the VM Algorithm exhibits significantly higher variability than the corresponding vector under our ARM Algorithm. We then consider larger numbers of demand classes in Section 8.3. Finally, in Section 8.4, we study the impact on the convergence rate of the ARM Algorithm under different step sizes.

### 8.1 Four Demand Classes

In this section, we consider the setting involving four demand classes, replicating the original experiments conducted in Van Ryzin and McGill (2000). The demand for each class follows a Gaussian distribution with given parameters. In Table 2, we show the mean and the standard deviation of the demand distributions, along with the corresponding fare and the optimal protection levels.

<table>
<thead>
<tr>
<th>Class</th>
<th>Fare</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Optimal Protection Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1,050</td>
<td>17.3</td>
<td>5.8</td>
<td>16.7</td>
</tr>
<tr>
<td>2</td>
<td>$567</td>
<td>45.1</td>
<td>15.0</td>
<td>44.6</td>
</tr>
<tr>
<td>3</td>
<td>$527</td>
<td>73.6</td>
<td>17.4</td>
<td>134.0</td>
</tr>
<tr>
<td>4</td>
<td>$350</td>
<td>19.8</td>
<td>6.6</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 2: Fares, demand distributions, and the optimal protection levels for each demand class.

As in the original experiments, we consider four different parameter settings, corresponding to different capacity levels and initial protection levels. The four settings are given in Table 3. Note that in Case II and IV, we set the initial protection level for class 3 to the value of the capacity.

<table>
<thead>
<tr>
<th>Capacity</th>
<th>Initial Protection Levels for Classes 1, 2 &amp; 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low: (0, 15, 65)</td>
<td>High: (35, 110, 210)</td>
</tr>
<tr>
<td>124</td>
<td>Case I</td>
</tr>
<tr>
<td>164</td>
<td>Case III</td>
</tr>
</tbody>
</table>

Table 3: Capacities and initial protection levels for each of the four cases considered in the experiment involving four demand classes.
Recall that the VM Algorithm and our ARM Algorithm generate a sequence of protection levels according to the description in Sections 6 and 7, respectively. For the step sizes, we use

$$\varepsilon_{VM,t} = \frac{1}{f_1} \cdot \frac{\sigma_A}{t + \sigma_B}$$

and

$$\varepsilon^t = \frac{C}{f_1 \cdot t},$$

where $C$ and $f_1$ denote the capacity and the most expensive fare, respectively. Note that in the definition of $\varepsilon_{VM,t}$, we use $\sigma_A = 200 \cdot f_1$ and $\sigma_B = 10$ as in the original experiment in Van Ryzin and McGill (2000) (see Section 6 for more details.)

Figure 2 shows the comparison between the running average revenue under the optimal protection levels, the VM Algorithm, and our ARM Algorithm over 1000 problem instances. For each problem instance, we consider 1000 time periods and plot the running average revenue over time. The dash lines above and below the solid lines represent the 95% confidence interval. As seen from the figures, in all four cases, the revenue generated by both the VM Algorithm and our ARM Algorithm are comparable, converging to the same value.

The performance of the VM Algorithm depends on the choice of constants in $\varepsilon_{VM,t}$. The particular choice of $\varepsilon_{VM,t}$ (chosen in the original experiment) is presumably selected such that the VM Algorithm performs well on these four problems. As noted in Section 6.1, when we do not know the demand distributions a priori, it is not clear how one should choose the step sizes for the VM Algorithm. We note that the step sizes $\varepsilon^t$ of our ARM Algorithm are the default step sizes suggested by Theorem 13, and they perform remarkably well without any tweaking of parameters.

8.2 Variability of Adjustment Vectors

Although Figure 2 shows that the revenue under both the VM Algorithm and ARM Algorithm are comparable, it turns out that the adjustment vectors under the VM Algorithm exhibit significantly larger variability than the adjustment vectors under our ARM Algorithm. For each of the four parameter settings considered in Section 8.1, we plot, in Figure 3, the estimated total standard deviation of the adjustment vectors over time. For each period $t$, let $\hat{\text{Var}} \left( H_{VM,t}^j \right)$ and $\hat{\text{Var}} \left( H_t^j \right)$ denote the sample variance of class $j$ adjustments from 1000 problem instances. We then compute the estimated total standard deviation using

$$\sqrt{\sum_{j=1}^{N-1} \hat{\text{Var}} \left( H_{VM,t}^j \right)}$$

and

$$\sqrt{\sum_{j=1}^{N-1} \hat{\text{Var}} \left( H_t^j \right)}$$
Running Average Revenue Under VM and ARM Algorithms
(1000 problem instances, 4 class, 124 capacity, low initial values)
NOTE: Dash lines correspond to 95% confidence intervals

Running Average Revenue Under VM and ARM Algorithms
(1000 problem instances, 4 class, 124 capacity, high initial values)
NOTE: Dash lines correspond to 95% confidence intervals

Running Average Revenue Under VM and ARM Algorithms
(1000 problem instances, 4 class, 164 capacity, low initial values)
NOTE: Dash lines correspond to 95% confidence intervals

Running Average Revenue Under VM and ARM Algorithms
(1000 problem instances, 4 class, 164 capacity, high initial values)
NOTE: Dash lines correspond to 95% confidence intervals

I: 124 capacity with low initial values
II: 124 capacity with high initial values
III: 164 capacity with low initial values
IV: 164 capacity with high initial values

Figure 2: Running average revenue under the VM and ARM ALGORITHMS with four demand classes under four different parameter settings.
Figure 3: Estimated total standard deviation of the adjustment vectors over time under the VM and ARM ALGORITHMS for the four parameter settings from Section 8.1.
and plot these values over time. Figure 3 shows that the adjustment vectors under VM ALGORITHM exhibits higher variability, with estimated total standard deviation being approximately 50% higher than the ARM ALGORITHM. Moreover, the variability seems to remain constant even as the time period increases and the protection levels converges to the optimal. This observation is consistent with our result in Section 6.2.

8.3 Larger Numbers of Demand Classes

In this section, we compare the performance of the VM and ARM ALGORITHMS when the number of demand classes is larger. We consider 8 and 12 demand classes. The demand distribution for each class is normally distributed. We generate the mean and standard deviation, along with the fare, of each class as follows. From the 4-class setting in Section 8.1, let $C_0 = 124$ and let $I_0 = \{(f_i, \mu_i, \sigma_i) : 1 \leq i \leq 4\}$ denote the collection of fare, mean, and standard deviation for each of the four demand classes considered in Section 8.1. Let

$$C_1 = 1.1 \times C_0 \quad \text{and} \quad I_1 = \{1.1 \times (f_i, \mu_i, \sigma_i) : 1 \leq i \leq 4\}$$

$$C_2 = 1.2 \times C_0 \quad \text{and} \quad I_2 = \{1.2 \times (f_i, \mu_i, \sigma_i) : 1 \leq i \leq 4\}.$$

We use the following parameters for the 8-class and 12-class settings.

<table>
<thead>
<tr>
<th>Settings</th>
<th>Capacity</th>
<th>Fares, Means, and Stdev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-class</td>
<td>$C_0 + C_1$</td>
<td>$I_0 \cup I_1$</td>
</tr>
<tr>
<td>12-class</td>
<td>$C_0 + C_1 + C_2$</td>
<td>$I_0 \cup I_1 \cup I_2$</td>
</tr>
</tbody>
</table>

Table 4: Parameters for 8-class and 12-class settings, respectively.

The initial protection levels for each class $i$ is set to the total expected demand from class 1 through $i$. We use the same step sizes for both algorithms given by $\varepsilon^{V_M,t} = \varepsilon^t = C/(f_1 \cdot t)$.

Figure 4 shows the running average revenue under both algorithms for 8-class and 12-class settings, respectively, averaged over 1000 problem instances. For each problem instance, we run both algorithms for 10,000 time periods, and plot the running average revenue. From the figure, the running average revenue under our ARM ALGORITHM is higher than the revenue under the VM ALGORITHM, and the difference appears to be statistically significant.
Figure 4: Running average revenue under the VM and ARM algorithms for 8-class and 12-class settings.

Figure 5: Running average revenue of the ARM ALGORITHM under different step sizes $\varepsilon^t = C/(f_1 \cdot t^\alpha)$ where $0.5 \leq \alpha \leq 1$. 
8.4 Impact of Different Step Sizes

We investigate the impact of different step sizes on the performance of the ARM Algorithm. In this experiment, we consider the 4-class setting with 124 seats capacity and high initial values. Figure 5 compares the running average revenue of the ARM Algorithm for step sizes of the form $\varepsilon^t = C/(f_1 \cdot t^\alpha)$ where $\alpha$ ranges from 0.5 to 1.0. The asymptotic performance of the ARM Algorithm improves as $\alpha \in [0.5, 1)$ increases, consistent with our discussion in Theorem 13 of Section 7.

9 Concluding Remarks

In this paper, we introduce the concept of a concave umbrella that provides a concave lower bound for a given function, while retaining the same maximizer and the same maximum value. We then apply the technique to the problem of adaptive capacity control with censored demand data, developing an adaptive algorithm for setting protection levels based on historical sales of each demand class. Our formulation assumes monotonic fare classes. We note that possible future extensions of this paper include developing adaptive algorithms for non-monotonic fare classes and multiple resources.
A Proofs of Theorem 2 and Theorem 3

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^n$. We first establish the following two lemmas.

**Lemma 14.** Suppose there exists $B \geq 0$ such that for any $z \in S$,

$$
\langle \nabla \Phi(z), z - z^* \rangle \geq \Phi(z) - \Phi(z^*) + \frac{(B/2)}{2} \| z - z^* \|^2.
$$

Then, for any $T \geq 1$,

$$
\frac{1}{T} \sum_{t=1}^{T} E[\Phi(z_t) - \Phi(z^*)] \leq \frac{E \| z_1 - z^* \|^2}{2\epsilon_1} + \frac{1}{2} \sum_{t=1}^{T-1} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B \right\} E \| z_{t+1} - z^* \|^2
$$

$$
+ \frac{B^2}{2} \sum_{t=1}^{T} \epsilon_t + diam(S) \sum_{t=1}^{T} E[\| \delta(z_t) \|].
$$

**Proof.** Since $z_{t+1}$ is recursively defined in terms of $z_t$ and the projection operator is non-expansive (i.e., does not increase the distance between two points), for any $t \geq 1$,

$$
E \| z_{t+1} - z^* \|^2 \leq E \| z_t - \epsilon_t \cdot H(z_t) - z^* \|^2
$$

$$
= E \| (z_t - z^*) - \epsilon_t \cdot H(z_t) \|^2
$$

$$
= E \| z_t - z^* \|^2 + \epsilon_t^2 E \| H(z_t) \|^2 - 2\epsilon_t \cdot E \langle H(z_t), z_t - z^* \rangle,
$$

which implies

$$
E[\langle H(z_t), z_t - z^* \rangle] \leq \frac{E \| z_t - z^* \|^2 - E \| z_{t+1} - z^* \|^2 + \epsilon_t^2 E \| H(z_t) \|^2}{2\epsilon_t}
$$

$$
= \frac{E \| z_t - z^* \|^2}{2\epsilon_t} - \frac{E \| z_{t+1} - z^* \|^2}{2\epsilon_t} + \frac{\epsilon_t}{2} E \| H(z_t) \|^2.
$$

Also,

$$
E[\langle H(z_t), z_t - z^* \rangle] = E [E \langle H(z_t), z_t - z^* \rangle | z_t]
$$

$$
= E \langle E[H(z_t) | z_t], z_t - z^* \rangle
$$

$$
= E \langle \nabla \Phi(z_t) + \delta(z_t), z_t - z^* \rangle
$$

$$
= E \langle \nabla \Phi(z_t), z_t - z^* \rangle + E \langle \delta(z_t), z_t - z^* \rangle.
$$

Combining the above two results,

$$
E[\langle \nabla \Phi(z_t), z_t - z^* \rangle]
$$

$$
= E[\langle H(z_t), z_t - z^* \rangle] - E[\langle \delta(z_t), z_t - z^* \rangle]
$$

$$
\leq \frac{E \| z_t - z^* \|^2}{2\epsilon_t} - \frac{E \| z_{t+1} - z^* \|^2}{2\epsilon_t} + \frac{\epsilon_t}{2} E \| H(z_t) \|^2 - E[\langle \delta(z_t), z_t - z^* \rangle].
$$
It follows from the hypothesis of Lemma 14 and the above inequality that for any $z \in \mathcal{S}$,

$$
\sum_{t=1}^{T} E [\Phi(z_t) - \Phi(z^*)]
$$

\[ \leq \sum_{t=1}^{T} \left\{ E \left[ \langle \nabla \Phi(z_t), z_t - z^* \rangle \right] - \frac{B}{2} \| z_t - z^* \|^2 \right\} \]

\[ \leq \sum_{t=1}^{T} \left\{ \frac{E \| z_t - z^* \|^2}{2\epsilon_t} - \frac{E \| z_{t+1} - z^* \|^2}{2\epsilon_t} - \frac{B}{2} \| z_t - z^* \|^2 \right\} + \sum_{t=1}^{T} \frac{\epsilon_t}{2} E \| H(z_t) \|^2 - \sum_{t=1}^{T} E [\langle \delta(z_t), z_t - z^* \rangle] \]

\[ \leq \sum_{t=1}^{T} \left\{ \frac{E \| z_t - z^* \|^2}{2\epsilon_t} - \frac{E \| z_{t+1} - z^* \|^2}{2\epsilon_t} - \frac{B}{2} \| z_t - z^* \|^2 \right\} + \bar{B}^2 \sum_{t=1}^{T} \frac{\epsilon_t}{2} + \text{diam}(\mathcal{S}) \cdot \sum_{t=1}^{T} E [\| \delta(z_t) \|] \]

\[ \leq \frac{E \| z_1 - z^* \|^2}{2\epsilon_1} + \frac{1}{2} \sum_{t=1}^{T-1} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B \right\} E \| z_{t+1} - z^* \|^2 \]

\[ + \bar{B}^2 \sum_{t=1}^{T} \frac{\epsilon_t}{2} + \text{diam}(\mathcal{S}) \cdot \sum_{t=1}^{T} E [\| \delta(z_t) \|] \]

where the third inequality follows from the hypothesis of Theorem 3 and from the Cauchy-Schwartz inequality, which shows that $|\langle \delta(z_t), z_t - z^* \rangle| \leq \| \delta(z_t) \| \| z_t - z^* \| \leq \text{diam}(\mathcal{S}) \cdot \| \delta(z_t) \|$. The final inequality is obtained by rearranging the terms.

Note that since $\Phi$ is a convex function, $\bar{B} = 0$ satisfies the requirement of Lemma 14. By imposing additional requirements on the objective function $\Phi$, we can establish a non-trivial lower bound on $\bar{B}$.

**Lemma 15.** Suppose $\Phi$ satisfies the requirements of Theorem 3. Then, for any $z \in \mathcal{S}$, (2) holds with $\bar{B} = \bar{\epsilon} \cdot \delta^2 / \text{diam}(\mathcal{S})^2$, which is strictly positive.

**Proof.** From the requirements of Theorem 3, there exist $\delta > 0$ and $\bar{\epsilon} > 0$ such that

$$
\langle \nabla \Phi(z), z - z^* \rangle \geq \Phi(z) - \Phi(z^*) + \frac{\bar{\epsilon}}{2} \| z - z^* \|^2
$$

for any $z \in \mathcal{S}$ satisfying $\| z - z^* \| \leq \delta$. Since $\bar{\epsilon} \geq \bar{B}$ by definition, (2) holds if $\| z - z^* \| \leq \delta$.

For any $z \in \mathcal{S}$ satisfying $\| z - z^* \| > \delta$, choose $z'$ such that $z'$ lies in the line segment connecting $z$ and $z^*$ and the distance between $z^*$ and $z'$ is exactly $\delta$, i.e., $z' = z^* + \delta \cdot (z - z^*) / \| z - z^* \|$. Then,
by convexity of $\Phi$ and the fact that $\|z' - z^*\| \leq \delta$, we have that
\[
\Phi(z) - \Phi(z^*) = [\Phi(z) - \Phi(z')] + [\Phi(z') - \Phi(z^*)] \\
\leq \langle \nabla \Phi(z), z - z' \rangle + \langle \nabla \Phi(z'), z' - z^* \rangle - \frac{\bar{\epsilon}}{2} \|z' - z^*\|^2 \\
\leq \langle \nabla \Phi(z), z - z' \rangle + \langle \nabla \Phi(z), z' - z^* \rangle - \frac{\bar{\epsilon}}{2} \|z' - z^*\|^2 \\
= \langle \nabla \Phi(z), z - z^* \rangle - \frac{\bar{\epsilon}}{2} \|z' - z^*\|^2
\]
where the second inequality follows from the convexity of $\Phi$ along the line segment connecting $z$ and $z^*$, which implies that $\langle \nabla \Phi(z'), z' - z^* \rangle \leq \langle \nabla \Phi(z), z' - z^* \rangle$. Therefore,
\[
\langle \nabla \Phi(z), z - z^* \rangle \geq [\Phi(z) - \Phi(z^*)] + \frac{\bar{\epsilon}}{2} \|z' - z^*\|^2 \\
\geq [\Phi(z) - \Phi(z^*)] + \frac{\bar{\epsilon} \cdot \delta^2}{2 \cdot \text{diam}(S)^2} \|z - z^*\|^2,
\]
where the last inequality follows from the fact that $\|z' - z^*\| = \delta \geq \delta \|z - z^*\| / \text{diam}(S)$.

**Proof of Theorem 2.** Let $\gamma > 0$ and $0 < \alpha < 1$. Since $\Phi$ is convex, Lemma 14 with $B = 0$ implies
\[
\sum_{t=1}^{T} E[\Phi(z_t) - \Phi(z^*)] \leq \frac{E \|z_1 - z^*\|^2}{2 \epsilon_1} + \frac{1}{2} \sum_{t=1}^{T-1} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} \right\} E \|z_{t+1} - z^*\|^2 \\
+ B^2 \sum_{t=1}^{T} \frac{\epsilon_t}{2} + \text{diam}(S) \cdot \sum_{t=1}^{T} E[\|\delta(z_t)\|].
\]
The first two terms of (3) is bounded above by
\[
\frac{\text{diam}(S)^2}{2} \left( \frac{1}{\epsilon_1} + \sum_{t=1}^{T-1} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} \right\} \right) = \frac{\text{diam}(S)^2}{2} \cdot \frac{1}{\epsilon_T} = \frac{\text{diam}(S)B}{2\gamma} T^\alpha.
\]
The last two terms of (3) is bounded above by
\[
\frac{\gamma \text{diam}(S)B}{2} \sum_{t=1}^{T} \frac{1}{t^\alpha} + \text{diam}(S) \cdot \sum_{t=1}^{T} E[\|\delta(z_t)\|] \\
\leq \frac{\gamma \text{diam}(S)B}{2} \cdot \frac{T^{1-\alpha}}{1-\alpha} + \text{diam}(S) \cdot \sum_{t=1}^{T} E[\|\delta(z_t)\|],
\]
where the inequality follows from the fact that, for $0 < \alpha < 1$, we have $\sum_{t=1}^{T} 1/t^\alpha \leq \int_{0}^{T} 1/t^\alpha = T^{1-\alpha}/(1-\alpha)$. Dividing the sum of the above bounds by $T$ yields the desired result.

**Proof of Theorem 3.** Let $B = \bar{\epsilon} \cdot \delta^2 / \text{diam}(S)^2 > 0$. From Lemma 14 and 15,
\[
\sum_{t=1}^{T} E[\Phi(z_t) - \Phi(z^*)] \leq \frac{E \|z_1 - z^*\|^2}{2 \epsilon_1} + \frac{1}{2} \sum_{t=1}^{T-1} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B \right\} E \|z_{t+1} - z^*\|^2 \\
+ B^2 \sum_{t=1}^{T} \frac{\epsilon_t}{2} + \text{diam}(S) \cdot \sum_{t=1}^{T} E[\|\delta(z_t)\|].
\]
From the definition of $B$ and the step size $\epsilon_t$, for any $t \geq 1$,
\[
\frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B = \frac{B}{\gamma \text{diam}(S)} [(t + 1)^\alpha - t^\alpha] - \frac{\bar{\epsilon} \cdot \delta^2}{\text{diam}(S)^2} \leq \frac{1}{\text{diam}(S)} \left[ \frac{B}{\gamma} \cdot \frac{\alpha}{t^{1-\alpha}} - \frac{\bar{\epsilon} \cdot \delta^2}{\text{diam}(S)} \right]
\]
where the inequality follows from the concavity of $f(x) = x^\alpha$, i.e., $f(x + 1) - f(x) \leq f'(x)$.

We will now consider the two cases.

**Case $\alpha = 1$ and $\gamma \geq \text{diam}(S)B/(\bar{\epsilon} \delta^2)$:** In this case, we have that $E \|z_1 - z^*\|^2/(2\epsilon_1) \leq \text{diam}(S) B/(2\gamma)$ by definition of $\epsilon_1$. Moreover, for $\gamma \geq \text{diam}(S) B/(\bar{\epsilon} \delta^2)$, we have that for any $t \geq 1$,
\[
\frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B \leq \frac{1}{\text{diam}(S)} \left[ \frac{B}{\gamma} - \frac{\bar{\epsilon} \cdot \delta^2}{\text{diam}(S)} \right] \leq 0,
\]
which implies that
\[
\sum_{t=1}^{T} E [\Phi(z_t) - \Phi(z^*)] \leq \frac{\text{diam}(S) B}{2\gamma} + \frac{\text{diam}(S) B}{2} \sum_{t=1}^{T} \frac{\epsilon_t}{t} + \text{diam}(S) \cdot \sum_{t=1}^{T} E \|\delta(z_t)\|
\]
\[
= \frac{\text{diam}(S) B}{2\gamma} + \frac{\gamma \text{diam}(S) B}{2} \sum_{t=1}^{T} \frac{1}{t} + \text{diam}(S) \cdot \sum_{t=1}^{T} E \|\delta(z_t)\|
\]
\[
\leq \frac{\text{diam}(S) B}{2\gamma} + \frac{\gamma \text{diam}(S) B}{2} (2 \ln T) + \text{diam}(S) \cdot \sum_{t=1}^{T} E \|\delta(z_t)\|,
\]
where the last inequality follows from the fact that $\sum_{t=1}^{T} 1/t \leq 2 \ln T$. Dividing both sides by $T$ yields the desired result.

**Case $\alpha \in (0, 1)$ and $\gamma > 0$:** Note that the expression
\[
\frac{1}{\text{diam}(S)} \left[ \frac{B}{\gamma} \cdot \frac{\alpha}{t^{1-\alpha}} - \frac{\bar{\epsilon} \cdot \delta^2}{\text{diam}(S)} \right]
\]
is the difference between two terms, where the first term decreases to zero at the rate of $O\left(1/t^{1-\alpha}\right)$ and the second expression is a positive constant. Thus, the difference is negative for all $t > T_o(\alpha)$, where
\[
T_o(\alpha) = \left[ \frac{\alpha \text{diam}(S) B}{\gamma \bar{\epsilon} \delta^2} \right]^{\frac{1}{1-\alpha}}.
\]
It follows that the first two terms of (4) are bounded above by

$$\frac{E \left\| z_1 - z^* \right\|^2}{2\epsilon_1} + \frac{1}{2} \sum_{t=1}^{T-1} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B \right\} E \left\| z_{t+1} - z^* \right\|^2$$

$$\leq \frac{E \left\| z_1 - z^* \right\|^2}{2\epsilon_1} + \frac{1}{2} \sum_{t=1}^{\min\{T-1, \lfloor T_0(\alpha) \rfloor \}} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} - B \right\} E \left\| z_{t+1} - z^* \right\|^2,$$

$$\leq \frac{\text{diam}(S)^2}{2} \left( \frac{1}{\epsilon_1} + \sum_{t=1}^{\min\{T-1, \lfloor T_0(\alpha) \rfloor \}} \left\{ \frac{1}{\epsilon_{t+1}} - \frac{1}{\epsilon_t} \right\} \right)$$

$$\leq \frac{\text{diam}(S)^2}{2} \cdot \frac{1}{\epsilon_{\lfloor T_0(\alpha) \rfloor}}$$

$$= \frac{\text{diam}(S)^2}{2} \cdot \frac{\overline{\beta} \cdot \lfloor T_0(\alpha) \rfloor^\alpha}{\gamma \cdot \text{diam}(S)}$$

$$\leq \frac{\text{diam}(S)^2}{2\gamma} \cdot \frac{\alpha \cdot \text{diam}(S) \cdot \overline{\beta}}{\gamma \cdot \eta \cdot \delta^2} \cdot \left[ \alpha \cdot \text{diam}(S) \cdot \overline{\beta} \right]^{\alpha - \frac{1}{2}}.$$

The second last two terms of (4) are bounded as in the proof of Theorem 2. Dividing these bounds by $T$, we obtain the required result.

Thus, we complete the proof of Theorem 3. 

\[\square\]

**B Proof of Lemma 4**

For part (i), if $y_{N-1} \geq \cdots \geq y_1$, then it follows that for any $j$, $x_j \geq y_{j-1}$ holds with probability one, where $x_j$ denotes the remaining capacity just before the arrival of class $j$ demand. Thus, the extended revenue function $\tilde{R}$ coincides with the original revenue function. Also, the result in part (iii) follows immediately from the result of part (i) and (ii) since the optimal protection levels are monotonic. Thus, it suffices to prove part (ii). To prove part (ii), it follows from part (i) that $\tilde{R}(\bar{y}_{N-1}, \ldots, \bar{y}_1|C) = R(\bar{y}_{N-1}, \ldots, \bar{y}_1|C)$ holds. Thus, it is sufficient to establish

$$\tilde{R}(y_{N-1}, \ldots, y_1|C) \leq \tilde{R}(\bar{y}_{N-1}, \ldots, \bar{y}_1|C).$$

(5)

Suppose we fix a sample path of demand realizations $(d_N, \ldots, d_1)$ of $(D_N, \ldots, D_1)$. For any $j$, let $x_j$ and $\bar{x}_j$ denote the remaining capacity under the extended revenue $\tilde{R}$ at the beginning of period $j$ (just before the arrival of class $j$ demand) given the protection levels $(y_{N-1}, \ldots, y_1)$ and
(\bar{y}_{N-1}, \ldots, \bar{y}_1), \text{ respectively. It follows from the definition of the extended revenue that for any } j
\bar{x}_{j-1} = \bar{x}_j - \min \{d_j, \bar{x}_j - \bar{y}_{j-1}\} = \max \{\bar{x}_j - d_j, \bar{y}_{j-1}\}, \text{ and }
x_{j-1} = x_j - \min \{d_j, x_j - y_{j-1}\} = \max \{x_j - d_j, y_{j-1}\}.

where } x_N = \bar{x}_N = C \text{ and } d_j \text{ denote the realized demand for class } j. \text{ Before we proceed to the proof of (5), we will show that for any } j, \text{ the following properties hold: (a) } \bar{x}_j \geq x_j; \text{ (b) if } \bar{y}_j = y_j, \text{ then } \bar{x}_j = x_j; \text{ and (c) if } \bar{x}_j > x_j, \text{ then } \bar{x}_j = \bar{y}_j.

We prove these three properties by induction on } j. \text{ The proof of Property (a) is straightforward and we omit the details. Suppose Properties (b) and (c) hold for } j. \text{ We wish to show that they also hold for } j - 1.

We will first prove Property (b). Suppose that } \bar{y}_{j-1} = y_{j-1}. \text{ If } \bar{x}_j = x_j, \text{ then it follows from the definition that } \bar{x}_{j-1} = x_{j-1}, \text{ which is the desired result. On the hand, suppose that } \bar{x}_j > x_j, \text{ then by our inductive hypothesis for Property (c) for } j, \text{ we conclude that } \bar{x}_j = \bar{y}_j, \text{ implying that } \bar{y}_j = \bar{x}_j > x_j \geq y_j. \text{ Since } \bar{y}_j = \max \{\bar{y}_{j-1}, y_j\} \text{ by definition and } \bar{y}_j > y_j, \text{ it follows that } y_j < \bar{y}_j = \bar{y}_{j-1} = y_{j-1}. \text{ Therefore, } \max \{\bar{x}_j, x_j\} \leq y_{j-1} = \bar{y}_{j-1}. \text{ It follows from the definition that } \bar{x}_{j-1} = \bar{y}_{j-1} = y_{j-1} = x_{j-1}, \text{ which is the desired result. Thus, Property (b) also holds for } j - 1.

We will now prove Property (c) for } j - 1. \text{ Suppose that } \bar{x}_{j-1} > x_{j-1}. \text{ Then, the contra-positive statement of Property (b) for } j - 1 \text{ implies } \bar{y}_{j-1} > y_{j-1}. \text{ There are two cases to consider: } \bar{x}_j = x_j \text{ and } \bar{x}_j > x_j. \text{ Suppose that } \bar{x}_j = x_j. \text{ Since } \bar{x}_{j-1} > x_{j-1} \text{ and } \bar{y}_{j-1} > y_{j-1}, \text{ it follows from the definition of } \bar{x}_{j-1} \text{ that we must have } \bar{x}_{j-1} = \bar{y}_{j-1}, \text{ which is the desired result. On the other hand, suppose that } \bar{x}_j > x_j. \text{ Then, by the inductive hypothesis for Property (c) for } j, \text{ we conclude that } \bar{x}_j = \bar{y}_j. \text{ However, by the contra-positive of Property (b) for } j, \text{ we also have that } \bar{y}_j > y_j. \text{ Since } \bar{y}_j = \max \{\bar{y}_{j-1}, y_j\} \text{ by definition, it must be the case that } \bar{y}_j = \bar{y}_{j-1}. \text{ So, we have that } \bar{x}_j = \bar{y}_j = \bar{y}_{j-1}, \text{ which implies that } \bar{x}_{j-1} = \bar{y}_{j-1}, \text{ which is the desired result. Since the result holds for both cases, Property (c) also hold for } j - 1, \text{ completing the proof.}

For (5), we will prove by induction that for any } j,
\bar{R}_j(\bar{x}_j, \bar{y}_{j-1}, \ldots, \bar{y}_1) - \bar{R}_j(x_j, y_{j-1}, \ldots, y_1) \geq f_{j+1} \cdot (\bar{x}_j - x_j).

When } j = 1, \text{ then by Property (b), } \bar{x}_1 = x_1 \text{ holds, implying that the above result is clearly true. Assume the result holds for } j. \text{ We wish to prove that it is also true for } j + 1. \text{ Consider any pair of
where the expectation is taken over \( D \).

\[
\bar{R}_{j+1}(\bar{x}_{j+1}, \bar{y}_j, \ldots, \bar{y}_1) - \bar{R}_{j+1}(x_{j+1}, y_j, \ldots, y_1)
= E \left[ f_{j+1} \cdot \{(\bar{x}_{j+1} - x_j) - (x_{j+1} - x_j)\} + \bar{R}_j(\bar{x}_j, \bar{y}_{j-1}, \ldots, \bar{y}_1) - \bar{R}_j(x_j, y_{j-1}, \ldots, y_1) \right],
\]

where the expectation is taken over \( D_{j+1} \), and the values of \( x_j \) and \( \bar{x}_j \) depend on the realizations of \( D_{j+1} \). We consider any possible realization of \( D_{j+1} \).

**Case I:** Suppose \( x_{j+1} = \bar{x}_{j+1} \). If \( x_j = \bar{x}_j \), then \( (\bar{x}_{j+1} - x_j) - (x_{j+1} - x_j) \) is zero, and the required result follows. Otherwise, if \( x_j < \bar{x}_j \), then Property (c) implies \( \bar{x}_j = \bar{y}_j \), and the above expression becomes

\[
f_{j+1} \cdot [-(\bar{x}_j - x_j)] + \bar{R}_j(\bar{x}_j, \bar{y}_{j-1}, \ldots, \bar{y}_1) - \bar{R}_j(x_j, y_{j-1}, \ldots, y_1)
\geq f_{j+1} \cdot [-(\bar{x}_j - x_j)] + f_{j+1} \cdot (\bar{y}_j - x_j) = f_{j+1} \cdot (\bar{y}_j - \bar{x}_j) = 0,
\]

where the inequality follows from the induction hypothesis.

**Case II:** Suppose \( x_{j+1} < \bar{x}_{j+1} \). Then, it follows from Property (c) that \( \bar{y}_{j+1} = \bar{x}_{j+1} \). Therefore, we have that \( \bar{y}_{j+1} = \bar{x}_{j+1} > x_{j+1} \geq y_{j+1} \). In this case, we must have \( \bar{y}_j = \bar{y}_{j+1} \), for otherwise (i.e., \( \bar{y}_j < \bar{y}_{j+1} \)), \( \bar{y}_{j+1} = y_{j+1} \) must hold. Since \( \bar{x}_{j+1} = \bar{y}_{j+1} = \bar{y}_j \), we obtain \( \bar{x}_{j+1} = \bar{x}_j \), implying \( (\bar{x}_{j+1} - \bar{x}_j) - (x_{j+1} - x_j) = -(x_{j+1} - x_j) \). Thus, it follows from the induction hypothesis that

\[
\bar{R}_{j+1}(\bar{x}_{j+1}, \bar{y}_j, \ldots, \bar{y}_1) - \bar{R}_{j+1}(x_{j+1}, y_j, \ldots, y_1)
\geq f_{j+1} \cdot [-(x_{j+1} - x_j)] + f_{j+1} \cdot (\bar{y}_j - x_j)
= f_{j+1} \cdot (\bar{y}_j - x_{j+1})
= f_{j+1} \cdot (\bar{x}_{j+1} - x_{j+1}),
\]

proving the result for \( j + 1 \).

## C Proof of Lemma 7

It follows from Lemma 4 that the extended revenue function \( \bar{R} \) achieves its maximum at the optimal protection levels \( y_{N_1}^* \geq \cdots \geq y_1^* \) under the original revenue function. Since the optimal protection levels are monotonic and the extension coincides with the original revenue function for monotonic protection levels, it follows immediately from the optimality condition of Brumelle and McGill (1993) that for any \( j < N \),

\[
\frac{f_{j+1}}{f_1} = \frac{\partial \bar{R}_j}{\partial x_j} (y_j^*, y_{j-1}^*, \ldots, y_1^*).
\]
Using the recursive definition of the partial derivative of \(\tilde{R}_j\) in Lemma 6 and the above optimality condition, it is easy to verify by induction that for any \(j\), the function \(\tilde{R}_j(\cdot, y_{j-1}^*, \ldots, y_1^*)\) is concave in \(\mathbb{R}_+\). Thus, it follows from the definition that \(V_j(\cdot, y_{j-1}^*, \ldots, y_1^*)\) is nonincreasing, and the function \(S_j(\cdot, y_{j-1}^*, \ldots, y_1^*)\) is concave and achieves its maximum at \(y_j^*\).

Therefore, it remains to prove parts (ii) and (iii) only. It follows from Lemma 6 that both \(\partial \tilde{R}_j(x_j, y_{j-1}, y_{j-2}^*, \ldots, y_1^*)/\partial y_{j-1}\) and \(V_{j-1}(y_{j-1}|y_{j-2}^*, \ldots, y_1^*)\) have the same sign and satisfy
\[
\left| \frac{\partial}{\partial y_{j-1}} \tilde{R}_j(x_j, y_{j-1}, y_{j-2}^*, \ldots, y_1^*) \right| \leq |V_{j-1}(y_{j-1}|y_{j-2}^*, \ldots, y_1^*)|
\]
for any \(y_{j-1}\), from which we obtain (ii). We prove (iii) inductively. Consider
\[
\tilde{R}_j(x_j, y_{j-1}^*, \ldots, y_1^*) - \tilde{R}_j(x_j, y_{j-1}, \ldots, y_1) = \left\{ \tilde{R}_j(x_j, y_{j-1}^*, \ldots, y_1^*) - \tilde{R}_j(x_j, y_{j-1}, y_{j-2}^*, \ldots, y_1^*) \right\} + \left\{ \tilde{R}_j(x_j, y_{j-1}, y_{j-2}^*, \ldots, y_1^*) - \tilde{R}_j(x_j, y_{j-1}, \ldots, y_1) \right\}.
\]
Apply (ii) to the first term, and use the induction hypothesis to the second term to complete the proof.

### D Proof of a Claim in the proof of Lemma 10

In this section, we prove the following claim: if \(\Delta S_T^T \leq \eta(T)\) and
\[
\Delta S_j^T \leq \eta(T) + diam(S) \cdot M \sum_{i=1}^{j-1} \max_{1 \leq k \leq i} \Delta S_k^T
\]
for any \(j \geq 2\), then
\[
\Delta S_j^T \leq \eta(T) \cdot (2 + diam(S) \cdot M)^{j-1}.
\]
This claim is used in the proof of Lemma 10.

Let \(\beta = diam(S) \cdot M\). For any \(j \geq 2\), we have
\[
\Delta S_j^T \leq \eta(T) + \beta \sum_{i=1}^{j-1} \sum_{k=1}^{i} \Delta S_k^T.
\]
Let \(u(1) = \eta(T)\), and for any \(j \geq 1\), define
\[
u(j + 1) = \eta(T) + \beta \sum_{i=1}^{j} \sum_{k=1}^{i} u(k).
\]
Clearly, \( u(j) \) is an upper bound on \( \Delta S^T_j \). Also,

\[
\begin{align*}
u(j + 1) &= \eta(T) + \beta \cdot \left[ u(j) + \sum_{i=1}^{j-1} \sum_{k=1}^{i} u(k) + \sum_{k=1}^{j-1} u(k) \right] \\
&\leq \eta(T) + \beta \cdot u(j) + 2 \cdot \left( \beta \sum_{i=1}^{j-1} \sum_{k=1}^{i} u(k) \right) \\
&= \eta(T) + \beta \cdot u(j) + 2 \cdot (u(j) - \eta(T)) \\
&= (2 + \beta) \cdot u(j) - \eta(T) \\
&\leq (2 + \beta) \cdot u(j).
\end{align*}
\]

Therefore, from the above recursive bound, we obtain \( u(j + 1) \leq (2 + \beta)^j \eta(T) \), completing the proof of the claim.

### E Proof of Lemma 11

The proof of Lemma 11 relies on the following result. The first result establishes a relationship between the \( V_j \) and \( V_i \) functions.

**Lemma 16.** If \( j \geq i \), then for any \( z \geq y^*_j \),

\[
0 \geq V_j(z|y^*_{j-1}, \ldots, y^*_1) \geq V_i(z|y^*_{i-1}, \ldots, y^*_1).
\]

**Proof.** The first inequality follows immediately from Lemma 7(i) which shows that \( V_j \) is nonincreasing and crosses zero at \( y^*_j \). Thus, it remains to prove the second inequality. Since \( y^*_{N-1} \geq \cdots \geq y^*_1 \), it suffices to show that for any \( i \) and \( z \geq y^*_{i+1} \),

\[
V_{i+1}(z|y^*_i, y^*_{i-1}, \ldots, y^*_1) \geq V_i(z|y^*_i, \ldots, y^*_1).
\]

However, it follows from the recursion in the first part of Lemma 9 that

\[
V_{i+1}(z|y^*_i, y^*_{i-1}, \ldots, y^*_1) = -f_{i+2} + f_{i+1} + E \left[ 1 (z - D_{i+1} > y^*_i) \cdot V_i(z - D_{i+1}|y^*_{i-1}, \ldots, y^*_1) \right] \\
\geq -f_{i+2} + f_{i+1} + V_i(z|y^*_{i-1}, \ldots, y^*_1) \\
\geq V_i(z|y^*_{i-1}, \ldots, y^*_1),
\]

which is the desired result. The first inequality follows from Lemma 7(i) which shows that \( V_i(z|y^*_{i-1}, \ldots, y^*_1) \) is weakly decreasing and crosses zero at \( y^*_i \). Thus, \( V_i(z|y^*_{i-1}, \ldots, y^*_1) \) is nonpositive for any \( z \geq y^*_{i+1} \geq y^*_i \). The final inequality follows from the fact that the fares are monotonic.
The following corollary follows immediately from the above lemma and calculus.

**Corollary 17.** If $j \geq i$, then for any $z \geq y_j^*$,

$$U_j(y_j^*, y_{j-1}^*, \ldots, y_1^*) - U_j(z, y_{j-1}^*, \ldots, y_1^*) \leq U_i(y_i^*, y_{i-1}^*, \ldots, y_1^*) - U_i(z, y_{i-1}^*, \ldots, y_1^*).$$

The following lemma establishes a bound on the difference between the derivative at the optimal protection levels and at any other monotonic protection levels.

**Lemma 18.** For any $x_j$ and $(y_{N-1}, \ldots, y_1)$ satisfying $0 \leq y_1 \leq y_2 \leq \cdots \leq y_{j-1} \leq x_j$, we have

$$0 \leq V_j(x_j | y_{j-1}, \ldots, y_1) - V_j(x_j | y_{j-1}^*, \ldots, y_1^*) \leq M \sum_{i=1}^{j-1} U_i(y_i^*, y_{i-1}^*, \ldots, y_1^*) - U_i(y_i, y_{i-1}, \ldots, y_1).$$

**Proof.** The result holds trivially if $j = 1$. We assume the result for $j - 1$ and proceed by induction to prove the result for $j$. Let $W = x_j - D_j$. Then,

$$\frac{\partial}{\partial x_j} \tilde{R}_j(x_j, y_{j-1}, \ldots, y_1) = f_j + E \left[ 1[D_j < x_j - y_j-1] \cdot \left\{ -f_j + \frac{\partial}{\partial x_j} \tilde{R}_{j-1}(x_j - D_j, y_{j-2}, \ldots, y_1) \right\} \right],$$

$$= f_j + E \left[ 1[D_j < x_j - y_j-1] \cdot V_{j-1}(x_j - D_j | y_{j-2}, \ldots, y_1) \right]$$

$$= f_j + \int_{y_{j-1}}^{x_j} V_{j-1}(w | y_{j-2}, \ldots, y_1) f_W(w) dw,$$

where the first equality follows from Lemma 6 and the second equality follows from the definition of $V_{j-1}$.

Now, consider the middle expression in the statement of Lemma 18:

$$V_j(x_j | y_{j-1}, \ldots, y_1) - V_j(x_j | y_{j-1}^*, \ldots, y_1^*)$$

$$= \frac{\partial}{\partial x_j} \tilde{R}_j(x_j, y_{j-1}, \ldots, y_1) - \frac{\partial}{\partial x_j} \tilde{R}_j(x_j, y_{j-1}^*, \ldots, y_1^*)$$

$$= \int_{y_{j-1}}^{x_j} V_{j-1}(w | y_{j-2}, \ldots, y_1) f_W(w) dw - \int_{y_{j-1}}^{x_j} V_{j-1}(w | y_{j-2}^*, \ldots, y_1^*) f_W(w) dw$$

$$= \int_{y_{j-1}}^{x_j} \{ V_{j-1}(w | y_{j-2}, \ldots, y_1) - V_{j-1}(w | y_{j-2}^*, \ldots, y_1^*) \} f_W(w) dw$$

$$- \int_{y_{j-1}^*}^{y_{j-1}} V_{j-1}(w | y_{j-2}^*, \ldots, y_1^*) f_W(w) dw,$$

where the first equality follows from the definition of $V_j$, and the second equality follows from both Lemma 6 and the definition of $V_{j-1}$. We consider each of the above integrals. For the first integral,
the induction hypothesis for \( j - 1 \) implies that for any \( w \in \{y_{j-1}, x_j\} \), \( V_{j-1}(w \mid y_{j-2}, \ldots, y_1) - V_{j-1}(w \mid y_{j-2}^*, \ldots, y_1^*) \) is nonnegative and satisfies
\[
V_{j-1}(w \mid y_{j-2}, \ldots, y_1) - V_{j-1}(w \mid y_{j-2}^*, \ldots, y_1^*) \leq M \sum_{i=2}^{j-1} \left[ U_{i-1}(y_{i-1}^* \mid y_{i-2}^*, \ldots, y_1^*) - U_{i-1}(y_{i-1} \mid y_{i-2}^*, \ldots, y_1^*) \right].
\]
Thus, the first integral is also bound by the right-hand side of the above expression. For the second integral, Lemma 7 implies that \( V_{j-1}(y_{j-2}^*, \ldots, y_1^*) \) is weakly decreasing, and crosses zero at \( y_{j-1}^* \). Thus, \( \int_{y_{j-1}^*}^{y_j} V_{j-1}(w \mid y_{j-2}^*, \ldots, y_1^*) f_W(w) dw \) is nonpositive. From Assumption 1,
\[
\int_{y_{j-1}^*}^{y_j} -V_{j-1}(w \mid y_{j-2}^*, \ldots, y_1^*) f_W(w) dw \leq -M \int_{y_{j-1}^*}^{y_j} V_{j-1}(w \mid y_{j-2}, \ldots, y_1) dw = M \cdot \left[ U_{j-1}(y_{j-1}^* \mid y_{j-2}, \ldots, y_1) - U_{j-1}(y_{j-1} \mid y_{j-2}^*, \ldots, y_1^*) \right],
\]
where the last equality follows from calculus and the definition \( U_{j-1} \). By combining the results for the first and second integrals, we prove the result for \( j \) and complete the induction step.

Finally, here is the proof of Lemma 11. From Lemma 18,
\[
V_j(y_j \mid y_{j-1}, \ldots, y_1) - V_j(y_j \mid y_{j-1}^*, \ldots, y_1^*) \leq M \sum_{i=1}^{j-1} U_i(y_i^* \mid y_{i-1}^*, \ldots, y_1^*) - U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*).
\]
Therefore, it suffices to prove an upper bound for each term in the summand, i.e., for any \( i = 1, 2, \ldots, j - 1 \), we show
\[
U_i(y_i^* \mid y_{i-1}^*, \ldots, y_1^*) - U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) \leq \max_{k=1,2,\ldots} U_k(y_k^* \mid y_{k-1}^*, \ldots, y_1^*) - U_k(y_k \mid y_{k-1}^*, \ldots, y_1^*).
\]
If \( \bar{y}_i = y_i \), then it follows \( U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) = U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) \), and the above result holds trivially. Suppose \( \bar{y}_i > y_i \), and consider the following two cases: (a) \( y_i^* \geq \bar{y}_i > y_i \), and (b) \( y_i^* > \bar{y}_i \). In Case (a), the concavity of \( U_i \) implies \( U_i(y_i^* \mid y_{i-1}^*, \ldots, y_1^*) \geq U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) \). It follows
\[
U_i(y_i^* \mid y_{i-1}^*, \ldots, y_1^*) - U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) \leq U_i(y_i^* \mid y_{i-1}^*, \ldots, y_1^*) - U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) ,
\]
implying the required result. In Case (b), by the construction of \( \bar{y}_i = \max\{y_1, y_2, \ldots, y_i\} \), there exists \( k < i \) such that \( \bar{y}_i = y_k \). If \( y_i^* < \bar{y}_i \), then Corollary 17 implies
\[
U_i(y_i^* \mid y_{i-1}^*, \ldots, y_1^*) - U_i(y_i \mid y_{i-1}^*, \ldots, y_1^*) \leq U_k(y_k^* \mid y_{k-1}^*, \ldots, y_1^*) - U_k(y_k \mid y_{k-1}^*, \ldots, y_1^*) = U_k(y_k^* \mid y_{k-1}^*, \ldots, y_1^*) - U_k(y_k \mid y_{k-1}^*, \ldots, y_1^*). 
\]
Thus, the required result also holds, completing the proof of Lemma 11.
References


