Lecture 4

- Multi-period Planning Models
- Cash-Flow-Matching LP
  - Project-funding example
- Summary and Preparation for next class

Multi-period Planning Models

In many settings we need to plan over a time horizon of many periods because
- decisions for the current planning period affect the future
- requirements in the future need action now

Examples include:
- Production / inventory planning
- Human resource staffing
- Investment problems
- Capacity expansion / plant location problems
National Steel Corporation

- National Steel Corporation (NSC) produces a special-purpose steel used in the aircraft and aerospace industries. The sales department has received orders for the next four months:

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand (tons)</td>
<td>2300</td>
<td>2000</td>
<td>3100</td>
<td>3000</td>
</tr>
</tbody>
</table>

- NSC can meet demand by producing the steel, by drawing from its inventory, or a combination of these. Inventory at the beginning of January is zero. Production costs are expected to rise in Feb and Mar.

Production and inventory costs are:

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production cost</td>
<td>3000</td>
<td>3300</td>
<td>3600</td>
<td>3600</td>
</tr>
<tr>
<td>Inventory cost</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>250</td>
</tr>
</tbody>
</table>

- Production costs are in $ per ton. Inventory costs are in $ per ton per month. For example, 1 ton in inventory for 1 month costs $250; for 2 months, it costs $500.

- NSC can produce at most 3000 tons of steel per month. What production plan meets demand at minimum cost?

NSC Production Model Overview

- What needs to be decided?
  A production plan, i.e., the amount of steel to produce in each of the next 4 months.

- What is the objective?
  Minimize the total production and inventory cost. These costs must be calculated from the decision variables.

- What are the constraints?
  Demand must be met each month. Constraints to define inventory in each month. Production-capacity constraints. Non-negativity of the production and inventory quantities.

- NSC optimization model in general terms:

\[
\begin{align*}
\text{min} & \quad \text{Total Production plus Inventory Cost} \\
\text{subject to:} & \\
& \quad \text{Production-capacity constraints} \\
& \quad \text{Flow-balance constraints} \\
& \quad \text{Nonnegative production and inventory}
\end{align*}
\]
NSC Multi-period Production Model

- **Index:** Let \( i = 1, 2, 3, 4 \) represent the months Jan, Feb, Mar, and Apr, respectively.
- **Decision Variables:** Let
  \[
  P_i = \text{# of tons of steel to produce in month } i \\
  I_i = \text{# of tons of inventory from month } i \text{ to } i+1
  \]

Note: The production variables \( P_i \) are the main decision variables, because the inventory levels are determined once the production levels are set. Often the \( P_i \)'s are called *controllable* decision variables and the \( I_i \)'s are called *uncontrollable* decision variables.

- **Objective Function:**
  The total cost is the sum of production and inventory cost.
  Total production cost, \( PROD \), is:
  \[
  PROD = 3000 P_1 + 3300 P_2 + 3600 P_3 + 3600 P_4.
  \]
  Total inventory cost, \( INV \), is:
  \[
  INV = 250 I_1 + 250 I_2 + 250 I_3 + 250 I_4.
  \]

Demand Constraints

- In order to meet demand in the first month, we want
  \[
  P_1 \geq 2300.
  \]
  Set
  \[
  I_1 = P_1 - 2300
  \]
  and note that \( P_1 \geq 2300 \) is equivalent to \( I_1 \geq 0 \).
- In order to meet demand in the second month, the tons of steel available must be at least 2000:
  \[
  I_1 + P_2 \geq 2000.
  \]
  Set
  \[
  I_2 = I_1 + P_2 - 2000
  \]
  and note that \( I_1 + P_2 \geq 2000 \) is equivalent to \( I_2 \geq 0 \).
- The inventory and non-negativity constraints:
  (Month 1) \[
  I_1 = P_1 - 2300, \quad I_1 \geq 0
  \]
  (Month 2) \[
  I_2 = I_1 + P_2 - 2000, \quad I_2 \geq 0
  \]
  (Month 3) \[
  I_3 = I_2 + P_3 - 3100, \quad I_3 \geq 0
  \]
  define the inventory decision variables and enforce the demand constraints.
NSC Production Model (continued)

- Another way to view the constraints: The inventory variables link one period to the next. The inventory definition constraints can be visualized as “flow balance” constraints:

\[
\begin{align*}
P_1 &= I_1 + 2300 \\
I_1 + P_2 &= I_2 + 2000 \\
I_2 + P_3 &= I_3 + 3100 \\
I_3 + P_4 &= I_4 + 3000
\end{align*}
\]

- Flow-balance constraints for each month

\[
\text{Flow in} = \text{Flow out}
\]

(Month 1) \[ P_1 = I_1 + 2300 \]
(Month 2) \[ I_1 + P_2 = I_2 + 2000 \]
(Month 3) \[ I_2 + P_3 = I_3 + 3100 \]
(Month 4) \[ I_3 + P_4 = I_4 + 3000 \]

- Are there any other constraints? Production cannot exceed 3000 tons in any month:

\[ P_i \leq 3000 \quad \text{for } i = 1, 2, 3, 4. \]

NSC Linear Programming Model

\[
\text{Min } PROD + INV
\]

subject to:

- Cost Definitions:

\[
\begin{align*}
(\text{PROD Def.}) \quad \text{PROD} &= 3000 \ P_1 + 3300 \ P_2 + 3600 \ P_3 + 3600 \ P_4. \\
(\text{INV Def.}) \quad \text{INV} &= 250 \ I_1 + 250 \ I_2 + 250 \ I_3 + 250 \ I_4.
\end{align*}
\]

- Production-capacity constraints:

\[ P_i \leq 3000, \quad i = 1, 2, 3, 4. \]

- Inventory-balance constraints:

\[
\begin{align*}
(\text{Flow in} & = \text{Flow out}) \\
(\text{Month 1}) \quad P_1 &= I_1 + 2300 \\
(\text{Month 2}) \quad I_1 + P_2 &= I_2 + 2000 \\
(\text{Month 3}) \quad I_2 + P_3 &= I_3 + 3100 \\
(\text{Month 4}) \quad I_3 + P_4 &= I_4 + 3000
\end{align*}
\]

- Nonnegativity: All variables \( \geq 0 \)
The optimal solution has a total cost of $35,340,000.

---

**Multi-period Models in Practice**

- Most multi-period planning systems operate on a rolling-horizon basis:

  - A T-period model is solved in January and the optimal solution is used to determine the plan for January. In February, a new T-period model is solved, incorporating updated forecasts and other new information. The optimal solution is used to determine the plan for February.
  - Often long-horizon models are used to estimate needed capacity and determine aggregate planning decisions (strategic issues). Then more detailed short-horizon models are used to determine daily and weekly operating decisions (tactical issues).
Project-Funding Problem

- A company is planning a 3-year renovation of its facilities and would like to finance the project by buying bonds now (in 2001). A management study has estimated the following cash requirements for the project:

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>2003</td>
<td>2004</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

- The investment committee is considering four government bonds for possible purchase. The price and cash flows of the bonds (in $) are:

<table>
<thead>
<tr>
<th>Bond Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 1</td>
</tr>
<tr>
<td>2001</td>
</tr>
<tr>
<td>2002</td>
</tr>
<tr>
<td>2003</td>
</tr>
<tr>
<td>2004</td>
</tr>
</tbody>
</table>

- What is the least expensive portfolio of bonds whose cash flows equal or exceed the requirements for the project?

Linear-Programming Formulation

- **Decision Variables**: Let \( X_j \) = # of bond \( j \) to purchase today (in millions of bonds)
- **Objective function**: Minimize the total cost of the bond portfolio (in $ million):
  \[
  \text{min } 1.04 X_1 + 1.00 X_2 + 0.98 X_3 + 0.92 X_4.
  \]
- **Constraints**:
  - In each year, the cash flow from the bonds should equal or exceed the project’s cash requirements:
    Cash flow from bonds \( \geq \) Requirement
  - This leads to three constraints:
    \[(\text{yr. } 2002) \quad 0.05 X_1 + 0.04 X_2 + X_3 \geq 20\]
    \[(\text{yr. } 2003) \quad 0.05 X_1 + 1.04 X_2 + X_4 \geq 30\]
    \[(\text{yr. } 2004) \quad 1.05 X_1 \geq 40\]
  - Finally, the nonnegativity constraints:
    \( X_j \geq 0, \quad j = 1, 2, 3, 4. \)
- In this formulation, what happens to any excess cash in a given year?
Surplus-Cash Modification

Now suppose that any surplus cash from one year can be carried forward to the next year with 1% interest. How can the LP formulation be modified?

The surplus cash in year 2002 is:
\[ 0.05 X_1 + 0.04 X_2 + X_3 - 20. \]
Multiplying this amount by 1.01 and adding to the cash available in 2003 gives:
\[ 0.05 X_1 + 1.04 X_2 + X_4 + 1.01(0.05 X_1 + 0.04 X_2 + X_3 - 20) \geq 30. \]
This can be simplified to
\[ 0.1005 X_1 + 1.0804 X_2 + X_4 + 1.01 C_1 \geq 50.2. \]
The surplus cash in 2003 is:
\[ 0.1005 X_1 + 1.0804 X_2 + X_4 + 1.01 C_1 - 50.2. \]
This amount could be multiplied by 1.01 and added to the cash available in 2004.

This is getting ugly. Is there a better way?

Surplus-Cash Modification (continued)

A better way is to define surplus cash variables:
\[ C_i = \text{surplus cash in year } i, \text{ in } \$ \text{ millions, where } i = 1 (2002), 2 (2003), 3 (2004). \]

Constraints:

In each year, the cash-balance constraints can be written as:
Cash in = Cash out
or, in more detail,
Cash from bonds + Surplus cash from previous year = Requirement + Cash for next year

This leads to three constraints:
(yr. 2002) \[ 0.05 X_1 + 0.04 X_2 + X_3 \geq 20 + C_1 \]
(yr. 2003) \[ 0.05 X_1 + 1.04 X_2 + X_4 + 1.01 C_1 \geq 30 + C_2 \]
(yr. 2004) \[ 1.05 X_1 + 1.01 C_2 \geq 40 + C_3 \]

And, as usual, we add the non-negativity constraints:
\[ C_i \geq 0, \quad i = 1, 2, 3. \]
Project-Funding Linear Program

- The complete modified linear program is:
  \[
  \min \quad 1.04X_1 + 1.00X_2 + 0.98X_3 + 0.92X_4
  \]
  subject to:
  - (yr. 2002) \quad 0.05X_1 + 0.04X_2 + X_3 = 20 + C_1
  - (yr. 2003) \quad 0.05X_1 + 1.04X_2 + X_4 + 1.01C_1 = 30 + C_2
  - (yr. 2004) \quad 1.05X_1 + 1.01C_2 = 40 + C_3
  - (Non-neg.) \quad X_i \geq 0, \quad i = 1, 2, 3.
  - (Non-neg.) \quad C_i \geq 0, \quad i = 1, 2, 3.

- The cash constraints can be visualized as “flow-balance equations” at each time period:

![Cash Flow Balance Diagram]

Project-Funding Optimized Spreadsheet

- Objective Function: \(=\text{SUMPRODUCT}(C6:F6,C7:F7)\)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>PROJFUND.XLS</td>
<td>Project Funding Spreadsheet</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Total cost</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td># to purchase (in millions)</td>
<td>Bond 1</td>
<td>Bond 2</td>
<td>Bond 3</td>
<td>Bond 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Bond price</td>
<td>1.04</td>
<td>1.00</td>
<td>0.98</td>
<td>0.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Year</td>
<td>Year</td>
<td>Cash flow per bond</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>2002</td>
<td>0.05</td>
<td>0.04</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td>2003</td>
<td>0.05</td>
<td>1.04</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td>2004</td>
<td>1.05</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Year</td>
<td>Cash from bonds prev year</td>
<td>Reinvest amount</td>
<td>Cash from bonds</td>
<td>Surplus cash</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>2002</td>
<td>20.00</td>
<td>0</td>
<td>20.00</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>2003</td>
<td>30.00</td>
<td>0.00</td>
<td>30.00</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>2004</td>
<td>40.00</td>
<td>0.00</td>
<td>40.00</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Cell D17 contains the value 0, since there is no surplus cash from the previous year.
Project-Funding Optimal Solution

<table>
<thead>
<tr>
<th>Bond</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond price:</td>
<td>1.04</td>
<td>1.00</td>
<td>0.98</td>
</tr>
<tr>
<td>Number to purchase (in millions):</td>
<td>38.10</td>
<td>0.00</td>
<td>18.10</td>
</tr>
<tr>
<td>Total cost: $83.20 million.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: $C_i = 0$, for $i = 1, 2, 3$, i.e., there is no surplus cash in any year.

Determining Discount Rates over Time using SolverTable

- What is the added cost (today, in 2001) of an increase in $1 million in the cash requirements a year from now (in 2002)? In 2003? In 2004?
- These are the discount rates over time.
- To determine these discount rates, we will need to solve a number of new problems where we increase, one by one, the requirement in each of the years.
- This can be done in a clever way using SolverTable.

Determining Discount Rates over Time using SolverTable

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash flow per bond</th>
<th>Reinvest</th>
<th>Req'mnt</th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Bond 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>0.05</td>
<td>0.04</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2003</td>
<td>0.05</td>
<td>1.04</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>1.05</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Input Cell

The trick: The IF() statements will add $1 to the requirement of the "current year" entered in Input Cell A17.
SolverTable Parameters

- In SolverTable, make a OneWay table. Enter the following parameters:

- The input cell (A17) will vary from 2001 to 2004, in increments of 1 year. We record the total cost and the optimal portfolio of bonds in the space below the current model.
- The IF() statements in E17:E19 will correctly add $1 to the requirement in the "current year" (entered in input cell A17).

SolverTable Output and Discount Rates

- The output from SolverTable as well as the calculations of the discount rates and the yield are:

<table>
<thead>
<tr>
<th>Year</th>
<th>Present Value of additional $1</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001</td>
<td>$0.98</td>
<td>2.04%</td>
</tr>
<tr>
<td>2002</td>
<td>$0.92</td>
<td>4.26%</td>
</tr>
<tr>
<td>2003</td>
<td>$0.90</td>
<td>3.57%</td>
</tr>
<tr>
<td>2004</td>
<td>$0.92</td>
<td>4.25%</td>
</tr>
</tbody>
</table>

Optimal Cost = B24 - $B$23
Optimal Portfolio of Bonds = $H$24 * (1 / ($A$23 - A24)) - 1

- The discount rates over time are:

<table>
<thead>
<tr>
<th>Year</th>
<th>Present Value of additional $1</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>$0.98</td>
<td>2.04%</td>
</tr>
<tr>
<td>2003</td>
<td>$0.92</td>
<td>4.26%</td>
</tr>
<tr>
<td>2004</td>
<td>$0.90</td>
<td>3.57%</td>
</tr>
</tbody>
</table>
Cash-Flow-Matching Linear Programs

The project funding LP is one example of a \textit{cash-flow-matching LP}, also called an \textit{asset-liability-matching LP}. The bonds purchased are \textit{assets} and the project requirements are \textit{liabilities}. The cash-flow-matching linear program is one approach to problems in \textit{asset-liability management}. Related applications are:

- Pension planning
  - Pension-fund assets are short term
  - Pension liabilities are long term
  - Determine the least-cost portfolio of bonds purchased today that can guarantee funding of future liabilities

- Municipal-bond issuance
  - Bonds issued are liabilities (long term)
  - Cash is raised today (short term)
  - Determine the maximum amount of funds that can be raised today given forecasts of future tax collections

Cash-Flow-Matching LPs (continued)

- Yield-curve estimation
  - Can generate discount factors over time

- Corporate debt defeasance
  - Bonds purchased today can be used to remove long-term liabilities from corporate balance sheets

- Cash-flow-matching LPs have been used on Wall Street to buy and sell (issue) trillions of dollars of government, corporate, and municipal bonds.
For next class

- Read Chapter 6.1 and 6.6 in the W&A text.
- Read pp. 375-376 and 382-384 in the W&A text.
- Optional reading: “Improving Gasoline Blending at Texaco.”