Branch and Bound (for minimization)

- Set a variable and branch
- Compute lower bounds at nodes by solving a relaxed problem.
- Use lower bounds to prune tree. A lower bound at a node is a lower bound on all children.
- Three ways to prune:
  1. If a node is infeasible
  2. If a node has a lower bound that is larger than a candidate solution
  3. If a node has a value that is larger than a candidate solution.
Example

<table>
<thead>
<tr>
<th>j</th>
<th>r_j</th>
<th>p_j</th>
<th>d_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

To compute a lower bound we use a polynomially solvable relaxation, \(1|r_j, p\text{mtn}|L_{\text{max}}\). This is solved by preemptive EDD.

\[
\begin{array}{cccccc}
0 & 1 & 3 & 4 & 10 & 15 & 17 \\
4 & 5 & & & & & \\
\end{array}
\]

\[L_1 = -4, L_4 = 0, L_3 = 4, L_2 = 5.\]

\[L_{\text{max}} = 5.\]

This means that for the real problem \(L_{\text{max}} \geq 5.\)
Begin branching

Branch on the choice of first job

Now for each branch we compute a new lower bound.
Computing lower bounds

1 first. Same as original lowerbound

\[
\begin{array}{cccccc}
0 & 1 & 3 & 4 & 3 & 2 \\
4 & 5 & 10 & 15 & 17 \\
\end{array}
\]

\[L_1 = -4, L_4 = 0, L_3 = 4, L_2 = 5.\]

\[L_{\text{max}} = 5.\]

2 first.

\[
\begin{array}{cccc}
0 & 2 & 1 & 4 \\
1 & 3 & 7 & 12 \\
4 & 18 \\
\end{array}
\]

\[L_2 = -9, L_1 = -1, L_4 = 2, L_3 = 7.\]

\[L_{\text{max}} = 7.\]

Note that this is an actual solution.
3 first.

\[
L_3 = -2, \quad L_1 = 5, \quad L_4 = 8, \quad L_2 = 8.
\]

\[
L_{\text{max}} = 8.
\]

But, this node can actually be pruned. It is clearly dominated by the following schedule

In other words, we know that starting with node three is a bad idea. For the same reason, we can prune starting at node 4.
Current tree

Choose a node to expand further. Can only choose 1.

Now, we check the lower bounds at the nodes by solving the preemptive EDD schedule.
1,2 first.

![Diagram showing schedule for 1,2 first]

This is a real schedule with $L_{\text{max}} = 6$.

1,3 first.

![Diagram showing schedule for 1,3 first]

This is a real schedule with $L_{\text{max}} = 5$.

We can actually stop now, because we have a schedule that matches our lower bound.
\((*,*,*)\) 
\((1,*,*,*)\)  
\((2,*,*,*)\)  
\((3,*,*,*)\)  
\((4,*,*,*)\) 

LB = 5  
LB = 7. Actual schedule  
can’t be better than 5.  
THIS IS OPTIMAL