Arbitrage through Linear Programming

We are given:

- Two securities: 1 and 2, and three scenarios: 1, 2, 3. Prices:

<table>
<thead>
<tr>
<th></th>
<th>today</th>
<th>Scen. 1</th>
<th>Scen. 2</th>
<th>Scen. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>-7</td>
<td>19</td>
<td>-9</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>6</td>
<td>-8</td>
<td>8</td>
</tr>
</tbody>
</table>

Is arbitrage possible?

- We **take a position** today, we **close it tomorrow**
- What is today’s value?
  The sum of the position values, using today’s data!
- What is tomorrow’s value?
  The sum of the position values, using tomorrow’s data!
- **Arbitrage**: < 0 today, and ≥ 0 tomorrow in *every* scenario
- (But we need a riskless security to make this notion more complete)
<table>
<thead>
<tr>
<th></th>
<th>today</th>
<th>Scen. 1</th>
<th>Scen. 2</th>
<th>Scen. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>−7</td>
<td>19</td>
<td>−9</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>6</td>
<td>−8</td>
<td>8</td>
</tr>
</tbody>
</table>

**Example:** **Short 1** security 1, and **Short 1** security 2

**Today:** value  −18

**Tomorrow’s values:**
<table>
<thead>
<tr>
<th></th>
<th>today</th>
<th>Scen. 1</th>
<th>Scen. 2</th>
<th>Scen. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td>10</td>
<td>-7</td>
<td>19</td>
<td>-9</td>
</tr>
<tr>
<td><strong>2</strong></td>
<td>8</td>
<td>6</td>
<td>-8</td>
<td>8</td>
</tr>
</tbody>
</table>

**Example:** Short 1 security 1, and Short 1 security 2

**Today:** value $-18$

**Tomorrow’s values:**

<table>
<thead>
<tr>
<th>Scen. 1</th>
<th>Scen. 2</th>
<th>Scen. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-11</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>today</td>
<td>Scen. 1</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>−7</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

Example: **Long** 1 security 1, and **Short** 2 security 2

So we will have:

<table>
<thead>
<tr>
<th>Today</th>
<th>Scen. 1</th>
<th>Scen. 2</th>
<th>Scen. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>−6</td>
<td>−19</td>
<td>35</td>
<td>−25</td>
</tr>
<tr>
<td></td>
<td>today</td>
<td>Scen. 1</td>
<td>Scen. 2</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>−7</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>6</td>
<td>−8</td>
</tr>
</tbody>
</table>

We also have **cash**: interest rate = 8% in every scenario
We also have **cash**: interest rate = 8% in every scenario

How about  **+5.5** cash,  **−2** security 1,  **1** security 2?
We also have **cash**: interest rate = 8% in every scenario

How about **+5.5** cash, **−2** security 1, **1** security 2?

```
<table>
<thead>
<tr>
<th></th>
<th>today</th>
<th>Scen. 1</th>
<th>Scen. 2</th>
<th>Scen. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>−7</td>
<td>19</td>
<td>−9</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>6</td>
<td>−8</td>
<td>8</td>
</tr>
</tbody>
</table>
```

E.g. (Scen. 1): 5.5 * 1.08 + (−7) * (−2) + 6 * 1 = 25.94
General case

We are given:

- \( n \) securities: \( S_1, \ldots, S_n \), plus cash: this is security \( S_0 \)
- A collection of \( K \) scenarios for what the price of each security will be tomorrow (or next month, etc.):

  In scenario \( i \), the price of security \( j \) will be \( \pi_{ij} \). \((1 \leq i \leq K, 1 \leq j \leq n)\)

Security 0: be \( \pi_{i0} = 1 + r \), where \( r \) = risk-free interest rate \((1 \leq i \leq K)\)

Scenario 0: today’s prices (for cash, price = 1)

- How do we investigate the existence of arbitrage?
Use Linear Programming

- Use variables $x_j = \text{position we take in security } j$ (incl. cash)

So today’s value of our positions is:

$$x_0 + \sum_{j=1}^{n} \pi_{0j} x_j$$

And in scenario $i$, tomorrow’s value of our positions will be:

$$(1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j$$

Two kinds of arbitrage:

**Type A.** Today’s position $< 0$, and in tomorrow’s $\geq 0$ in every scenario

**Type B.** Zero cash flow today, tomorrow’s position $\leq 0$ in every scenario and $< 0$ in at least one scenario
Use Linear Programming

- Use variables $x_j = \text{position we take in security } j \text{ (incl. cash)}$

So today’s value of our positions is:

$$x_0 + \sum_{j=1}^{n} \pi_{0j} x_j$$

And in scenario $i$, tomorrow’s value of our positions will be:

$$(1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j$$

Two kinds of arbitrage:

**Type A.** Today’s position $< 0$, and in tomorrow’s $\geq 0$ in every scenario

**Type B.** Zero cash flow today, tomorrow’s position $\leq 0$ in every scenario and $< 0$ in at least one scenario

**Exercise.** Convince yourself that (because of the riskless security, cash) this is equivalent to the standard notion of arbitrage
\[ x_j = \text{position in security } j \]

Today’s value: \( x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \)

Tomorrow’s in scenario \( i \): \( (1 + r) x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \)

Consider the linear program

\[ \mathbf{V}^* = \text{Minimize} \quad x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Subject to:

\[ (1 + r) x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1 \]

\( x_j \) unrestricted in sign, for every \( j \)
\[ x_j = \text{position in security } j \]

Today’s value: \[ x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Tomorrow’s in scenario \( i \): \[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \]

Consider the linear program

\[ V^* = \text{Minimize } x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Subject to:

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1 \]

\[ x_j \text{ unrestricted in sign, for every } j \]

**Type A: Today’s cash flow} > 0, and in tomorrow’s} \leq 0 \text{ in every scenario}**

This happens if, and only if, \( V^* < 0 \)

Why?
\[ x_j = \text{position in security } j \]

Today’s value: \[ x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Tomorrow’s in scenario \( i \): \[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \]

\[ V^* = \text{Minimize } x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Subject to:

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1 \]

\( x_j \) unrestricted in sign, for every \( j \)

What if \( V^* = 0 \) ?
\( x_j = \text{position in security } j \)

Today’s value: \( x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \)

Tomorrow’s in scenario \( i \): \( (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \)

\[ V^* = \text{Minimize } x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Subject to:

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1 \]

\( x_j \) unrestricted in sign, for every \( j \)

What if \( V^* = 0 \) ?

Type B: can we find a vector \( x \), such that

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1 \]

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j > 0 \quad \text{for at least one scenario } i \geq 1 \]

\[ x_0 + \sum_{j=1}^{n} \pi_{0j} x_j = 0 \]
\( V^* = \text{Minimize } x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \) 

Subject to:

\[
(1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1
\]

\( x_j \) unrestricted in sign, for every \( j \)

If no Type A or Type B arbitrage exist, then \( V^* = 0 \) and every optimal solution to the LP satisfies:

\[
(1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j = 0 \quad \text{for every scenario } i \geq 1
\]
\[ V^* = \text{Minimize } x_0 + \sum_{j=1}^{n} \pi_{0j} x_j \]

Subject to:

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0 \quad \text{for every scenario } i \geq 1 \]

\[ x_j \text{ unrestricted in sign, for every } j \]

If no Type A or Type B arbitrage exist, then ... \( V^* = 0 \) and every optimal solution to the LP satisfies:

\[ (1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j = 0 \quad \text{for every scenario } i \geq 1 \]

Dual LP:

\[ V^* = \text{Maximize } \sum_{i=1}^{K} P_i \]

Subject to:

\[ (1 + r)\sum_{i=1}^{K} P_i = 1 \]

\[ \sum_{i=1}^{K} \pi_{ij} P_i = \pi_{0j} \quad \text{for every security } j \geq 1 \]

\[ P_i \geq 0 \quad \text{for every scenario } i \geq 1 \]
$V^* \text{=Minimize } x_0 + \sum_{j=1}^{\pi_0} x_j$

Subject to:

$(1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j \geq 0$ \text{ for every scenario } i \geq 1

$x_j$ unrestricted in sign, for every $j$

If no Type A or Type B arbitrage exist, then ... \text{ } V^* = 0 and every optimal solution to the LP satisfies:

$(1 + r)x_0 + \sum_{j=1}^{n} \pi_{ij} x_j = 0$ \text{ for every scenario } i \geq 1

dual: $V^* \text{=Maximize } \sum_{i=1}^{K} P_i$

Subject to:

$(1 + r)\sum_{i=1}^{K} P_i = 1$

$\sum_{i=1}^{K} \pi_{ij} P_i = \pi_{0j}$ \text{ for every security } j \geq 1

$P_i \geq 0$ \text{ for every scenario } i \geq 1

...And an appropriate optimal dual solution gives us risk-neutral probabilities