DB method, SL method and switching rules

We will be using the following notation:
\( N \) = Asset lifetime,
\( P \) = Initial cost of the asset,
\( F \) = The salvage value of the asset at the end of \( N \) years,
\( D_n \) = Depreciation expense taken in year \( n \),
\( BB_n \) = The book balance of the asset at the end of \( n \) years,
\( D_j^{SL} \) = Depreciation expense taken in year \( j \) when using straight line method,
\( D_j^{DB} \) = Depreciation expense taken in year \( j \) when using declining balance method.

Recall that under declining balance method (DB), the salvage value of the asset, \( F \), is not considered in calculating the depreciation schedule. Therefore, the implied salvage value of the asset, \( BB_N \), can be higher or lower than the real value \( F \).

1. First we will consider the case \( BB_N < F \). This means that if we use the DB method directly, the total depreciation that will be claimed over \( N \) years is \( (P - BB_N) \), which is larger than the total allowable amount \( (P - F) \). Hence the DB method should be modified.

Let \( \hat{n} \) be the smallest integer \( n \) such that \( BB_n < F \). The depreciation schedule under DB method is modified as:

\[
D_n = \begin{cases} 
\alpha BB_{n-1} & n = 1, 2, \ldots, \hat{n} - 1, \\
BB_{n-1} - F & n = \hat{n}, \\
0 & n = \hat{n} + 1, \ldots, N,
\end{cases}
\]

where \( \alpha = \sigma/N \). Assume that \( \sigma = 1.5 \) (150\% DB) or \( \sigma = 2 \) (double DB).

Suppose now that we have the option to switch to straight-line method (SL) in any year \( j \).

Under this option, with switching at time \( j \), the depreciation schedule \( D'_n \) is obtained as

\[
D'_n = \begin{cases} 
D_n & n = 1, 2, \ldots, j - 1, \\
\frac{BB_{n-1} - F}{N - j + 1} & n = j, \ldots, N.
\end{cases}
\]

For any interest rate \( i \), we can show (cf. Appendix) that:

\[
PV(i) \text{ of } \{D_n, n = 1, \ldots, N\} \geq PV(i) \text{ of } \{D'_n, n = 1, \ldots, N\}.
\]

This shows that if \( BB_N < F \), it is not optimal to switch to SL method.

2. Consider now the scenario \( BB_N > F \). That is, if we use the DB method, we would not be able to claim all of the depreciation allowance \( (P - F) \).

In such a case, we can show (cf. Appendix) that it is optimal to switch to SL method in year \( n \), when the depreciation under DB falls below the depreciation that would result from switching to SL. Mathematically, the switching rule is the following: We switch to SL the first time

\[
D_n^{DB} < D_n^{SL}.
\]

We know from class that:

\[
D_n^{DB} = \alpha BB_{n-1} = \alpha P(1 - \alpha)^{n-1},
\]

and \( D_n^{SL} \) is given by:

\[
D_n^{SL} = \frac{BB_{n-1} - F}{N - n + 1} = \frac{P(1 - \alpha)^{n-1} - F}{N - n + 1}.
\]
Therefore, optimal switching time, \( n^* \), is the smallest integer \( n \) such that

\[
D_n^{DB} < D_n^{SL}
\]

\[
\alpha P(1 - \alpha)^{n-1} < \frac{P(1 - \alpha)^{n-1} - F}{N - n + 1}
\]

\[
(1 - \alpha)^{n-1}(\alpha(N - n + 1) - 1) < -F/P.
\]

Examples

Many assets are depreciated following the half-year convention (HY). Under HY, the asset is assumed to be placed in the middle of the first year. Therefore, at the end of year 1, 1/2 of the normal depreciation allowance is claimed and the remaining lifetime of the asset is reduced by only 1/2. Therefore, under HY the depreciation schedule for DB is modified as follows:

\[
D_1^{DB} = P \alpha/2 \quad (\rightarrow BB_1 = P(1 - \alpha/2))
\]

\[
D_2^{DB} = P \alpha(1 - \alpha/2) \quad (\rightarrow BB_2 = P(1 - \alpha/2)(1 - \alpha))
\]

\[
\vdots
\]

\[
D_n^{DB} = P \alpha(1 - \alpha/2)(1 - \alpha)^{n-2} \quad (\rightarrow BB_n = P(1 - \alpha/2)(1 - \alpha)^{n-1})
\]

For the SL method, we just modify the depreciation schedule to take into account that in the first year the asset’s useful life is decreased by 1/2. Hence

\[
D_1^{SL} = \frac{P - F}{2N}
\]

\[
\vdots
\]

\[
D_n^{SL} = \frac{BB_{n-1} - F}{N - n + 1.5}
\]

\[
= \frac{P(1 - \alpha/2)(1 - \alpha)^{n-2} - F}{N - n + 1.5}.
\]

Since it is not optimal to switch to SL in year 1, we can consider the years \( n, \ n \geq 2 \). The optimal switching rule is the same is in Problem 2, but with the redefined depreciation schedules. So, \( n^* \) is the smallest integer \( n \), that satisfies

\[
D_n^{DB} < D_n^{SL}
\]

\[
(1 - \alpha/2)(1 - \alpha)^{n-2}(\alpha(N - n + 1.5) - 1) < -F/P.
\]

1. Suppose that \( F = 0 \). We will simplify the above equation to obtain an exact equation for \( n^* \) (\( n^* \) becomes a simple function of \( N \) and \( \sigma \)). We will calculate \( n^* \) for (a) \( N = 7, \ \sigma = 2 \) and (b) \( N = 15, \ \sigma = 1.50 \).

2. We will then calculate the resulting depreciation percentages for each year (i.e. calculate the depreciation percentages under DB method with switching to SL in year \( n^* \)). These are the MACRS percentages for 7 year and 15 yr property.

Now letting \( F = 0 \), we see that \( n^* \) is the smallest integer that satisfies

\[
\alpha(N - n + 1.5) - 1 < 0
\]

\[
N - n + 1.5 < 1/\alpha \quad (1/\alpha = N/\sigma)
\]

\[
n > N^{\sigma - 1}/\sigma + 1.5.
\]
In closed form we have,

\[ n^* = \left\lfloor N \frac{\sigma - 1}{\sigma} + 1.5 \right\rfloor, \]

where \([a]\) is defined as the smallest integer greater than \(a\).

1. (a) \(N = 7\) and \(\sigma = 2\) results in \(n^* = \left\lfloor 5.0 \right\rfloor = 6\).
   (b) \(N = 15\) and \(\sigma = 1.5\) results in \(n^* = \left\lfloor 6.5 \right\rfloor = 7\).

2. \(N = 7, \sigma = 2, \) and \(n^* = 6\). Therefore the depreciation schedule is

\[
D_n = \begin{cases} 
D_n^{DB} & n = 1, 2, 3, 4, 5, \\
D_n^{SL} & n = 6, 7, \\
\frac{1}{2}D_6^{SL} & n = 8.
\end{cases}
\]

Using the formulas for \(D_n^{DB}\) and \(D_n^{SL}\), we get the following percentages:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_n) (%)</td>
<td>14.29</td>
<td>24.49</td>
<td>17.49</td>
<td>12.49</td>
<td>8.92</td>
<td>8.92</td>
<td>8.92</td>
<td>4.46</td>
</tr>
</tbody>
</table>

3. \(N = 15, \sigma = 1.5, \) and \(n^* = 7\). Therefore the depreciation schedule is

\[
D_n = \begin{cases} 
D_n^{DB} & n = 1, 2, ..., 6, \\
D_n^{SL} & n = 7, 8, ..., 15, \\
\frac{1}{2}D_7^{SL} & n = 16.
\end{cases}
\]

Using the formulas for \(D_n^{DB}\) and \(D_n^{SL}\), we get the following percentages:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7-15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_n) (%)</td>
<td>5.0</td>
<td>9.5</td>
<td>8.5</td>
<td>7.69</td>
<td>6.93</td>
<td>6.23</td>
<td>5.90</td>
<td>2.95</td>
</tr>
</tbody>
</table>
Appendix

We will prove that in 1. we have:

\[ PV(i) \text{ of } \{D_n, n = 1, ..., N\} \geq PV(i) \text{ of } \{D'_n, n = 1, ..., N\}. \]

Let \( j, j \leq \hat{n} \leq N \), be the year that we switch to SL method. In comparing the PV’s we need to consider only the periods \( \{j, j+1, ..., N\} \), since the depreciation expenses for the years \( \{1, 2, ..., j-1\} \) is the same under two cases. Recall that if we switch to SL in period \( j \), the \( D'_n, n = j, ..., N \), is a constant and is given by

\[ \frac{BB_{j-1} - F}{N - j + 1} = D_j^{SL}. \]

First we need to show that \( D_j = \alpha BB_{j-1} \geq D_j^{SL} \). To show this, suppose the contrary is true and that \( D_j < D_j^{SL} \). Since under DB, \( D_n \) is monotonically decreasing, it must be that \( D_n < D_j^{SL} \) for all \( n, n = j, j+1, ..., N \). But this implies:

\[ \sum_{n=j}^{N} D_n < \sum_{n=j}^{N} D_j^{SL} = \frac{BB_{j-1} - F}{N - j + 1} (N - j + 1) = BB_{j-1} - F. \]

This a contradiction since under DB, with our modification, we have

\[ \sum_{n=j}^{\hat{n}} D_n = \sum_{n=j}^{N} D_n = BB_{j-1} - F. \]

We have now shown that \( D_j > D_j^{SL} \). Now, let \( \Delta_n = D_n - D_j^{SL} \), for \( n = j, ..., N \). We have \( \Delta_j \geq 0 \) and \( \Delta_n \) decreasing in \( n \). Moreover, since

\[ \sum_{n=j}^{N} D_n = \sum_{n=j}^{N} D_j^{SL} = BB_{j-1} - F, \]

we have \( \sum_{n=j}^{N} \Delta_n = 0 \). Therefore, the present value of \( \{\Delta_n\} \)

\[ PV = \sum_{n=j}^{N} \frac{\Delta_n}{(1+i)^n} \geq 0. \]

The above is always true, but we can also show this mathematically. Let \( k \) be the year in which \( \Delta_n \) becomes negative. That is, \( \Delta_n \geq 0 \) for \( n = j, j+1, ..., k-1 \) and \( \Delta_n < 0 \) for \( n = k, k+1, ..., N \). Then,

\[ PV = \sum_{n=j}^{k-1} \frac{\Delta_n}{(1+i)^n} + \sum_{n=k}^{N} \frac{\Delta_n}{(1+i)^n}. \]

It is easy to verify that

\[ \sum_{n=j}^{k-1} \frac{\Delta_n}{(1+i)^n} \geq \frac{\sum_{n=j}^{k-1} \Delta_n}{(1+i)^k}, \]

since \( \Delta_n \geq 0 \) for \( n = j, ..., k-1 \) and

\[ \sum_{n=k}^{N} \frac{\Delta_n}{(1+i)^n} \geq \frac{\sum_{n=k}^{N} \Delta_n}{(1+i)^k}, \]
since $\Delta_n < 0$ for $n = k, \ldots, N$. Therefore,

$$PV \geq \frac{\sum_{n=k}^{n-1} \Delta_n}{(1+i)^k} + \frac{\sum_{n=N}^{n-N} \Delta_n}{(1+i)^k}$$

$$= \frac{\sum_{n=k}^{n-1} \Delta_n + \sum_{n-N}^{n-N} \Delta_n}{(1+i)^k}$$

$$= 0.$$  

Arguments for 2.: Suppose that $BB_N > F$. This means that, if we use the DB, we won’t be able to claim all the depreciation allowance $(P - F)$ over $N$ years. Therefore, it is always better to switch to SL (at least in the last period $N$), so that all the depreciation is claimed. However switching in the last year may not be the optimal switching policy. Recall that, our aim is to depreciate the asset as soon as possible. This means that, we would like to claim as high depreciation as possible in the early years. To avoid confusion, let $D_n^{DB}$ denote the depreciation in yr $n$ under DB and $D_n^{SL}$ as the depreciation in yr $n$ if we switch to SL. When $\sigma = 1.5$ or 2.0, we have

$$D_1^{DB} = \alpha P = \frac{P}{N} \frac{\sigma}{N} > D_1^{SL} = \frac{P - F}{N}.$$  

Since we want to depreciate the asset as fast as possible, we would want to use the DB method and not to switch to SL in yr 1. However, $D_n^{DB}$ is decreasing in $n$ and there will be a period where $D_n^{DB} < D_n^{SL}$. In view of our objective, we would keep the DB as long as $D_n^{DB} \geq D_n^{SL}$, and switch to SL in the year that $D_n^{DB} < D_n^{SL}$.