1 Real Options

So far we have ignored from consideration the possibility of postponing the investment decision. To illustrate why this is an important option we will introduce a simple example where waiting may be the best option. Let $I$ be the cost of investing on a widget factory. We assume an inflation-free scenario, so that if we wait an invest a year from now the cost still be $I$. Let $i$ be the inflation-free and risk-free interest rate, and assume that investing in the widget factory results in the revenue stream $\{P_0, P_1, P_2, \ldots\}$ where $P_n = 1.5P_o, n = 1, \ldots$ with probability $q$ and $P_n = 0.5P_o, n = 1, \ldots$ with probability $1 - q$. Suppose we must make the decision at time 0. Assume further that the price of the widget is uncorrelated with the market so that discounting the expected value of the cash flow at the risk-free rate makes sense. Then, the expected net present value is given by

$$\text{ENPV}_0 = -I + P_o + q1.5P_o \sum_{n=1}^{\infty} (1 + i)^{-n} + (1 - q)0.5P_o \sum_{n=1}^{\infty} (1 + i)^{-n}$$

$$= -I + P_o \left[ 1 + q \frac{1.5}{i} + (1 - q) \frac{0.5}{i} \right],$$

and invest if this quantity is positive. That is, we invest now if

$$I \leq P_o \left[ 1 + q \frac{1.5}{i} + (1 - q) \frac{0.5}{i} \right].$$

For example, if $P_o = 200$, $q = 0.5$, $i = 10\%$ and $I = 1600$ we have

$$\text{ENPV}_0 = -1,600 + 2,200 = 600.$$  

More generally, it is optimal to invest now whenever

$$I \leq 2200.$$

Now consider the option of waiting for one year. The net present value of investing at the end of year one is

$$\frac{1.5P_o}{i} - \frac{I}{1+i}$$

if the price goes up and

$$\frac{0.5P_o}{i} - \frac{I}{1+i}$$

if the price goes down. Clearly, we would only make the investment if the resulting value is positive. So the expected net present value when we wait is given by

$$\text{ENPV}_w = q \left[ \frac{1.5P_o}{i} - \frac{I}{1+i} \right]_+$$

$$+ (1 - q) \left[ \frac{0.5P_o}{i} - \frac{I}{1+i} \right]_+$$

where $x_+ = \max(x, 0)$. Of course, the optimal decision is to wait only when the expected value is larger than investing now. Consequently, the optimal expected net present value is given by

$$\text{ENPV}^* = \max(\text{ENPV}_0, \text{ENPV}_w).$$

To illustrate, let us continue with our numerical example with $P_o = 200, q = 0.5$, and $i = 10\%$. For values of $I \in \{1,100,3,300\}$ we have

$$\text{ENPV}_w = 0.5 \left[ 3000 - \frac{I}{1.1} \right]$$

$$= 1500 - \frac{I}{2.2}.$$
Comparing to $2200 - I$ we see that $\text{ENPV}_w > \text{ENPV}$ for values of $I \in (1, 283.33, 3, 300]$ with the opposite inequality holding for $(1, 000, 1, 283.33, 1, 300].$ This results in $\text{ENPV}^* = \text{ENPV}_0 = 2200 - I$ on $I \leq 1, 283.33$ and $\text{ENPV}^* = \text{ENPV}_w = 1500 - I/2.2$ on $I \in (1, 283.33, 3, 300].$

Thus, for example, if $I = 1, 600$ we see that $\text{ENPV}_0 = 600 > 0,$ and yet it is better to wait, since $\text{ENPV}_w = 772.73.$ Here, we are better off postponing the investment even though the present value is positive. On the other hand, suppose that $I = 2, 500.$ Then, $\text{ENPV}_0 = -300$ so the present value is negative, yet $\text{ENPV}_w = 363.64 > 0.$ Here, if we were force to make the decision now we would reject the project. Yet, the expected present value of the project is positive when we have the option of waiting!

In summary, we would invest now if $I < 1, 283.33$ and never invest if $I > 3, 300.$ For intermediate values it is best to wait one year and investment if the price goes up.

A similar analysis can be carried out to determine values of $P_0$ for which it is best to invest now, wait, or not invest at all. Indeed, if $q = 0.5$ then

$$\text{ENPV}_0 = -I + P_0(1 + i)/i,$$

so $\text{ENPV}_0 \geq 0$ whenever

$$P_0 \geq I \frac{i}{1 + i}.$$

Let us now consider the option of waiting. If

$$\frac{1}{1.5} I \frac{i}{1 + i} \leq P_0 \leq 2I \frac{i}{1 + i},$$

then we would only invest if at time 2 the price goes up, so

$$\text{ENPV}_w = 0.5 \left[ \frac{1.5P_0}{i} \frac{I}{1 + i} \right].$$

Again,

$$\text{ENPV}^* = \max(\text{ENPV}_0, \text{ENPV}_w).$$

Here $\text{ENPV}_0 \leq \text{ENPV}_w$ on

$$P_0 \leq 2I \frac{1 + 2i}{1 + i} \frac{i}{1 + 4i}.$$

Thus if $i = 10\%$ and $I = 1600$ waiting is better than investing now if

$$97 \leq P_0 \leq 249.$$

An investing now is better than waiting if

$$P_0 > 249.$$

On the other hand, it is best not to invest at all if

$$P_0 < 97.$$

2 Bad News Principle

Let us assume now that at the end of time 1 the price of the widget goes up to $(1 + u)P_0$ with probability $q$ and down to $(1 - d)P_0$ with probability $1 - q.$ As before we assume that $P_n = P_1$ for all $n \geq 2.$ The inflation-free, risk-free rate is assumed to be $i.$ If we invest now the expected net present value is

$$\text{ENPV}_0 = P_0 \left[ 1 + \frac{1}{i}(q(1 + u) + (1 - q)(1 - d)) \right] - I.$$
If the data is such that if we wait we invest only if the price goes up then the expected net present value of waiting is

$$\text{ENPV}_w = \frac{q}{1 + i} \left[ (1 + u)P_0 \frac{1 + i}{i} - I \right].$$

The break even point between investing and not investing is the solution to the equation

$$P_0[1 + \frac{1}{i}(1 - q)(1 - d)] = I - \frac{q}{1 + i},$$

or

$$P_0 = I \frac{1 - \frac{q}{1 + i}}{1 + \frac{1}{i}(1 - q)(1 - d)}.$$

Notice that this equation is independent of $u$ (the good news), but depends on $d$ (the bad news).

For example, if $i = 10\%$, $d = u = q = 0.5$ and $I = 1600$ then

$$P_0 = 249$$

is the break even point.

3 Analogy to Financial Options

The current price of the widget is $P_0$ and it goes up to $P_0(1 + u)$ with probability $q$ and goes down to $P_0(1 - d)$ with probability $1 - q$. Assume for now that $u = d = 0.5$.

Let $F_0$ be the value of the opportunity to invest. At the end of period one, if $P_0(1 - d)/(1 + i)/i < I$ we invest only if the price of the widget goes up. Thus the value at time one is

$$F_u = P_0(1 + u) \frac{1 + i}{i} - I$$

if the price goes up and

$$F_d = 0$$

if the price goes down. We want to form a portfolio of the opportunity to invest and $n$ short positions on the widget. If we borrow $n$ widgets now we get a payoff of $-nP_0$ at time zero. Since the expected value of the widget a year from now is

$$P_0[q(1 + u) + (1 - q)(1 - u)] = P_0[q + 0.5].$$

Thus, the capital gain is $q - 0.5$. A rational investor would only take a long position if the sum of the capital gain and the dividend equals the risk-free rate. Thus an investor would require a dividend payment of

$$i - q + 0.5.$$

Consequently, to take a short position you need to pay

$$nP_0(1 + u + i - q + 0.5)$$

if the price goes up and

$$nP_0(1 - d + i - q + 0.5)$$

if the price goes down.

The value of the portfolio at time 1 is

$$\Phi_u = P_0(1 + u) \frac{1 + i}{i} - I - nP_0(1 + u + i - q + 0.5)$$
if the price goes up, and
\[ \Phi_d = -nP_0(1 - d + i - q + 0.5). \]
The idea is to make this portfolio risk free by selecting \( n \) so that \( \Phi_u = \Phi_d \). The choice of \( n \) is
\[ n = \frac{1 + u 1 + i}{u + d} \frac{I}{P_0(u + d)} \]
Remark: \( n \) is independent of \( q \).
For example, if \( u = d = 0.5 \), \( I = 1600 \), and \( i = .10 \) we find
\[ n = 16.5 - 1600/P_0, \]
and if \( P_0 = 200 \) then \( n = 8.5 \).
The value of the portfolio at time zero is
\[ \Phi_0 = F_0 - nP_0. \]
Since the portfolio is risk-free it must be that
\[ \Phi_1 = \Phi_0(1 + i). \]
Solving for \( F_0 \) we find
\[ F_0 = \frac{nP_0(d + q - 0.5)}{1 + i}. \]
Substituting for \( n \) we obtain for the case \( q = 0.5 \) we find
\[ F_0 = F_0 1 + u d\frac{1}{u + d} I d \frac{I}{(u + d)(1 + i)}. \]
For \( P_0 = 200, u = d = 0.5, I = 1600 \), and \( i = .10 \) and \( q = 0.5 \) we find
\[ F_0 = 1500 - 800/1.1 = 773. \]
On the other hand, if \( P_0 \) is unknown then
\[ F_0 = 7.5P_0 - 727. \]
We have calculated the value of the investment assuming that we would only want to invest if the price goes up next year. However, if \( P_0 \) is low enough we might never want to invest, and if \( P_0 \) is high enough it might be better to invest now than to wait. Below which price would we never invest? We see that \( F_0 = 0 \) when \( P_0 = $97 \). If \( P_0 \) is less than \$97, we would not invest even if the price goes up by 50%.
For what values of \( P_0 \) should we invest now rather than wait? We should invest now if ENPV_0 = \( 11P_0 - 1600 \) is greater or equal to \( F_0 \). The critical price satisfies
\[ 11P_0 - 1600 = 7.5P_0 - 727, \]
or \( P_0 = $249 \).
Sumarizing: We invest now if \( P_0 > $249 \). If \$97 < \( P_0 \leq $249 \) invest in period 1 only if price goes up. If \( P_0 \leq $97 \) never invest.