Solution to Assignment 10

April 28, 1999

1 Problem 1:

1.1 Part a:
The density is:

\[ f(x) = \begin{cases} \frac{n}{\theta} \left( \frac{x}{\theta} \right)^{n-1}, & \text{if } 0 < x < \theta \\ 0, & \text{otherwise} \end{cases} \]

1.2 Part b:
\[ E[\hat{\theta}] = \frac{n}{n+1} \theta, \ Var[\hat{\theta}] = \theta^2 \left( \frac{n}{n+2} - \frac{n^2}{(n+1)^2} \right). \]

1.3 Part c:
Since \( b = \theta - E[\hat{\theta}] = \frac{\theta}{n+1} \), we know \( b \) goes to zero as \( n \to \infty \).

1.4 Part d:
\[ MSE(\hat{\theta}) = \theta^2 \left( \frac{2n+2}{n+2} - \frac{n}{n+1} \right). \]

2 Problem 2:

2.1 Part a:
We have to assume that \( E[\hat{\theta}] = \theta \). Under this assumption, we know:
\[ E[g(\hat{\theta})] \approx g(\theta) + g'(\theta)(E[\hat{\theta} - \theta]) = g(\theta) \]
2.2 Part b:
Since \((g(\hat{\theta}) - g(\theta))^2 \approx g(\theta)^2(\hat{\theta} - \theta)^2\), we have:
\[
Var[g(\theta)] = E[(g(\theta) - g(\theta))^2] \approx g(\theta)^2E[(\hat{\theta} - \theta)^2] = g(\theta)^2Var[\hat{\theta}]
\]

2.3 Part c:
From Part b, we know the standard deviation of \(g(\hat{\theta})\) is about \(|g(\theta)|\sqrt{Var[g(\theta)]}\).

2.4 Part d:
It is easy to know that the confidence interval is:
\[
(g(\hat{\theta}) - z_{\frac{a}{2}}\sqrt{Var[g(\theta)]}, g(\hat{\theta}) + z_{\frac{a}{2}}\sqrt{Var[g(\theta)]})
\]

2.5 Part e:
Following Part d, we know the confidence interval is:
\[
(e^{-\hat{\theta}} - z_{\frac{a}{2}}\sqrt{Var[e^{-\hat{\theta}}]}, e^{-\hat{\theta}} + z_{\frac{a}{2}}\sqrt{Var[e^{-\hat{\theta}}]})
\]

3 Problem 7.69
\[
L(\beta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{x_i^{2-\alpha} e^{-x_i/\beta}}{\beta^{\alpha} \Gamma(\alpha)} = \beta^{-n\alpha} (\Gamma(\alpha))^{-n} e^{-\sum_{i=1}^{n} (x_i/\beta)} \prod_{i=1}^{n} x_i^{\alpha-1}.
\]
To simplify the calculations, we maximize
\[
lnL(\beta) = -nln(\Gamma(\alpha)) + (\alpha - 1) \sum_{i=1}^{n} ln(x_i) - n\alpha ln(\beta) - \sum_{i=1}^{n} \frac{x_i}{\beta}.
\]
Thus,
\[
\frac{\partial lnL(\beta)}{\partial \beta} \bigg|_{\beta=\hat{\beta}} = -\frac{n\alpha}{\beta} + \sum_{i=1}^{n} \frac{x_i}{\beta^2} = 0
\]
Solving for \(\hat{\beta}\) results in \(\hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha} = \bar{x}_\alpha\).

4 Problem 8.1
Hypothesis: \(H_0 : \mu = 130, H_a : \mu < 130\)
Test statistics:
\[
z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \approx \frac{128.6 - 130}{2.1/\sqrt{10}} = -4.22.
\]
Rejection region: \( z < -z_{0.05} = -1.645 \). Conclusion: Reject \( H_0 \) at \( \alpha = 0.05 \); i.e., there is sufficient evidence to conclude that the mean output voltage is less than 130 at \( \alpha = 0.05 \).

5 Problem 8.2

\[
\beta = P(fail - to - reject \ H_0 \ given \ that \ H_a \ is \ - true) = P\left(\frac{\bar{x} - 130}{2.1/\sqrt{40}} > -1.645 | \mu = 129\right) = P(\bar{x} > 129.4538 | \mu = 129) = P\left(\frac{\bar{x} - 129}{2.1/\sqrt{40}} > \frac{129.4538 - 129}{2.1/\sqrt{40}}\right) = P(Z > 1.37) = 0.0853.
\]

6 Problem 8.3

\[
n \geq \frac{(z_{0.05} + z_{0.1})^2 \sigma^2}{(\mu_a - \mu_n)^2} = \frac{(1.645 + 2.33)(2.1)^2}{(129 - 130)^2} = 69.68 \text{ or } n = 70.
\]

7 Problem 8.31

Summary Statistics:

<table>
<thead>
<tr>
<th></th>
<th>1: Native</th>
<th>2: Nonnative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \bar{x} )</td>
<td>87</td>
<td>77.3</td>
</tr>
<tr>
<td>( s )</td>
<td>3.2318</td>
<td>4.0291</td>
</tr>
</tbody>
</table>

Hypothesis: \( H_0 : \mu_1 - \mu_2 = 0, H_a : \mu_1 - \mu_2 > 0 \)

Test statistics: Assume independent samples and a common normal population,

\[
s_p = \left(\frac{(3.2318^2 + 4.0291^2)}{10 + 10 - 2}\right)^{1/2} = 3.6522
\]

\[
t = \frac{87 - 77.3}{3.6522 \sqrt{1/10 + 1/10}} = 5.94
\]

Rejection Region: \( t < -t_{0.05} = 1.734 \) (degree of freedom = 18)

Conclusion: Reject \( H_0 \) at \( \alpha = 0.05 \); i.e., there is sufficient evidence to conclude that the nonnative English speakers have a significantly smaller mean percentage of correct response at \( \alpha = 0.05 \).
8 Problem 8.36

Hypothesis: $H_0: \mu_2 - \mu_1 \leq 8, H_a: \mu_2 - \mu_1 > 8$

Test statistics: 
\[ z = \frac{5.55 - 5.45 - 8}{\sqrt{\frac{.01}{4} + 1.5}} = 0.4714. \]

Rejection region: $z > 1.645$.

Conclusion: Fail to reject $H_0$ at $\alpha = 0.05$; i.e., there is insufficient evidence to conclude that the mean tensile strength for 12-mm-diameter stell rods is at least 8 units ($N/mm^2$) higher than that for 10-mm-diameter stell rods at $\alpha = 0.05$. 