1. **Two correlated Assets.** Assume that the correlation $\rho$ between two assets A and B is $-0.2$, the mean returns are $\bar{R}_A = 10\%$, $\bar{R}_B = 20\%$, and the standard deviations are $\sigma_A = 15\%$, $\sigma_B = 25\%$.

   (a) Find the minimum variance portfolio.
   
   (b) What is the mean and the standard deviation of the minimum variance portfolio?
   
   (c) Trace the efficient frontier of portfolios that invest in assets A and B only.
   
   (d) Assume that the risk free rate is $\bar{R}_f = 5\%$. Find the portfolio of risky assets (A and B) with largest Sharpe ratio. (The Sharpe ratio of a portfolio is the ratio of the expected excess return of the portfolio divided by its standard deviation.) Determine the mean return and the standard deviation of this portfolio.
   
   (e) Find the efficient frontier with riskless lending and borrowing.

2. Consider two risky securities with variance $\sigma_1^2$, $\sigma_2^2$ and correlation coefficient $\rho$. Show that the minimum variance portfolio has variance

$$\sigma_m^2 = \frac{\sigma_1^2 \sigma_2^2 (1 - \rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2}.$$ 

Show that $\sigma_m^2 < \min(\sigma_1^2, \sigma_2^2)$ unless $\rho = \min(\sigma_1 / \sigma_2, \sigma_2 / \sigma_1)$.

3. **Finding the minimum variance portfolio of uncorrelated securities with equal expected returns.** Suppose that $\bar{R}_i = \bar{R}$ (a constant) for all $i$, that $\sigma_i = \sigma_i^2 > 0$ for $i = 1, \ldots, n$, and that $\sigma_{ij} = 0$ for all $i \neq j$.

   (a) Plot the set of portfolios in the standard deviation-mean return space.
   
   (b) Find the weights of the minimum variance portfolio.
   
   (c) What is the variance of the minimum variance portfolio?

4. **General Betting Wheel.** Consider a betting wheel with $n$ segments. The payoff for a $\$1$ bet on segment $i$ is $A_i > 0$. Suppose you bet an amount $B_i = 1/A_i$ on segment $i$ for each $i$.

   (a) What is the mean and variance of the return for this betting strategy?
   
   (b) Apply this to a wheel with three segments with payoffs $A_1 = 3$, $A_2 = 4$, $A_3 = 7$, with probabilities $p_1 = 0.5$, $p_2 = 1/3$, and $p_3 = 1/6$.

5. **Portfolio Tracking.** Suppose that it is impractical to form a portfolio that invests in all assets in a given universe of risky assets. One alternative is to find a portfolio, made up of a given subset of $n$ risky assets, that tracks a target portfolio that invest in all risky assets. Specifically, suppose that the target portfolio has random return $R_M$. Suppose that the $n$ assets have random returns $R_1, \ldots, R_n$. We wish to find weights $X_1, \ldots, X_n$ such that the resulting portfolio $R_p = \sum_{i=1}^n X_i R_i$, with $\sum_{i=1}^n X_i = 1$ minimizes $\text{Var}(R_p - R_M)$.

   (a) Find a set of equations for the weights $X_1, \ldots, X_n$.
   
   (b) Although the resulting portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. A logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the required mean return is varied, this results in a family of portfolios that are efficient in a new sense, say tracking efficient. Find the equation of the $X_i$'s that are tracking efficient.