1. **Two correlated Assets.** Assume that the correlation \( \rho \) between two assets A and B is \(-0.2\), the mean returns are \( R_A = 10\% \), \( R_B = 20\% \), and the standard deviations are \( \sigma_A = 15\% \), \( \sigma_B = 25\% \).

   (a) Find the minimum variance portfolio.

   Answer: Let \( X_A \) and \( X_B \) denote the weights. Solving for \( x = (X_A, X_B) \) results in

   \[
   X_A = \frac{\sigma_B^2 - \sigma_A \sigma_B \rho}{\sigma_A^2 - 2 \sigma_A \sigma_B \rho},
   \]

   and

   \[
   X_B = \frac{\sigma_A^2 - \sigma_A \sigma_B \rho}{\sigma_A^2 - 2 \sigma_A \sigma_B \rho}.
   \]

   Notice that \( X_A + X_B = 1 \). Evaluating, we obtain \( X_A = 0.70 \), and \( X_B = 0.30 \).

   (b) What is the mean and the standard deviation of the minimum variance portfolio?

   Answer: The mean is simply \( X_A R_A + X_B R_B = 13.0\% \). The standard deviation is \( \sigma_p = \sqrt{X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2X_A X_B \sigma_A \sigma_B \rho} = 11.6\% \).

   (c) Trace the efficient frontier of portfolios that invest in assets A and B only.

   Answer: See excel file.

   (d) Assume that the risk free rate is \( R_f = 5\% \). Find the portfolio of risky assets (A and B) with largest Sharpe ratio. (The Sharpe ratio of a portfolio is the ratio of the expected excess return of the portfolio divided by its standard deviation.) Determine the mean return and the standard deviation of this portfolio.

   Answer: Let \( X_A = \alpha \) and \( X_B = 1 - \alpha \), then

   \[
   S(\alpha) = \frac{\alpha R_A + (1 - \alpha) R_B - R_f}{\sqrt{\alpha^2 \sigma_A^2 + (1 - \alpha)^2 \sigma_B^2 + 2\alpha(1 - \alpha) \rho \sigma_A \sigma_B}}.
   \]

   Maximizing over \( \alpha \) results in \( \alpha = 53.1\% \), and \( S(\alpha) = 0.758 \). The portfolio \( X_A = 53.1\% \), \( X_B = 46.9\% \) has mean return 14.69\% and standard deviation 12.79\%.

   (e) Find the efficient frontier with riskless lending and borrowing.

2. Consider two risky securities with variance \( \sigma_1^2 \), \( \sigma_2^2 \) and correlation coefficient \( \rho \). Show that the minimum variance portfolio has variance

   \[
   \sigma_m^2 = \frac{\sigma_1^2 \sigma_2^2 (1 - \rho^2)}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}.
   \]

   Show that \( \sigma_m^2 < \min(\sigma_1^2, \sigma_2^2) \) unless \( \rho = \min(\sigma_1 / \sigma_2, \sigma_2 / \sigma_1) \).

   Answer: See class notes.

3. **Finding the minimum variance portfolio of uncorrelated securities with equal expected returns.** Suppose that \( R_i = \bar{R} \) (a constant) for all \( i \), that \( \sigma_{i,i} = \sigma_i^2 > 0 \) for \( i = 1, \ldots, n \), and that \( \sigma_{i,j} = 0 \) for all \( i \neq j \).

   (a) Plot the set of portfolios in the standard deviation-mean return space.

   All portfolios have the same expected return. The plot is a straight horizontal line starting from the minimum variance portfolio. Clearly only the minimum variance portfolio is efficient, so the efficient frontier consists of only one point.

   (b) Find the weights of the minimum variance portfolio.

   From \( x_e = V^{-1}e / e'V^{-1}e \), we see that the weights are \( x_i = \frac{1 / \sigma_i^2}{\sum_{j=1}^n 1 / \sigma_j^2} \).
(c) What is the variance of the minimum variance portfolio?
The minimum variance portfolio has variance equal to $\sigma^2 = \frac{1}{\sum_{i=1}^{n} 1/A_i}$.

4. General Betting Wheel. Consider a betting wheel with $n$ segments. The payoff for a $1$ bet on segment $i$ is $A_i > 0$. Suppose you bet an amount $B_i = 1/A_i$ on segment $i$ for each $i$.

(a) What is the mean and variance of the return for this betting strategy?
Answer: The payoff of this strategy is $V_1 = 1$ with probability one. Since $V_0 = \sum_{i=1}^{n} 1/A_i$, the return is $R = \frac{V_1 - V_0}{V_0} = \frac{1}{\sum_{i=1}^{n} 1/A_i} - 1$ with probability one, so it has variance equal to zero.

(b) Apply this to a wheel with three segments with payoffs $A_1 = 3$, $A_2 = 4$, $A_3 = 7$, with probabilities $p_1 = 0.5$, $p_2 = 1/3$, and $p_3 = 1/6$.
Answer: In this case $V_0 = 1/3 + 1/2 + 1/7 = 41/42$, so the return is $1/41$ with probability one. Notice that the probabilities are irrelevant for this strategy.

5. Portfolio Tracking. Suppose that it is impractical to form a portfolio that invests in all assets in a given universe of risky assets. One alternative is to find a portfolio, made up of a given subset of $n$ risky assets, that tracks a target portfolio that invest in all risky assets. Specifically, suppose that the target portfolio has random return $R_M$. Suppose that the $n$ assets have random returns $R_1, \ldots, R_n$. We wish to find weights $X_1, \ldots, X_n$ such that the resulting portfolio $R_p = \sum_{i=1}^{n} X_i R_i$, with $\sum_{i=1}^{n} X_i = 1$ minimizes $\text{Var}(R_p - R_M)$.

(a) Find a set of equations for the weights $X_1, \ldots, X_n$.
The problem is to minimize
$$\text{Var}[R_p - R_m] = x'Vx - a'x + \sigma_m^2,$$
subject to $e'x = 1$, where $V$ is the variance-covariance matrix and $a$ is a column vector with components $a_i = 2\text{Cov}(R_i, R_m)$. Notice that we can drop $\sigma_m^2$ from the problem since it is independent of the portfolio $x$ that we select. The Lagrangean is given by
$L(x, \lambda) = x'Vx - a'x - \lambda(e'x - 1)$.
The first order conditions are
$$2Vx - \lambda e = a$$
and
$$e'x = 1.$$
This is a system of $n + 1$ equations in $n + 1$ unknowns.

(b) Although the resulting portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. A logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the required mean return is varied, this results in a family of portfolios that are efficient in a new sense, say tracking efficient. Find the equation of the $X_i$'s that are tracking efficient.
If we add the constraint $x'R = \bar{R}$ then the Lagrangean is modified to read
$L(x, \lambda, \mu) = x'Vx - a'x - \lambda(e'x - 1) - \mu(x'R - \bar{R})$,
and the system of equations is now
$$2Vx - \lambda e - \mu x = a$$
e'x = 1,$$
and
$$x'R = \bar{R}.$$ This is a system of $n + 2$ equations in the same number of unknowns.