Consider $n$ risky securities with expected returns $\mathbf{R}_1, \ldots, \mathbf{R}_n$. Let $\mathbf{r}$ be the column vector of expected returns. Let $\mathbf{R}_f$ be the risk-free rate, let $\mathbf{e}$ be an $n \times 1$ vector of ones, and $q = \mathbf{r} - \mathbf{R}_f e$ be the vector of expected excess returns. Let $V$ denote the variance-covariance matrix. For any portfolio $x_p$ let $q_p = q^T x_p$ denote the expected excess return of portfolio $x_p$, let $\sigma^2_p = x_p^T V x_p$ denote the variance of the return of portfolio $x_p$, and let $S_p = q_p / \sigma_p$ denote the Sharpe ratio of portfolio $x_p$. Finally, for any two portfolios $x_p$ and $x_s$, let $\sigma_{ps} = x_p^T V x_s$ denote the covariance between the returns of these portfolios.

1. Let

$$x_e = \frac{V^{-1} e}{e^T V^{-1} e}$$

and

$$x_q = \frac{V^{-1} q}{e^T V^{-1} q}.$$

You should recognize, from your class notes, that $x_e$ is the minimum variance portfolio, and $x_q$ is the portfolio with maximum Sharpe ratio. Assume, in what follows that $q \neq e$.

(a) Find $q_e$ and $q_q$. Assume in what follows that $q_e > 0$.

(b) The Two-Fund Theorem states that we can express any efficient portfolio of risky securities as a convex combination of two efficient portfolios. Consider the portfolio

$$x_p = \frac{q_q - q_p}{q_q - q_e} x_e + \frac{q_p - q_e}{q_q - q_e} x_q.$$

Verify that the expected excess return of $x_p$ is $q_p$, and notice that portfolio $x_p$ is indeed a convex combination of portfolios $x_e$ and $x_q$.

(c) Find $\sigma_{eq}$.

(d) Find $\sigma^2_q = \sigma_{e e}$ and $\sigma^2_q = \sigma_{q q}$.

(e) Use parts (c) and (d) to find $\sigma^2_p$ for portfolio $x_p$ of part (b).

(f) Let $\kappa = \frac{\sigma^2_q - \sigma^2_e}{(q_q - q_e)^2}$. Show that

$$\sigma^2_p = \sigma^2_e + \kappa (q_p - q_e)^2.$$

Notice that this formula gives us the variance of a portfolio with expected excess return $q_p$.

2. For any portfolio $x_p$, let $\beta_p = \frac{\sigma_{e p}}{\sigma_{e e}}$. This is the beta of portfolio $x_p$ relative to portfolio $x_q$.

(a) Show that $\beta_p = \frac{q_p}{q_q}$, and conclude that

$$q_p = \beta_p q_q. \tag{1}$$

Interpret equation (1). What happens if $\beta_p > 1$? What if $\beta_p < 1$?

(b) Show that

$$\beta_e = \frac{\sigma_e}{\sigma_q}.$$

(c) Show that

$$\frac{S_e}{S_q} = \frac{\sigma_e}{\sigma_q} \leq 1.$$
(d) Show that for any portfolio $x_p$

$$\frac{S_p}{S_q} = \rho_p$$

where

$$\rho_p = \frac{\sigma_{pq}}{\sigma_p \sigma_q}$$

is the correlation coefficient between the returns of portfolios $x_p$ and $x_q$. Use this to compute $\rho_c$.

(e) Suppose that $S_q = 0.5$, and that for some portfolio $x_p$, you know that $\overline{R}_p = 15\%$, and $\sigma_p = 20\%$. Furthermore, you know that $\overline{R}_f = 5\%$. Compute $S_p$. What is the correlation of portfolios $x_p$ and $x_q$?

3. Consider the utility function

$$u(x) = r'x - \frac{1}{\tau}x'Vx$$

where $\tau \geq 0$ is a risk tolerance measure, i.e., the larger $\tau$ the more risk you are willing to tolerate. The objective is to maximize $u(x)$ subject to $e'x = 1$ and $\underline{x} \leq x \leq \overline{x}$ where $\underline{x}$ and $\overline{x}$ are lower and upper bounds on portfolio holdings.

(a) Show that the marginal utility is given by $r - \frac{2}{\tau}Vx$.

(b) Assume that the initial portfolio satisfies $e'x = 1$ and security $i$ (resp., $j$) has the largest (resp., smallest) marginal utility among securities whose current allocation is not at their upper (resp., lower) bound. Let $s = e_i - e_j$. Find the value of $\alpha$, say $\alpha^*$, that maximizes $u(x + \alpha s)$.

(c) What might prevent you from taking a step of size $\alpha^*$ in the direction of $s$ in part (b)?

(d) What is the optimal step size taking into account your answer to parts (b) and (c)?

(e) We say that asset $i$ is “down” if $x_i = \underline{x}_i$, is “in” if $\underline{x}_i < x_i < \overline{x}_i$ and is “up” if $x_i = \overline{x}_i$. What can you say about the marginal utility of “down”, “in”, and “up” assets at the optimal solution?

(f) What is the optimal portfolio if there are no lower and upper bound on portfolio holdings? Hint: All assets must be “in”.

(g) Show that you can write the solution to part (f) as $x = x_e + \tau z$ where $x_e$ is the minimum variance portfolio and $z$ is a swap vector. Find $z$ and verify that $e'z = 0$.

(h) Find the expected return of portfolio $x_e + \tau z$.

(i) Assume that $x_a$ and $x_b$ are optimal portfolios for $\tau = a$ and $\tau = b$ with $a \neq b$. Assume that $x_a$ and $x_b$ are the portfolios offered by two mutual funds. How might an investor with risk tolerance $\tau$ us such funds optimally?

(j) How might an investor use these funds optimally to obtain expected return $\overline{R}$?