Consider \( n \) risky securities with expected returns \( \overline{R}_1, \ldots, \overline{R}_n \). Let \( \overline{r} \) be the column vector of expected returns. Let \( \overline{R}_f \) be the risk-free rate, let \( e \) be an \( n \times 1 \) vector of ones, and \( q = \overline{r} - \overline{R}_f e \) be the vector of expected excess returns. Let \( V \) denote the variance-covariance matrix. For any portfolio \( x_p \) let \( q_p = q^t x_p \) denote the expected excess return of portfolio \( x_p \), let \( \sigma_p^2 = x_p^t V x_p \) denote the variance of the return of portfolio \( x_p \), and let \( S_p = q_p / \sigma_p \) denote the Sharpe ratio of portfolio \( x_p \). Finally, for any two portfolios \( x_p \) and \( x_s \), let \( \sigma_{ps} = x_p^t V x_s \) denote the covariance between the returns of these portfolios.

1. Let

\[
x_e = \frac{V^{-1}e}{e^t V^{-1} e}
\]

and

\[
x_q = \frac{V^{-1}q}{e^t V^{-1} q}
\]

You should recognize, from your class notes, that \( x_e \) is the minimum variance portfolio, and \( x_q \) is the portfolio with maximum Sharpe ratio. Assume, in what follows that \( q \neq e \).

(a) Find \( q_e \) and \( q_q \). Assume in what follows that \( q_e > 0 \).

Answer: \( q_e = q^t V^{-1} e / e^t V^{-1} e \), and \( q_q = q^t V^{-1} q / e^t V^{-1} q \).

(b) The Two-Fund Theorem states that we can express any efficient portfolio of risky securities as a convex combination of two efficient portfolios. Consider the portfolio

\[
x_p = \frac{q_p - q_e}{q_q - q_e} x_e + \frac{q_p - q_e}{q_q - q_e} x_q.
\]

Verify that the expected excess return of \( x_p \) is \( q_p \).

Answer: \( q_p = \frac{q_p - q_e}{q_q - q_e} q_e + \frac{q_p - q_e}{q_q - q_e} q_q \).

(c) Find \( \sigma_{eq} \).

Answer:

\[
\sigma_{eq} = x_e^t V x_q = \frac{1}{e^t V^{-1} e}.
\]

(d) Find \( \sigma_e^2 = \sigma_{ee} \) and \( \sigma_q^2 = \sigma_{qq} \).

Answer:

\[
\sigma_e^2 = \frac{1}{e^t V^{-1} e},
\]

\[
\sigma_q^2 = \frac{q^t V^{-1} q}{(e^t V^{-1} e)^2}.
\]

(e) Use parts (c) and (d) to find \( \sigma_p^2 \) for portfolio \( x_p \) of part (b).

Answer:

\[
\sigma_p^2 = \left[ \frac{q_p - q_e}{q_q - q_e} \right]^2 \sigma_e^2 + 2 \left[ \frac{q_p - q_e}{q_q - q_e} \right] \left[ \frac{q_p - q_e}{q_q - q_e} \right] \sigma_{eq} + \left[ \frac{q_p - q_e}{q_q - q_e} \right]^2 \sigma_q^2.
\]

(f) Let \( \kappa = \frac{\sigma_q^2 - \sigma_e^2}{(q_q - q_e)^2} \). Show that

\[
\sigma_p^2 = \sigma_e^2 + \kappa (q_p - q_e)^2.
\]

Answer: Add and subtract \( \left[ \frac{q_p - q_e}{q_q - q_e} \right]^2 \sigma_e^2 \) and use the fact that \( \sigma_e^2 = \sigma_{eq} \).

2. For any portfolio \( x_p \), let \( \beta_p = \frac{\sigma_p}{\sigma_{eq}} \). This is the beta of portfolio \( x_p \) relative to portfolio \( x_q \).
(a) Show that \( \beta_p = \frac{q_p}{q_q} \), and conclude that
\[
q_p = \beta_p q_q.  
\]  
Answer: From the definition we have
\[
\beta_p = \frac{\sigma_{q_p}}{\sigma_{q_q}} = \frac{q'_q V x_p}{x'_q V x_q}. 
\]
Now notice that
\[
x'_q V = \frac{q' V^{-1} e}{e' V^{-1} e}, 
\]
which implies that
\[
\beta_p = \frac{q'_q x_p}{q' x_q} = \frac{q_p}{q_q}. 
\]
Interpretation: The equation can be written as
\[
\bar{R}_p = \bar{R}_f + \beta_p (\bar{R}_q - \bar{R}_f). 
\]
The expected return of portfolio \( p \) is linearly related to the expected excess return of portfolio \( q \). And this relationship is with respect to the nondiversifiable component of the risk of portfolio or security \( x_p \).

(b) Show that
\[
\beta_e = \frac{\sigma_e^2}{\sigma_q^2} 
\]
Answer: Recall that \( \sigma_e^2 = [e' V^{-1} e]^{-1} \) and notice that \( \sigma_q^2 \) can be written as \( q_q/(q' V^{-1} e) \).
From this we see that
\[
\frac{\sigma_e^2}{\sigma_q^2} = \frac{q' V^{-1} e}{e' V^{-1} e} = \frac{q_e}{q_q}. 
\]

(c) Show that
\[
\frac{S_e}{S_q} = \frac{\sigma_e}{\sigma_q} \leq 1. 
\]
Answer:
\[
\frac{S_e}{S_q} = \frac{q_e}{q_q} \frac{\sigma_e}{\sigma_q} = \frac{\sigma_e}{\sigma_q} \leq 1. 
\]
The inequality follows from the fact that \( \sigma_e \) is the smallest possible variance of any portfolio.

(d) Show that for any portfolio \( x_p \)
\[
\frac{S_p}{S_q} = \rho_p 
\]
where
\[
\rho_p = \frac{\sigma_{pq}}{\sigma_p \sigma_q} 
\]
is the correlation coefficient between the returns of portfolios \( x_p \) and \( x_q \). Use this to compute \( \rho_e \).
Answer:
\[
\frac{S_p}{S_q} = \frac{q_p}{q_q} \frac{\sigma_p}{\sigma_q} = \beta_q \frac{\sigma_q}{\sigma_e} = \frac{\sigma_{pq}}{\sigma_q \sigma_p}. 
\]
We conclude that \( \rho_e = \sigma_e/\sigma_q \) if the expected excess return of the minimum variance portfolio is positive, and \( \rho_e = -\sigma_e/\sigma_q \) if the expected excess return of the minimum variance portfolio is negative.
(e) Suppose that $S_q = 0.5$, and that for some portfolio $x_p$, you know that $\overline{F}_p = 15\%$, and 
$\sigma_p = 20\%$. Furthermore, you know that $\overline{F}_f = 5\%$. Compute $S_p$. What is the correlation
of portfolios $x_p$ and $x_q$?
Answer: $S_q = \frac{15 - 0.05}{2} = 4$, and $\rho_p = S_q/S_p = 4/1.5 = 8.33$.

3. Consider the utility function

$$u(x) = r'x - \frac{1}{\tau}x'Vx$$

where $\tau \geq 0$ is a risk tolerance measure, i.e., the larger $\tau$ the more risk you are willing to tolerate. The objective is to maximize $u(x)$ subject to $e'x = 1$ and $\underline{x} \leq x \leq \overline{x}$ where $\underline{x}$ and $\overline{x}$ are lower and upper bounds on portfolio holdings.

(a) Show that the marginal utility is given by $r - \frac{2}{\tau}Vx$.
Answer: We have seen before that the gradient of $r'x$ is $r$ and the gradient of $x'Vx$ is $2Vx$ from which the result follows.

(b) Assume that the initial portfolio satisfies $e'x = 1$ and security $i$ (resp., $j$) has the largest (resp., smallest) marginal utility among securities whose current allocation is not at their upper (resp., lower) bound. Let $s = e_i - e_j$. Find the value of $\alpha$, say $\alpha^*$, that maximizes $u(x + \alpha s)$.
Answer: 

$$u(x + \alpha s) = r'x + \alpha r's - \frac{1}{\tau}[x'Vx + 2\alpha s'Vx + \alpha^2 s'Vs].$$

This is a quadratic in $\alpha$ and the solution is

$$\alpha^* = \frac{\tau r's - 2s'Vs}{2s'Vs}.$$

(c) What might prevent you from taking a step of size $\alpha^*$ in the direction of $s$ in part (b)?
Answer: The upper bound on $x_i$ and the lower bound on $x_j$.

(d) What is the optimal step size taking into account your answer to parts (b) and (c)?
Answer: The optimal step size is $\min\{\alpha^*, x_i - x_j, x_j - x_q\}$.

(e) What might prevent you from taking a step of size $\alpha^*$ in the direction of $s$ in part (b)?
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Answer: All “in” assets must have the same marginal utility, say $\lambda$ since otherwise we could improve on the optimal solution by increasing the position of an “in” asset and decreasing the position of another “in” asset with smaller marginal utility. Similarly, all “down” (resp., “up”) assets must have marginal utilities at most (resp., at least) $\lambda$.

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for $\lambda$ results in

$$\lambda = \frac{\tau e'V^{-1}r - 2}{\tau e'V^{-1}e}.$$  

Substituting $\lambda$ into

$$x = \frac{\tau}{2}V^{-1}(r - \lambda e)$$

and simplifying we obtain

$$x = x_e + \tau z$$

where

$$x_e = \frac{V^{-1}e}{e'V^{-1}e}$$

and

$$z = \frac{1}{2}[V^{-1}r - \frac{e'V^{-1}r}{e'V^{-1}e}V^{-1}e].$$

(g) Show that you can write the solution to part (f) as $x = x_e + \tau z$ where $x_e$ is the minimum variance portfolio and $z$ is a swap vector. Find $z$ and verify that $e'z = 0$.

Answer: The form of the solution has already been established in part (f). Notice that

$$e'z = \frac{1}{2}[e'V^{-1}r - \frac{e'V^{-1}r}{e'V^{-1}e}e'V^{-1}e] = 0.$$  

(h) Find the expected return of portfolio $x_e + \tau z$.

Answer:

$$r'[x_e + \tau z] = r'x_e + \tau e'z.$$  

(i) Assume that $x_a$ and $x_b$ are optimal portfolios for $\tau = a$ and $\tau = b$ with $a \neq b$. Assume that $x_a$ and $x_b$ are the portfolios offered by two mutual funds. How might an investor with risk tolerance $\tau$ us such funds optimally?

Answer: If he invests $\alpha$ on $x_a$ and $(1 - \alpha)$ on $x_b$ he obtains

$$x_e + [\alpha a + (1 - \alpha)b]z$$

which is optimal for risk tolerance $\alpha a + (1 - \alpha)b$. To obtain risk tolerance $\tau$ simply solve $\alpha a + (1 - \alpha)b = \tau$. This results in

$$\alpha = \frac{b - \tau}{b - a}.$$  

(j) How might an investor use these funds optimally to obtain expected return $\overline{R}$?

Find $\tau$ such that $r'x_e + \tau e'z = \overline{R}$, and then use part (i).