1. **Capital Market Line** Assume the expected rate of return on the market portfolio is 12% and the rate of return on T-bills is 5%. The standard deviation of the market is 14%. Assume the market portfolio is efficient.

(a) What is the equation of the capital market line.
Answer: The Sharpe Ratio is 0.5, so the Capital Market Line is given by

\[ R = 5\% + .5\sigma. \]

(b) If an expected return of 15% is desired, what is the standard deviation of this position?
If you had $1,000 to invest, how should you allocate it to achieve the above position if you can borrow and lend at 5%?
Answer: To get a return of 15% we need \( \sigma = 20\% \). On the other hand, if we invest \( \alpha\% \) in the market our return is given by

\[ 5\% + 7\%\alpha = 15\%, \]

which result in \( \alpha = 10/7 \). Hence we would invest $1,428.57 in market and borrow $428.57 at the risk free rate.

(c) If you invest $300 in the risk-free asset and $700 in the market portfolio, how much money would you expect to have at the end of the year.
Here \( \alpha = .7 \), so your expected return is 9.9%, so at the end of the year you expect to have 1099.

2. **Two Asset Economy** Consider an economy with only two risky stocks, A and B. There are 100 shares outstanding of Stock A, and 150 shares outstanding of Stock B. The price per share of stock A is $1.50, while that of stock B is $2.00.

(a) Find the market capitalization of each stock, and form the market portfolio using market capitalization weights.
Answer: The market capitalizations are respectively 150 and 300 for securities A and B. The market portfolio is therefore \( x_m = (1/3, 2/3) \).

(b) The expected return of stock A is 15%, and that of stock B is 12%. Find the expected return of the market portfolio.
Answer: The expected return of the market portfolio is \( (1/3)15\% + (2/3)12\% = 5\% + 8\% = 13\% \).

(c) The standard deviation of returns are 15% and 9% respectively for stock A and stock B. Assume that the correlation coefficient between the returns of stocks A and B is \( \rho = 1/3 \). Find the standard deviation of the market portfolio.
Answer: The market portfolio has variance equal to

\[ \sigma_m^2 = (1/3)^2 15^2 + 2(1/3)(2/3)(1/3)(15)(9) + (2/3)^2 9^2 = 81, \]

so \( \sigma_m = 9 \).

(d) Find the Beta of stock A.
Answer: By definition \( \beta_A = \sigma_{Am}/\sigma_m^2 \). Now \( \sigma_{Am} = (1/3)\sigma_A^2 + (2/3)\sigma_{AB} = 75 + 30 = 105 \), so \( \beta_A = 105/81 = 1.297 \).

(e) Find the risk-free rate assuming that the CAPM holds exactly.
We have

\[ R_f = (\overline{R}_A - \beta_A \overline{R}_m)/(1 - \beta_A) = 6.25\%. \]
(f) Find the Beta of stock B. Compute the expected return of stock B using the CAPM.
Answer: We could use the same technique as we did to find \( \beta_A \). However, we know that
\[
5.75\% = \overline{R}_B - R_f = \beta_B(\overline{R}_m - R_f) = \beta_B 6.75\% ,
\]
so \( \beta_B = 5.75/6.75 = 0.852 \).

3. Residual Risk under the CAPM
Under the CAPM excess returns are given by
\[
R_t - R_f = \beta(R_m - R_f) + \epsilon_t ,
\]
where \( \beta_i = \sigma_{m}/\sigma_m^2 \), and the random variable \( \epsilon_t \) is uncorrelated with the market portfolio.

(a) Show that you can write the variance-covariance matrix as
\[
V = \sigma_m^2 \beta \beta' + T
\]
where \( T \) is the variance-covariance matrix of residual risk, i.e., \( T_{ij} = \text{Cov}(\epsilon_i, \epsilon_j) \).
Answer: Under the CAPM we have
\[
\text{Cov}(R_t, R_s) = \text{Cov}(\beta R_m + \epsilon_t, \beta_j R_m + \epsilon_j) = \beta_i \beta_j \sigma_m^2 + \text{Cov}(\epsilon_i, \epsilon_j) ,
\]
so \( V \) is of the desired form.

(b) Show that \( Tx_m = 0 \).
Answer: We can write \( T = V - \sigma_m^2 \beta \beta' \). Then
\[
T x_m = V x_m - \sigma_m^2 \beta.
\]
This last quantity is zero because \( \beta = V x_m / \sigma_m^2 \).

(c) Show that for any portfolio \( x_p \) the residual variance can be written as \( \omega_p^2 = x_p' T x_p = \sigma_p^2 - \sigma_m^2 \beta_p^2 \), where \( \sigma_p^2 = x_p' V x_p \) and \( \beta_p = \beta x_p \).
Answer:
\[
x_p T x_p = x_p' (V - \sigma_m^2 \beta \beta') x_p = \sigma_p^2 - \sigma_m^2 \beta_p^2 .
\]

(d) Argue that the market portfolio is the minimum variance portfolio with beta equal to one.
Answer: Since \( x_p' T x_p \geq 0 \) it follows that
\[
\sigma_p^2 \geq \sigma_m^2 \beta_p^2 .
\]
If \( x_p \) has \( \beta_p = 1 \) then
\[
\sigma_p^2 \geq \sigma_m^2 ,
\]
so the benchmark is the portfolio with smallest variance and beta equal to one.

4. Multi-Period Investments
Consider a security with annual total return \( TR = 0.5 \) with probability 0.5 and \( TR = 1.8 \) with probability 0.5.

(a) Compute the mean \( \mu \) and the standard deviation \( \sigma \) of the return \( R = TR - 1 \).
Answer: \( \mu = 0.5(0.8 - .5) = .15 = 15\% , \sigma = .65 = 65\% \)
(b) Let $TR_n$ be the total return in year $n = 1, 2, \ldots, N$. Assume the returns over different years are independent and have the same distribution. The return over $N$ periods is given by $TR_1TR_2 \ldots TR_N - 1$. Find a formula for the mean and standard deviation of the return over $N$ years in terms of $\mu$ and $\sigma$.

Answer:

$$E[TR_1 \ldots TR_N - 1] = E[TR_1] \ldots E[TR_N] - 1 = (1 + \mu)^N - 1.$$ 

$$\text{Var}[TR_1 \ldots TR_N - 1] = \text{Var}[TR_1 \ldots TR_N] = E[TR_1^2 \ldots TR_N^2] - E[TR_1 \ldots TR_N]^2.$$ 

Since $E[TR_n^2] = \sigma^2 + (1 + \mu)^2$ we have

$$\text{Var}[TR_1 \ldots TR_N - 1] = (\sigma^2 + (1 + \mu)^2)^N - (1 + \mu)^{2N}.$$ 

(c) Use the formula obtained in part (b) to compute the ratio of the mean return to the standard deviation of return for $N = 1, 2, \ldots, 30$ years. Is the ratio increasing or decreasing over time?

Answer: Let $\theta = 1 + \mu$ then the ratio is given by

$$\frac{\theta^N - 1}{\sqrt{\sigma^2 + \theta^2}|N - \theta^N|}.$$ 

The ratio first increases from 0.231 to 0.301 and then decreases monotonically to 0.015.

(d) Let $V = \ln(TR)$. Compute the mean $\nu$ and the standard deviation $\tau$ of $V$.

Answer: The random variable $V$ takes values $\ln(5) = -0.693$ with probability 0.5 and value $\ln(1.5) = 0.588$ with probability 0.5, resulting in mean $\nu = -0.0527$ and standard deviation $\tau = 0.6405$.

(e) Write $TR_1TR_2 \ldots TR_N = \exp(V_1 + V_2 + \ldots V_N)$. By the central limit theorem, for large $N$,

$$V_1 + V_2 + \ldots V_N$$

is approximately normal with mean $N\nu$ and variance $N\tau^2$. Use this approximation to compute the probability that the return over $N$ periods is negative, i.e.,

$$\text{Prob}(TR_1TR_2 \ldots TR_N \leq 1).$$

In particular, compute this probability for $N = 30$.

Answer:

$$\text{Prob}((TR_1TR_2 \ldots TR_N \leq 1) = \text{Prob}(V_1 + \ldots + V_N \leq 0) \approx \text{Prob}(Z \leq -\sqrt{N\nu}/\tau).$$

For $N = 30$ we have $\sqrt{30\nu}/\tau = -0.4502$, and $\text{Prob}(Z \leq 0.4502) = 0.674$.

(f) What happens to

$$\text{Prob}(TR_1TR_2 \ldots TR_N \leq 1)$$

as $N \to \infty$?

Answer: Since $\nu < 0$, $-\sqrt{N\nu}/\tau$ increases to infinity as $N$ increases to infinity, so the probability goes to one.

(g) Would you invest in this security? Why or why not?

Answer: I would not invest in this security by itself because in the long run I would lose money with probability one. (Think what happens to your investment if after a long period of time you have an equal number of good and bad years). Notice that mean-variance analysis would recommend this security (if it was the only risky security) for single and multiple periods as long as the risk-free rate is below 15%. Only in the limit does mean-variance analysis recognize that this is a bad investment. (The Sharpe ratio goes to zero and the risk-free rate becomes the only efficient portfolio).
5. **Portfolio Rebalancing** Assume there are two independent securities with the same return characteristics as in problem 2. Consider the two investment strategies:

- **Buy-and-hold** Invest half your money on each of the two securities and hold this portfolio indefinitely.
- **Yearly Rebalance** Invest half your money on each of the two securities. At the end of each year rebalance your portfolio so that half the money is invested in each of the securities.

(a) Find the mean and variance of the return under the buy-and-hold strategy at the end of 30 years.
Answer: Let $TR_n$ be the total year $n$ return of security one and let $T\bar{R}_n$ be the total year $n$ return of security two. The buy and hold strategy has $N$ period return equal to

$$R_N = 0.5TR_1\ldots TR_N + 0.5T\bar{R}_1\ldots T\bar{R}_N - 1$$

with mean

$$E[R_N] = 0.5\theta^N + 0.5\theta^N - 1 = \theta^N - 1$$

and variance

$$\text{Var}[R_N] = 0.25\text{Var}[TR_1\ldots TR_N] + 0.25\text{Var}[T\bar{R}_1\ldots T\bar{R}_N] = 0.50[(\sigma^2 + \theta^2)^N - \theta^N].$$

Thus $E[R_{30}] = 65.21$ and $\text{Var}[R_{30}] = 8,968,644.54$.

(b) Find the mean and variance of the return under the yearly rebalance strategy at the end of 30 years.
Answer: The rebalance strategy has period $N$ total return $T\bar{R}_n = \frac{TR_n + T\bar{R}_n}{2}$ with mean 0.15 and standard deviation $\hat{\sigma} = \sigma/\sqrt{2} = 0.4596$. Thus $R_N = T\bar{R}_1\ldots T\bar{R}_N - 1$, so

$$E[R_{30}] = \theta^{30} - 1 = 65.21.$$ 

and

$$\text{Var}[R_{30}] = (0.5\sigma^2 + \theta^2)^{30} - \theta^{30} = 369,414.25.$$ 

(c) Which of these strategies, if any, would you recommend?
Both strategies have the same expected return over 30 years. The rebalancing strategy has much lower variance (about 25 times smaller). In fact, it is easy to see that $V = \ln(T\bar{R})$ has positive expected value, so

$$\text{Prob}(T\bar{R}_1\ldots T\bar{R}_N \leq 1) \to 0$$

as $N \to \infty$. Thus combining two long term sure losers (by rebalancing every year) we now have a long term sure winner. For example, $\text{Prob}(T\bar{R}_1\ldots T\bar{R}_{30} \leq 1) \simeq 0.30$. 