1. **Pricing and Probabilities** A security with current price $P$ has random value $F$ a period later. Assume $F$ takes value $F_j$ with probability $\tilde{q}_j$, $j = 1, \ldots, S$. If $\psi_j > 0$, $j = 1, \ldots, S$ are the state prices and $r = 1 + R_f$ is the total return of a risk-free investment we know that

$$P = \frac{1}{r} \hat{E}[F] = \frac{1}{r} \sum_{j=1}^{S} F_j q_j$$

where $\hat{E}$ represents expectation with respect to the risk-neutral probabilities $q_j = r^{\psi_j}$, $j = 1, \ldots, S$. Let $\pi$ be a random variable taking value $\pi_j = q_j / \tilde{q}_j$ with probability $\tilde{q}_j$, $j = 1, \ldots, S$.

(a) Show that $E[\pi] = 1$. (Here $E$ denotes expectation with respect to the true probability mass function.)

Answer: $E[\pi] = \sum_{j=1}^{S} \pi_j \tilde{q}_j = \sum_{j=1}^{S} q_j = 1$.

(b) Show that

$$P = \frac{1}{r} E[F \pi].$$

Answer: It is enough to show that $E[F \pi] = \hat{E}[F]$. We have

$$E[F \pi] = \sum_{j=1}^{S} \tilde{q}_j F_j \pi_j = \sum_{j=1}^{S} q_j F_j = \hat{E}[F].$$

(c) Show that $\text{cov}(F, \pi) = \hat{E}[F] - E[F]$. 

Answer: $\text{cov}(F, \pi) = E[F \pi] - E[F]E[\pi] = \hat{E}[F] - E[F]$ by parts (a) and (b).

2. **Alternative Derivation of the Price of a Call Option** Suppose that there are two states (up and down) and two securities (a bond and a stock). The bond sells for $1 and the stock for $S > 0$. The bond pays $r = 1 + R_f > 1$ in both states, while the stock pays $uS$ in the upstate and $dS$ in the down state. A call option with strike price $K$ pays $C_u = (uS - K)_+$ in the up state and $C_d = (dS - K)_+$ in the down state. The object is to find the current price $C_0$ of the call option.

(a) Find the payoff (in the up and down states) of a portfolio consisting of $\alpha > 0$ shares and $-1$ calls.

Answer: Payoff up state is $\alpha uS - C_u$, down state $\alpha dS - C_d$.

(b) Find the payoff of the portfolio (in the up and down states) if $\alpha = \frac{C_u - C_d}{S(u - d)}$.

Answer: At this level of $\alpha$ both payoffs are equal to

$$\frac{d}{u - d} C_u - \frac{u}{u - d} C_d.$$

(c) What is the cost of the resulting portfolio?

Answer: The cost is $\frac{C_u - C_d}{u - d} - C_0$.

(d) Is the resulting portfolio risk-free?

Answer: Yes.

(e) Argue that the portfolio must earn the risk-free rate in the absence of arbitrage, and use this to determine the cost $C_0$ of the call option.

Answer: If the portfolio does not return the risk-free rate you can engage in arbitrage by shorting it if its return is below the risk-free rate and by holding long otherwise.

We must have

$$\frac{C_u - C_d}{u - d} - C_0 = \frac{1}{r} \left( \frac{d}{u - d} C_u - \frac{u}{u - d} C_d \right).$$
Solving for $C_0$ yields

$$C_0 = \frac{1}{r} \left( \frac{r - d}{u - d} C_u + \frac{u - r}{u - d} C_d \right).$$

3. Suppose there are three states of the world: Good (G), Fair (F), and Poor (P). Security I pays $45 in state G, $30 in state F, and $30 in stage P. Security II pays $25 regardless of the state of the world. Assume that security I trades for $36 and security II trades for $22.5.

(a) Notice that both securities have the same payment under states F and P. Collapse the states F and P into state FP. Use this to find the value of an elementary security that pays $1 if the state is G and $0 if the state is FP, and of an elementary security that pays $1 if the state is FP and $0 if the state is G.

Answer: The payoff matrix is given by

$$D = \begin{pmatrix} 45 & 30 \\ 25 & 25 \end{pmatrix}$$

and the security prices are

$$p = \begin{pmatrix} 36.0 \\ 22.5 \end{pmatrix}.$$

The state prices satisfy $D\psi = p$ and are given by

$$\psi = \begin{pmatrix} 0.6 \\ 0.3 \end{pmatrix}.$$

(b) Suppose you work for an investment firm and a client wants a product that will pay $40 in state G, $30 in state F and $20 in state P. It is clearly impossible to exactly replicate this payoff vector with securities I and II. It is quite possible, however, to create a portfolio of securities I and II with a payoff at least as large as the one required by the client. Find the minimum cost portfolio with payoff at least as large as that required by the client. What will the payoff of this portfolio be? How much would it cost?

Answer: The problem is to minimize $p'x$ subject to $D'x \geq c$ where

$$c = \begin{pmatrix} 40 \\ 30 \end{pmatrix}.$$

is the required payoff of the product in states G and F. The optimal portfolio is

$$x = \begin{pmatrix} 0.667 \\ 0.4 \end{pmatrix}.$$

and has payoff 40 in state G and 30 in both states F and P. The cost of this portfolio is $33.

(c) Suppose the cost of the portfolio is $33 and it pays $40 in state G, and $30 in states F and P. By selling this portfolio at $33 the investment firm makes $10 if the state is P. The firm fears, however, that the client will not be willing to pay $33 and may go to another investment firm. Suppose that your firm agrees to “make a market” and buy or sell security III with payoff $40 in G, $30 in F, and $20 in P at $31. Use this third security to complete the market and find the prices of the three elementary securities (these securities pay $1 in one and only one state). Are your results consistent with part (a).
Answer: Because the firm is willing to reduce the price by $2 and the instrument pays an additional $10 in state P, the state price $\phi_P = 0.20$. Indeed, it is possible to verify that the new state prices are given by

$$
\psi = \begin{pmatrix}
0.6 \\
0.1 \\
0.2
\end{pmatrix}.
$$

This is consistent with part (a) because the state price $\psi_G$ did not change and because the sum $\psi_P + \psi_G$ did not change either.

4. Consider a two period model involving a bond and a stock. In each time period the bond increases by 5%. In each time period the stock goes up (u) 30% or goes down (d) 10%. Consider the following securities involving the investment of $1 today or at the end of period one.

- b: buy bond today; sell it at the end of period 1
- s: buy stock today; sell it at the end of period 1
- ub: At period 1, if the stock went up, buy a bond; sell it at the end of period 2
- us: At period 1, if the stock went up, buy a stock; sell it at the end of period 2
- db: At period 1, if the stock went down, buy a bond; sell it at the end of period 2
- ds: At period 1, if the stock went down, buy a stock; sell it at the end of period 2

Answer: See the excel file named Prob123-4 for the solution to this problem.

(a) Write the payoff matrix under all possible future states (u, d, uu, ud, du, dd) for each of the six securities. Examples: Security b has cost $1 and payoff 1.05 in states u and d, and payoff equal to zero otherwise. Security ub has cost $0 and payoff -$1 in state u (you buy the bond) and payoff $1.05 in states uu and ud (you sell the bond).

(b) Use the current prices of the six securities to find the prices of the six corresponding elementary securities. (The elementary securities pay $1 in one and only one state.)

(c) An investment firm offers the following investment product. At time 2, the investment firm will pay the holder an amount equal to $1.50 if the stock is worth more than $1.50, and $1 if the stock is worth less than $1.00. Use the prices of the elementary securities to compute the value of this “collar” around the price of the stock.

(d) Find the replicating “portfolio” that has a payoff equal to the new investment product. Interpret this portfolio as a dynamic strategy of buying and selling bonds and stocks.

(e) Suppose that our model for the stock price is in error in one respect. If the stocks do poorly the first period, then the stock goes up (u) 20% or goes down (d) 20% in the second period. What is the payoff at the end of period 2 of the portfolio found in part (d).

(f) Solve parts (c) and (d) under the new payoff matrix.

(g) Suppose that the payoff matrix obtained in part (a) is completely accurate, but that the market moves too fast to make any trades at the end of the first period. Instead the positions established at the outset must be held until the end of the second period. What are the payoffs under the states \{uu, ud, du, dd\} of the initial positions of the portfolio found in part (d)?