1 Arbitrage Pricing Theory (APT)

The arbitrage pricing theory (APT) is about the structure imposed by the market on a multi-index return generation process. Consider a multi-index return generation process

$$R_i = a_i + \sum_{k=1}^{K} b_{ik} I_k + \epsilon_i$$  \hspace{1cm} (1)$$

where $a_i$ and $b_{ik}$ $k = 1, \ldots, K$ are constants, $I_k$ $k = 1, \ldots, K$ are the random factor values, and $\epsilon_i$ is a mean zero random variable denoting the idiosyncratic risk of security $i$. Here $b_{ik}$ is the exposure of security $i$ to factor $k$ (also known as the sensitivity of security $i$ to factor $k$). We can think of $a_i = b_{i0}$ as the exposure of security $i$ to factor zero.

Taking expectations in equation (1) we obtain

$$r_i = E[R_i] = a_i + \sum_{k=1}^{K} b_{ik} E[I_k].$$  \hspace{1cm} (2)$$

The APT tells us that $a_i$'s cannot be independent of the $b_{ik}$'s and concludes that

$$r_i = E[R_i] = \lambda_0 + \sum_{k=1}^{K} b_{ik} \lambda_k.$$  

As we shall see this is a powerful result. However the APT is silent about which factors to include in the model and about how many factors should be included.

1.1 Portfolio Factor Exposures

Suppose there are $n$ stocks. Let $a = (a_1, \ldots, a_n)'$ denote the column vector of exposures to factor zero, and let $b_k = (b_{1k}, \ldots, b_{nk})'$ be the column vector of exposures to factor $k = 1, \ldots, K$.

For any portfolio $x_p$ we can write the return as

$$R_p = a_p + \sum_{k=1}^{K} b_{kp} I_k + \epsilon_p$$

where $a_p = a' x_p$, $b_{kp} = b_{k}' x_p$ and $\epsilon_p = \epsilon' x_p$. Here $a_p$ is the portfolio exposure to factor zero, while $b_{kp}$ is the portfolio exposure to factor $k = 1, \ldots, K$. As we know, for well diversified portfolios $\text{Var}[\epsilon_p] = \sum_{i=1}^{n} x_i^2 \epsilon_i^2 \simeq 0$.

1.2 Arbitrage

We now seek the possibility of making risk-free profits without investing any money. Consider a swap vector $x$ such that $\epsilon' x = 0$ and $b_{k}' x = 0$ for $k = 1, \ldots, K$. This swap vector will have cost zero and zero exposure to factors $k = 1, \ldots, K$. The return is given by

$$R = a' x + \epsilon' x.$$  

If $x$ is well diversified then $\text{Var}[x'] \simeq 0$ is negligible. If we ignore $x' \epsilon$ our portfolio will have a risk-free return $a' x$. To find the best such portfolio we use the following linear programming formulation

$$\text{max } a' x$$

subject to

$$\epsilon' x = 0$$
\[ b'_1 x = 0 \]
\[ \vdots \]
\[ b'_K x = 0 \]

If there is a solution to this linear program with \( d'x > 0 \) we have the recipe to make money with zero investment at essentially no risk. The idea of the APT is that investors will exploit these opportunities and that their actions will force prices to adjust to close these opportunities. The APT states that in equilibrium \( d'x = 0 \).

Corresponding to every linear program there is a dual program. Let \( \gamma_0 \) be the dual variable to the constraint \( e'x = 0 \) and for \( k = 1, \ldots, K \), let \( \gamma_k \) be the dual variable to the constraint \( b'_k x = 0 \). The dual program is

\[
\begin{align*}
\min & \sum_{k=0}^{K} \gamma_k \\
\text{subject to} & \gamma_0 e + \sum_{k=1}^{K} \gamma_k b_k = a.
\end{align*}
\]

Notice that the coefficients of the objective function are all zero, so any feasible solution is an optimal solution for the dual problem.

The main result of linear programming, called strong duality, is that the objective values of the primal and dual programs agree at optimality if and only if both programs are feasible. If there is no arbitrage then \( d'x = 0 \) so the dual must be feasible, so there exists numbers \( \gamma_0, \gamma_1, \ldots, \gamma_K \) such that

\[
a = \gamma_0 e + \sum_{k=1}^{K} \gamma_k b_k. \tag{3}
\]

This tells us that the zero factor \( a \) depends linearly in the other \( K \) factors. In particular

\[ a_i = \gamma_0 + \sum_{k=1}^{K} \gamma_k b_{ik}. \]

Substituting in

\[ r_i = a_i + \sum_{k=1}^{K} b_{ik} E[I_k] \]

we obtain

\[ r_i = \gamma_0 + \sum_{k=1}^{K} \gamma_k b_{ik} + \sum_{k=1}^{K} b_{ik} E[I_k] \]
\[ = \gamma_0 + \sum_{k=1}^{K} b_{ik} (\gamma_k + E[I_k]). \]

Now letting \( \lambda_0 = \gamma_0 \) and \( \lambda_k = \gamma_k + E[I_k] \), we obtain

\[ r_i = \lambda_0 + \sum_{k=1}^{K} b_{ik} \lambda_k. \]

The constant \( \lambda_k \) is often called the factor price of index \( I_k \). As the exposure of a portfolio to index \( k \) is increased the expected return grows at rate \( \lambda_k \).
What about residual risk? If there is a large number of securities then residual risk can be diversified. If the number of securities \( n \) is not very large, then the APT holds only approximately. There will be small pricing errors for many securities and possibly large pricing errors for a few securities.

1.3 Implications

What if we select a fully invested portfolio \( x \) with zero exposure to factors \( k = 1, \ldots, K \)? This portfolio will have return given by

\[
R_{(0)} = a' x + e' x,
\]

but on account of

\[
a = \gamma_0 e + \sum_{k=1}^{K} \gamma_k b_k
\]

we see that

\[
a' x = \gamma_0 e' x = \gamma_0 = \lambda_0.
\]

The expected return of this portfolio is

\[
\overline{R}_{(0)} = \lambda_0.
\]

If \( x \) is well diversified the risk is negligible so \( \lambda_0 = R_f \) the risk-free rate.

Now suppose we design a fully invested portfolio to have unit exposure to factor \( j \) and zero exposure to all other factors. Here \( e' x = 1 \), \( b'_j x = 1 \) and \( b'_k x = 0 \) for \( k \neq j \). The return of this portfolio is

\[
R_{(j)} = a' x + I_j + e' x.
\]

On account of

\[
a = \gamma_0 e + \sum_{k=1}^{K} \gamma_k b_k
\]

we have

\[
d' x = \gamma_0 + \gamma_j
\]

This portfolio has expected return

\[
\overline{R}_{(j)} = \gamma_0 + \gamma_j + E[I_j] = R_f + \lambda_j.
\]

Consequently, for \( k = 1, \ldots, K \)

\[
\lambda_k = \overline{R}_{(k)} - R_f,
\]

which allow us to write for any portfolio \( x_p \)

\[
\overline{R}_p = R_f + \sum_{k=1}^{K} b_{pk} (\overline{R}_{(k)} - R_f).
\]

1.4 Identifying the Factors

As stated before the APT is silent about the number and nature of the factors that are prices, i.e., that have values of lambda that are statistically different from zero. Several studies have found that including a handful of factors can explain a large portion of the returns. Typical factors included are:

- Growth rate in industrial production
- Rate of inflation (both expected and unexpected)
- Spread between long-term and short-term interest rates
- Spread between low-grade and high-grade bonds
- Size: the natural logarithm of equity capitalization
- Return on equity: (earnings divided by book)

1.5 The Tangent Portfolio and the APT

Let $F$ be $K \times K$ factor variance-covariance matrix, and let $B = (b_i)$ be the $n \times K$ matrix of factor exposures. Finally, let $T$ be the $n \times n$ diagonal matrix with elements $T_{ii} = \tau_i^2$, $i = 1, \ldots, n$. Then it is possible to show that

$$V = BF B' + T.$$ 

On the other hand, we know that

$$r = R_f e + B\lambda$$

where $\lambda = (\lambda_1, \ldots, \lambda_K)$. Thus

$$q = B\lambda.$$ 

Consequently,

$$x_q = \frac{(BF B' + T)^{-1}B\lambda}{\epsilon' (BF B' + T)^{-1}B\lambda}.$$ 

1.6 The APT vs CAPM

+ No need for the market portfolio.
- APT does not specify the factors.

Can both the APT and the CAPM hold at the same time? Yes. If there is a single factor and that factor is the market portfolio we recover the form of the CAPM.

$$\bar{R}_p = R_f + b_p (\bar{R}_m - R_f)$$

and in this case $b_p = \beta_p.$