1. (20 points) **Two Fund Theorem and Corner Portfolios** Suppose $x_p$ and $x_h$ are two portfolios in the efficient frontier of risky portfolios. Let $r_p$ and $r_h$ denote respectively the expected returns of portfolios $x_p$ and $x_h$ and let $\sigma^2_p$, $\sigma^2_h$, and $\sigma_{ph}$ denote respectively the variance of portfolio $x_p$, the variance of portfolio $x_h$ and the covariance between portfolios $x_p$ and $x_h$. Suppose you invest $\alpha$ in portfolio $x_p$ and $(1 - \alpha)$ in portfolio $x_h$.

(a) Find the expected return of the combined portfolio.

(b) Find the variance of returns of the combined portfolio.

(c) True or False: The two fund theorem states that if there are no bounds on holdings then the combined portfolio will be efficient for all values of $\alpha$.

(d) True or False: If there are bounds on holdings, but portfolios $x_p$ and $x_h$ have active positions in the same securities (they both have the same ‘in’ variables) then the combined portfolio will be efficient for all values of $\alpha \in [0, 1]$.
2. (15 points) **Constant Correlations** Assume the constant correlation model

\[ \sigma_{ij} = \rho \sigma_i \sigma_j \quad \text{for all} \quad i \neq j. \]

We know (problem 2 assignment 3) that security \( i \) is held long in the tangent portfolio if and only if

\[ S_i \geq \frac{\rho \sum_{j=1}^{n} S_j}{n \rho + 1 - \rho}, \]

where \( S_i = q_i / \sigma_i \) denotes the Sharpe ratio of security \( i \).

(a) Show that all securities are held long in the tangent portfolio if

\[ \min_i S_i \geq \left( \frac{n \rho}{n \rho + 1 - \rho} \right) \overline{S} \]

where \( \overline{S} \) is the average Sharpe-ratio over all the securities.

(b) Recall (problem 2 assignment 2) that \( S_i = \rho_{iq} S_q \) where \( \rho_{iq} \) is the correlation between security \( i \) and the tangency portfolio. Assume that \( \rho_{iq} = \sqrt{\rho} \) for all \( i \). Show that all securities are held long in the tangent portfolio.

(c) Show that security \( i \) is held long in the *minimum variance portfolio* if

\[ \frac{1}{\sigma_i} \geq \left( \frac{n \rho}{n \rho + 1 - \rho} \right) \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\sigma_j}. \]

Hint: Use what we known for \( x_q \) to obtain the desired results for \( x_i \).
3. (15 points) **Residual Variance Under CAPM** Recall that under the CAPM
\[
\sigma_i^2 = \beta_i \sigma_m^2 + \tau_i^2.
\]

(a) Show that \(\beta_i = \rho_{im} \frac{\sigma_i}{\sigma_m}\).

(b) Use part (a) to show that \(\beta_i^2 \sigma_m^2 = \rho_{im}^2 \sigma_i^2\).

(c) Use part (b) and the formula for \(\sigma_i^2\) to conclude that \(\tau_i^2 = (1 - \rho_{im}^2) \sigma_i^2\).
4. (25 points) **Active Portfolio Management** Consider the utility function

\[ u(x) = q'x - \frac{1}{\tau}x'\Sigma x. \]

We know (problem 3 assignment 2) that \( u(x) \) is maximized at

\[ x_e + \tau z \]

where \( x_e \) is the minimum variance portfolio and

\[ z = \frac{\epsilon'\Sigma^{-1}q}{2}(x_q - x_e). \]

(a) Find \( \tau^* \) such that

\[ x_e + \tau^* z = x_q. \]

Fact: \( \tau^* \) can also be written as

\[ \tau^* = 2 \frac{\sigma_q^2}{q \epsilon}. \]

This choice of \( \tau^* \) specifies the utility function to be used in part (d) below.

(b) Suppose that you disagree with the consensus estimate, \( q \), of excess expected returns and assume your forecast of excess expected returns is \( g \). Then the tangency portfolio based on \( g \) is \( x_g = \Sigma^{-1}g / \epsilon'\Sigma^{-1}g \). Argue that

\[ g_q = g' x_q = \frac{\epsilon'\Sigma x_g}{\sigma_g^2} g' x_g = \frac{\epsilon'\Sigma x_g}{\sigma_g^2} g. \]

Hint: This is not difficult but you may want to solve the other parts first.
(c) Let $\beta = \frac{V_p}{\sigma_g^2}$. Use part (b) and $\beta_g = \beta' x_g$ to show that

$$g_q = \beta_g \frac{\sigma_g^2}{\sigma_g^2 g_q}.$$ 

Hint: This is not difficult but you may want to solve the other parts first.

(d) The combination of the risk-free rate and portfolio $x_g$ that maximizes

$$u(x) = g' x - \frac{1}{\tau^*} x' V x$$

invests fraction

$$a = \frac{\tau^* g_g}{2 \sigma_g^2}$$

on portfolio $x_g$. Let $x_p = ax_g$. Use parts (a), (b), and (c) to show the second equality in

$$\beta_p = a \beta_g = \frac{g_q}{g_q}.$$ 

Note: Although investing fraction $a$ on $x_g$ is optimal for an investor with risk tolerance $\tau^*$, $\beta_p$ may be significantly different from $\beta_q = 1$ exposing active portfolio managers to significant residual risk relative to the benchmark $x_q$. Active managers loath taking the risk of significantly under performing the benchmark (they may get fired).
(e) Assume \( g = q + \alpha \), and that \( \alpha' x_q = 0 \), indicating that the benchmark portfolio is alpha-neutral. Rather than investing in portfolio \( x_q \) let us consider taking active positions in the portfolio that maximizes the ratio of active return \( \alpha' x \) to active risk \( \sqrt{\alpha' V \alpha} \). Following the same logic used to determine the portfolio with the maximum Sharpe ratio it can be shown that the this portfolio is given by

\[
x_{\alpha} = \frac{V^{-\frac{1}{2}} \alpha}{\sqrt{\alpha' V^{-1} \alpha}}.
\]

Show that \( \beta_\alpha = \beta' x_{\alpha} = 0 \).

Note: Portfolios of the form \( a x_q + (1 - a)x_{\alpha} \) have the advantage of being beta-neutral (they have the same beta as the benchmark). This reduces the business risk exposure of the active portfolio manager, but gives rise to a principal-agent problem where the agent (the manager) is not necessarily doing what is best for the principal (the investor).
5. (25 points) **Log Optimal Pricing** Your objective is to allocate current wealth $W_0$ among $n$ securities to maximize the expected utility $EU(W_1) = E \ln(W_1)$ of wealth $W_1$ a period later. Let $P_i$ be the known current price of security $i$, and let $F_i$ be the (possibly) random price of security $i$ at the end of period 1. You want to select $X_i$ the number of shares of security $i$ to maximize

$$E \ln \left( \sum_{i=1}^{n} X_i F_i \right)$$

subject to

$$\sum_{i=1}^{n} X_i P_i = W_0.$$ 

As usual we proceed by studying the Lagrangian

$$L(X_1, \ldots, X_n, \lambda) = E \ln \left( \sum_{i=1}^{n} X_i F_i \right) - \lambda \left( \sum_{i=1}^{n} X_i P_i - W_0 \right).$$

(a) Show that the first order conditions of optimality are:

$$E \left[ \frac{F_i}{W_1^*} \right] - \lambda P_i = 0 \quad \text{for all } i = 1, \ldots, n$$

and

$$\sum_{i=1}^{n} X_i^* P_i = W_0,$$

where

$$W_1^* = \sum_{j=1}^{n} X_j^* F_j.$$ 

Hint: $\frac{d}{dx} \ln(x) = u'(x)/u(x)$. 


(b) Suppose there is a risk-free asset such that $P_j = 1$ and $F_j = 1 + R_f$ with probability one. Use the conditions obtained in part (a) to conclude that

$$\lambda = (1 + R_f)E \left[ \frac{1}{W_1^*} \right].$$

(c) Use the fact that

$$\lambda W_0 = \lambda \sum_{i=1}^{n} P_i X_i^*$$

and part (a) to show that $\lambda = \frac{1}{W_0}$ and conclude that

$$E \left[ \frac{1}{W_1^*} \right] = \frac{1}{W_0(1 + R_f)}.$$
(d) Use parts (a) and (c) to argue that if \( F_i \) is deterministic then

\[
P_i = \frac{F_i}{1 + R_f}
\]

and interpret this result.

(e) Suppose there are only two securities with \( (P_1, P_2) = (1,1) \) and future payoffs \( F_1, F_2 \) having joint distribution \( (F_1, F_2) = (1,2) \) with probability 0.5 and \( (F_1, F_2) = (1,0.8) \) with probability 0.5. Write down the first order conditions for \( W_0 = 1 \), and verify that \( (X_1^*, X_2^*) = (-1, 2) \) is the optimal allocation.