1. **Reverse CAPM** Recall that the tangent portfolio is given by

\[
x_q = \frac{V^{-1}q}{e'V^{-1}q}
\]

where \(V\) is the variance-covariance matrix, \(q\) is the vector of excess expected returns, and \(e\) is the vector of ones. Let \(p\) be the vector of market capitalization of the risky securities, so \(p_i\) is equal to the share price of security \(i\) multiplied by the number of outstanding shares of security \(i\). Consequently, the market portfolio (for this universe of securities) is given by

\[
x_m = \frac{p}{e'p}.
\]

Notice that \(x_m\) depends on observable data (share prices and number of shares outstanding).

(a) Assume that \(V\) is known and does not change with time, that the market portfolio is equal to the tangency portfolio, \(x_q = x_m\), and that we know (or have a good estimate of) the expected excess return of the market portfolio \(q_m = q'x_m = q'x_q = q_q\). Assume, however, that we do not know \(q\). Use equation (1) and \(q_q = q_m\) to solve for \(q\) and to show that

\[
q = \frac{Vx_m}{x_m'Vx_m}q_m.
\]

Use \(x_m = p/e'p\) to conclude that

\[
q = e'p \frac{Vp}{p'Vp}q_m.
\]
(b) Consider an economy consisting of two risky securities with variance-covariance matrix

\[
V = \begin{pmatrix} 225 & 45 \\ 45 & 81 \end{pmatrix}
\]

and current market capitalization vector

\[
p = \begin{pmatrix} 150 \\ 300 \end{pmatrix}.
\]

Suppose that our estimate of the excess expected return of the market portfolio is \( q_m = 8.5\% \). Use (2) to estimate \( q \). How does your estimate of \( q \) changes if a day later

\[
p = \begin{pmatrix} 140 \\ 310 \end{pmatrix},
\]

and your estimate of \( q_m \) remains equal to 8.5%?
2. **Arbitrage Pricing Theory** Based on a one-factor model, two portfolios, A and B, have equilibrium expected returns of 9.8% and 11.0% respectively. If the factor sensitivity of portfolios A and B is 0.80 and 1.00, respectively, what must be the risk-free rate?
3. Spot Rates, Forward Rates, and Expectations Dynamics

(a) Recall that spot rates are usually calculated for six-month intervals and then annualized by doubling the six month rate. The spot rate for a zero-coupon bond that matures $6n$ months from now is given by

$$S_{0,n} = 2\left(\frac{(F/P)^{1/n} - 1}{F} \right),$$

where $F$ is the face value of the bond and $P$ is its current value. Determine $S_{0,20}$, the spot rate of a 10 year zero-coupon bond, by forming a portfolio of the following bonds: Bond A is 10-year bond with face value $F_A = 100$, coupon rate $r = 10\%$, $m = 2$ (so it pays $5$ every six months), and current price $P_A = 98.72$. Bond B is a 10-year bond with face value $F_B = 100$, coupon rate $r = 8\%$, $m = 2$ (so it pays $4$ every six months), and current price $P_B = 85.89$.

(b) The forward rates are given by

$$f_{s,t} = 2\left[ \left( \frac{1 + S_{0,t}/2}{1 + S_{0,s}/2} \right)^{1/(t-s)} - 1 \right].$$

Under the expectations dynamics theory the forward rate $f_{s,t}$ is equal to the market expectation of what the $t - s$ spot rate will be after $6s$ months, i.e., $E[S_{s,t}] = f_{s,t}$. Use the spot rates $S_{0,1} = 6\%$, $S_{0,2} = 7\%$ and $S_{0,3} = 8\%$ to estimate $E[S_{1,2}]$, $E[S_{1,3}]$. Are the spot rates expected to increase?
4. **Duration** Rank the following bonds in terms of duration. Explain the rationale behind your rankings. (You do not have to actually calculate the bonds' durations. Logical reasoning will suffice.)

<table>
<thead>
<tr>
<th>Bond</th>
<th>Term-to-Maturity</th>
<th>Coupon Rate</th>
<th>Yield-to-Maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30 years</td>
<td>10.0%</td>
<td>10.0%</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>0.0</td>
<td>10.0</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>10.0</td>
<td>7.0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>10.0</td>
<td>10.0</td>
</tr>
</tbody>
</table>
5. **Static Analysis of Call Options** Recall the Black-Scholes formula

\[ C_t = S_t \Phi(d_1) - K_t \Phi(d_2) \]

for the current (time \( t \)) price of a call option of a non-dividend paying stock with current price \( S_t \), exercise price \( K_0 \), expiration date \( T > t \), and continuously compounded risk-free rate \( r \). Here \( K_t = e^{-r(T-t)} K_0 \),

\[ d_1 = \frac{\ln(S_t/K_0) + \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}} \]

and

\[ d_2 = \frac{\ln(S_t/K_0) - \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}} \]

Determine whether the following statements are true (T) or false (F).

(a) The higher the price of the stock \( S_t \), the higher the value of the call option. ( )

(b) The higher the strike price \( K_0 \), the higher the value of the call option. ( )

(c) The higher the risk-free rate \( r \), the lower the value of the call option. ( )

(d) The greater the risk \( \sigma \) of the common stock, the higher the value of the call option. ( )

(e) The longer the time to expiration date \( T - t \), the higher the value of the call option. ( )
6. **Risk Neutral Pricing** Assume that security prices are given by \( p = D \psi \) where \( \psi \) is the state-price vector. Let \( \pi \) be a random variable taking values \( \pi_j = \psi_j / \bar{q}_j, \ j = 1, \ldots, S \), where \( \bar{q} \) is the vector of state probabilities. A portfolio\(^1\) \( x \) has payoff \( D' x \) and current cost \( p' x \). For any portfolio with \( p' x \neq 0 \), let

\[
R^x = \frac{D' x}{p' x}.
\]

denote its total return vector.

(a) Show that expected total return of portfolio \( x \) is given by

\[
E[R^x] = \frac{\bar{q} D' x}{\psi' D' x}.
\]

(b) Assume that \( \pi_1 \leq \pi_2 \leq \ldots \leq \pi_S \) and assume complete markets so that it is possible to invest directly in elementary securities. Show that an investment in elementary security \( j \) has expected total return \( 1/\pi_j \), and conclude that elementary security \( 1 \) has the highest expected total return.

\(^1\)Here \( x_i \) is the number of units of security \( i = 1, \ldots, n \).
(c) The log utility function suggests allocating \( a \) units of capital among the \( S \) elementary securities in proportions \( a \tilde{q}_j \), so that the payoff on state \( j \) is \( a \tilde{q}_j / \psi_j \). Let \( R^n \) denote the total return of this portfolio. Show that

\[
R^n_j = 1 / \pi_j \quad j = 1, \ldots, S,
\]

and conclude that

\[
E[R^n] = \sum_{j=1}^{S} \frac{\tilde{q}_j}{\pi_j}.
\]

(d) Let \( r = 1 + R_f \) be the risk-free total return. Recall that \( q = r \psi \) is the vector of risk-neutral probabilities. Use part (a) to show that

\[
E[R^*] = r \frac{q^T D' x}{q^T D' x}.
\]

and conclude that if \( \tilde{q} = q \) then all portfolios with \( p' x \neq 0 \) have total expected return equal to \( r \).