1 Put-Call Parity

In a geometric Brownian motion model we also have put-call parity. In fact, in any market where risk-neutral pricing exists, we have put-call parity for European puts and calls. In particular, let \( \{S_t\}_{t \in [0,T]} \) be the underlying stock price process, then for any \( K > 0 \) we have

\[
(S_T - K)^+ - (K - S_T)^+ = S_T - K. \tag{1}
\]

By taking conditional expectation given \( \mathcal{F}_t \) we have that

\[
C(t, S_t) - P(t, S_t) = F(t, S_t), \tag{2}
\]

where the time-\( t \) prices of call, put, and forward are given by

\[
C(t, S_t) = \tilde{E}\{e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t}\}, \tag{3}
\]

\[
P(t, S_t) = \tilde{E}\{e^{-r(T-t)}(K - S_T)^+|\mathcal{F}_t}\}, \tag{4}
\]

\[
F(t, S_t) = \tilde{E}\{e^{-r(T-t)}(S_T - K)|\mathcal{F}_t\} = S_t - e^{-r(T-t)}K. \tag{5}
\]

Alternatively, one can also prove (2) using PDE approach. Because any European derivative price process \( \{v(t, S_t)\}_{t \in [0,T]} \) satisfies Black-Scholes equation:

\[
v_t + rxv_x + \frac{1}{2}\sigma^2x^2v_{xx} = rv, \ t \in [0,T). \tag{6}
\]

The terminal condition \( v(T, x) \) uniquely determines the price function \( v(t, x) \) for any \( (t, x) \in [0,T) \times (0, \infty) \). Since we have linear relationship between \( C(T, x) \), \( P(T, x) \) and \( F(T, x) \), see (1). By uniqueness of the solution of PDE in (6), we must always have

\[
C(t, x) - P(t, x) = F(t, x), \ (t, x) \in [0,T) \times (0, \infty). \tag{7}
\]

2 The Greeks

Greek letters are rates of change of option price with respect to model parameters. In this section the option is an European call option struck at \( K > 0 \). By Black-Scholes formula we know that the time-\( t \) price of the call is given by

\[
C(t, S_t) = S_tN(d_+(T - t, S_t)) - Ke^{-r(T-t)}N(d_-(T - t, S_t)), \tag{8}
\]

where

\[
d_{\pm}(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left\{ \log \frac{x}{K} + \left( r \pm \frac{1}{2}\sigma^2\right)\tau \right\}. \tag{9}
\]
The first Greek letter is called delta:

\[ \Delta(t, S_t) := \frac{\partial C}{\partial x}(t, S_t) = N(d_+(T - t, S_t)). \]  

It measures the sensitivity of call price to the small moves of the underlying stock price. For example, if \( \Delta = 0.5 \) now, then a 0.01 move of stock price would result in a price move of 0.005 for the call. \( \Delta \) is always less than one, so call price seems to be less volatile than stock price. However, do not think stock is more risky.

**Example 1.** Suppose you have \( N \) dollars and you want to either invest in stocks or invest in options only. If you use the money to long stocks only, then you can purchase \( \frac{N}{S_0} \) shares, and a tiny movement \( \epsilon \) in stock price would result in a value change in your all-stock portfolio of \( \epsilon \cdot \frac{N}{S_0} \). On the other hand, if you purchase calls only, then you can purchase \( \frac{N}{C(0, S_0)} \), and a tiny movement \( \epsilon \) in stock price would result in a value change in your all-option portfolio of \( \Delta(0, S_0)\epsilon \cdot \frac{N}{C(0, S_0)} \). To compare which strategy is more risky, let us compute

\[ \frac{\epsilon \cdot N/S_0}{\Delta(0, S_0)\epsilon \cdot N/C(0, S_0)} = \frac{C(0, S_0)}{\Delta(0, S_0)S_0} = 1 - \frac{Ke^{-rT} N(d_-(T, S_0))}{S_0 N(d_+(T, S_0))} < 1. \]  

So call option is actually more risky.

The second Greek letter is the second partial derivative with respect to the underlying stock price, which is denoted by \( \Gamma \):

\[ \Gamma(t, S_t) := \frac{\partial^2 C}{\partial x^2}(t, S_t) = \frac{1}{\sigma S_t \sqrt{T-t}} \phi(d_+(T - t, S_t)), \]  

where \( \phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2} \) is the standard normal density. Note that both \( \Delta \) and \( \Gamma \) are positive for any \( (t, S_t) \in [0, T) \times (0, \infty) \), so for a fixed \( t \), the call price \( C(t, S_t) \) is an increasing convex function of \( S_t \). It is also worth pointing out that, the positivity of \( \Delta \) and \( \Gamma \) are observable in real market, although the real stock market does not follow geometric Brownian motion exactly.
The third Greek letter is the first partial derivative with respect to $t$, which is denoted by $\Theta$:

$$\Theta(t,S_t) := \frac{\partial C}{\partial t}(t,S_t) = -r Ke^{-r(T-t)} N(d_-(T-t,S_t)) - \frac{\sigma S_t}{2\sqrt{T-t}} \phi(d_+(T-t,S_t)).$$

(13)

$\Theta$ for a call is always negative. As $t$ approaches maturity $T$, the call option loses value. This phenomena is also seen in real life. It can be interpreted as follows: an American call has a negative $\Theta$ as the less time remains to maturity, the fewer optionalsity left to the holder; American call has the same value as European call, so European call also has negative $\Theta$.

The above three Greeks are related by Black-Scholes equation. In fact, from (6) we have that

$$\Theta(t,S_t) + r S_t \Delta(t,S_t) + \frac{1}{2} \sigma^2 S_t^2 \Gamma(t,S_t) = r C(t,S_t).$$

(14)

If the market follows a Black-Scholes model, consider a strategy of short selling a call and buying $\Delta(t,S_t)$ shares of stock, then the value process $\{X_t\}_{t \in [0,T]}$ follows

$$dX_t = -dC(t,S_t) + \Delta(t,S_t) dS_t + r(X_t + C(t,S_t) - \Delta(t,S_t) S_t) dt$$

$$= \left\{ r X_t + r C(t,S_t) - \Theta(t,S_t) - \frac{1}{2} \sigma^2 S_t^2 \Gamma(t,S_t) - r \Delta(t,S_t) S_t \right\} dt. \quad (15)$$

Using (14) we have

$$dX_t = r X_t dt. \quad (16)$$

This means if we delta-hedge a short position in call, our portfolio value will grow at the risk-free rate. The result is the same as the one in binomial market. However, because the real market is not a geometric Brownian motion, the above delta-hedging strategy do outperform money market sometimes.
The last Greek letter is the first partial derivative with respect to $\sigma$, which is denoted by $\nu$:

$$
\nu(t, S_t) := \frac{\partial C}{\partial \sigma} = Ke^{-r(T-t)} \phi(d_-(T - t, S_t)) \sqrt{T - t}.
$$

Note that vega $\nu$ is always positive. With all the other parameters fixed, the call price is an increasing function of volatility $\sigma$. For this reason traders usually quote the implied volatile instead of the actual option price.

**Remark 1.** The volatility smile effect implies that Black-Scholes model underestimates the options that are deep out of money (DOOM). It can be shown that if there is a jump in price (the price suddenly becomes volatile), then the European call option would be higher than the one produced by Black-Scholes formula.

## 3 Delta-neutral Trading strategy

We study delta-neutral trading in practice. For simplicity we assume interest rate $r = 0$, and we will only consider at-the-money European options for liquidity consideration. First, for at-the-money call $K = S_0$, we know that its delta is

$$
\frac{\partial C}{\partial x}(0, S_0) = \Delta(0, S_0) = N(0.5\sigma\sqrt{T}) \approx N(0) = 0.5.
$$

Using put-call parity (2) we have that at any time $t \in [0, T)$

$$
\frac{\partial C}{\partial x}(t, S_t) - \frac{\partial P}{\partial x}(t, S_t) = 1.
$$

So the delta of an at-the-money put at time zero is approximately $-0.5$.

If we hold one at-the-money call and one at-the-money put at time zero, with the same maturity, then the total delta of our portfolio is

$$
\Delta^C(0, S_0) + \Delta^P(0, S_0) = 0.5 - 0.5 = 0.
$$

We thus have a delta-neutral portfolio, which is insensitive to instantaneously small moves of stock price. The combination of a call and a put with the same strike and maturity is called a straddle. A straddle is betting on volatility: both the call and the put has the same gamma, which is positive, so the portfolio benefits from the enhanced convexity. If the trader can predict that the volatility will increase within the next few days, but he cannot predict the direction of the price movement, then he can setup a straddle now. Later the price movement will be volatile if he is right, and he can make a profit despite the direction of price move. There is a catch, however, in this strategy, the theta is always negative, so the time-decay may eat up all the profit the trader can make. To delta with it, the trader can purchase long-maturity straddle, so the time-decay is tolerable in the next few profitable days.

If the market is stagnant, like a range market. This can be regarded as a low volatility case, the trader can still make a profit by short selling the above straddle.

![Graph of Straddle vs Stock Price](image-url)