1 Introduction

Like those in binomial markets, American options are derivatives which can be exercised at any time prior to maturity $T$. It is obvious that an American option will be more expensive than its European counterpart. It is obvious that holders of American options tend to make use of the additional optionality and exercise options optimally.

We only consider American calls and puts with finite maturity. The standard assumption of no arbitrage is still the basis of our pricing theory. For simplicity, we will also assume that the underlying stock has no dividend. However, models with dividend can be analyzed if one is careful enough.

Under above assumptions, we can immediately kill American calls, since we known that, the optimal exercise time of an American call under no arbitrage and no dividend is at the maturity of this option. So effectively the price of an American call can be priced using Black-Scholes formula.

We will thus only focus on American puts in what follows.

2 Pricing in a binomial model

This optimal exercise time will not be a constant in general, rather, it will be a stopping time. So pricing of an American option boils down to finding a optimal exercise stopping time.

Recall that in the binomial model, the pricing of an American put is conducted as follows:

- At maturity $N$, the value of a put is given by its intrinsic value $V_N = (K - S_N)^+$;
- If the value of put is known at time $k +1$, then its value at time $k$ is given by a nonlinear back recursion equation:

$$V_k = \max\{K - S_k, (1 + r)^{-1} \times \hat{E}\{V_{k+1}|\mathcal{F}_k}\},$$

(1)

Pricing of an American put is done backwardly. And the optimal exercise time for the above American put is a stopping time defined by

$$\tau^* = \min\{k \geq 0 | K - S_k \geq V_k\},$$

(2)

where we set $\tau^* = \infty$ if the set on the right hand side of (2) is empty. And $\tau^* = \infty$ means that we do not exercise the option.

On the other hand, the stopping time $\tau^*$ has a probabilistic interpretation. We know that the discounted price process of an American put is not a martingale, but a supermartingale\(^1\). This is because that, if we miss the optimal stopping time, then on average, the payoff we receive in the future will not be as much as that received at the optimal exercise time. So intuitively, the optimal stopping time is a time after which the put option will lose value on average. What happens before $\tau^*$? We can check this from the definition of the optimal stopping time $\tau^*$. In particular, we see from (2) that, before $\tau^*$, the put price is fully determined by conditional expectation. This means that if $k + 1 \geq \tau^*$ on a path of stock price, then

$$V_k = \frac{1}{(1 + r)} \hat{E}\{V_{k+1}|\mathcal{F}_k\},$$

(3)

\(^1\)A stochastic process $\{X_t\}$ is a supermartingale if $\hat{E}\{X_t|\mathcal{F}_s\} \leq X_s$ for all $t \geq s$.  

November 17, 2010
So the discounted value process before $\tau^*$ is a martingale. In other words, the stopped process
\[ \{(1+r)^{-\tau^*}V_{\tau^*}\} \]
is a martingale.

From above discussion, we extract two pieces of important information:

- At the optimal time $\tau^*$, $K - S_\tau = V_{\tau^*}$; Although the value $K - S_\tau$ is not known in advance, we know from (2) that informally, at $k = \tau^*$,
  \[ \frac{\partial V_k}{\partial S_k} = -1. \]  
  \[ (4) \]
- Before the optimal time $\tau^*$, the discounted put price is a martingale.

We will use these two fact in price American puts in continuous models.

3 Pricing in a geometric Brownian motion model

We consider the standard geometric Brownian motion model for the stock price process. That is, under the risk-neutral measure, we have
\[ dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 > 0, \]  
\[ (5) \]
where $W_t$ is a standard Brownian motion, $r, \sigma > 0$ are constants, representing risk-free interest rate and volatility, respectively. Our objective here is to price an American put struck at $K > 0$.

Let us try to use the idea of backward recursion in this continuous-time model. At maturity $T$, the value of the option is of course $(K - S_T)^+$. And the exercise boundary is given by $K$, which means, exercise the option if $S_T < K$ and do not exercise the option if $S_T > K$.

It will be great if we can find an explicit optimal exercise boundary at any time $t \in [0, T]$. To this effect, let us denote by $L(T - t)$ the optimal exercise boundary of the put if time $t$. This means that,
\[ \begin{cases} 
  \text{exercise}, & \text{if } S_t < L(T - t), \\
  \text{do not exercise}, & \text{if } S_t > L(T - t). 
\end{cases} \]  
\[ (6) \]
Moreover, we know that $L(0) = K$.

Unfortunately, there is no analytical formula for the optimal exercise boundary $L(T - t)$. However, we could heuristically draw a smooth curve which ends at $K$ for $t = T$. And assume that this curve is the optimal exercise boundary. Now we are in a situation in which we have two disjoint region in set $\{(t, x)|0 \leq t \leq T, x > 0\}$:

\[ \mathcal{C} = \{(t, x)|x > L(T - t)\}, \]  
\[ (7) \]
\[ \mathcal{S} = \{(t, x)|0 < x \leq L(T - t)\}. \]  
\[ (8) \]

Here the set $\mathcal{C}$ is called the continuation set, and the set $\mathcal{S}$ is called the exercise set. So what is the optimal exercise time? Well, by the way we define the optimal exercise boundary, the optimal exercise time of this put is simply the first hitting time to the optimal exercise boundary.

In set $\mathcal{C}$, that is, before the optimal exercise time, we know that the discounted price process $\{e^{-rt}P(t, S_t)\}$ is a martingale. So in $\mathcal{C}$, we have Black-Scholes PDE hold. That is,
\[ \frac{\partial P}{\partial t} + r x \frac{\partial P}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} = r P, \quad \text{for any } (t, x) \in \mathcal{C}. \]  
\[ (9) \]
Moreover, in set $\mathcal{S}$, the put price should be strictly greater than its intrinsic value (see (2)). That is,
\[ P(t, x) > (K - x), \quad \text{for any } (t, x) \in \mathcal{S}. \]  
\[ (10) \]

For any $(t, x) \in \mathcal{S}$, we have the put price $P(t, x)$:
\[ P(t, x) = (K - x), \quad \text{for any } (t, x) \in \mathcal{S}. \]  
\[ (11) \]
We can also write down a PDE like (9), which is satisfied by all \((t, x) \in S\). This equation can be easily shown to be

\[
\frac{\partial P}{\partial t} + rx \frac{\partial P}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} = r(P - K), \text{ for any } (t, x) \in S.
\]

(12)

We can now discuss in detail the computation of the price function \(P(t, x)\). First we have the terminal condition

\[
P(t, x) = (K - x)^+.
\]

(13)

Then we use this terminal conditional to solve for \(P\) in set \(C\). Since both the boundary \(L(T - t)\) and the value of \(P\) on this boundary are unknown, this problem will not be easy. But we can use the fact that, on the unknown boundary \(L(T - t)\), we have that

\[
\frac{\partial P}{\partial x} = -1.
\]

(14)

This comes from (4), which is called the smooth-pasting condition. Finally, as the stock price \(x \to \infty\), we must also have

\[
\lim_{x \to \infty} P(t, x) = 0.
\]

(15)

Using (9), (11), (13), (14) and (15) we can fully determine the function \(P(t, x)\) numerically. The details of the algorithm used for this computation will be discussed next time.

4 Probabilistic characterization of \(\tau^*\)

In the last section we point out that \(\tau^*\) is the first hitting time to the moving boundary \(L(T - t)\). We justify this claim in this section. To do so, we need to show that, before hitting the boundary \(L(T - t)\), the discounted put price is a martingale, whereas after hitting \(L(T - t)\), the discounted put price will be a supermartingale (conditional expectation will become smaller).

Let us compute the differential of \(e^{-rt}P(t, S_t)\).

\[
d(e^{-rt}P(t, S_t)) = e^{-rt}\left( -rP + \frac{\partial P}{\partial t} + rS_t \frac{\partial P}{\partial x} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt + e^{-rt} \sigma S_t \frac{\partial P}{\partial x} dW_t.
\]

Before entering the exercise region \(S\), the equation in (9) holds, so there is no drift in (16). Thus the stopped process \(X_{t \wedge \tau^*}\) is a martingale, where \(X_t = e^{-rt}P(t, S_t)\) is the discounted put price. After enter the exercise region \(S\), in view of (12), the drift in (16) becomes \(-rK e^{-rt} < 0\). This negative drift will make the discounted put price smaller, and thus, a supermartingale. Can the price increase to what it used to be if the price process revisit the continuation region \(C\)? The answer is no, as the drift in (16) is non-positive no matter what.

In conclusion we see that the first hitting time \(\tau^*\) is indeed the optimal exercise time.

5 How to hedge?

The hedging of an American put is straightforward. We know that the price evolution in region \(C\) is Black-Scholes. So when shorting a put, we just need to buy \(P_x(t, S_t)\) shares of stock. Let \(X_t\) be the value of the portfolio consisting of -1 put, \(P_x(t, S_t)\) shares of stock and money market, then we have that

\[
dX_t = -dP(t, S_t) + \frac{\partial P}{\partial x} dS_t + r\left( X_t + P(t, S_t) \right) dt = rX_t dt,
\]

(17)

before entering the exercise region \(S\). So the hedging works in \(C\).

Once we enter the exercise region \(S\). The put can still be hedged by buying \(\frac{\partial P}{\partial x}\) shares of stock and withdrawing cash at rate \(rK \mathbb{I}_{S_t < L(T-t)}\). We withdraw cash in order to reduce the value of the portfolio, hoping it will replicate the declining put price.