Lecture 20: Change of Numéraire

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1 Numéraire and change of measure

A numéraire is a positively priced asset which denominate other asset. A typical example of a numéraire is the currency of a country, because people usually measure other asset’s price in terms of the unit of currency. One can also use other numéraire when it is more convenient. For example, let us denote by $R_t$, $0 \leq t \leq T$ an adapted interest rate process, then the price per share of a money market account at time $t$ is

$$M_t = e^{\int_0^t R_u du}. \tag{1}$$

We can use $\{M_t\}$ as the numéraire to denominate other assets, then the value of one unit of currency at time $t$ will be

$$D_t = \frac{1}{M(t)} = e^{-\int_0^t R_u du}, \tag{2}$$

which is the discounted factor; the value of a stock $S_t$ at time $t$ will be

$$S_t^{(M)} = \frac{S_t}{M_t} = D_t S_t, \tag{3}$$

which is the discounted stock price. It is known that under the risk neutral measure $\tilde{P}$, the process $\{S_t^{(M)}\}$ is a martingale. In this lecture, our objective is to generalize this result to any numéraire that pays no dividend.

**Main result:** Let $\{N_t\}$ be a numéraire that pays no dividend, and $\{S_t\}$ any asset price process, then there exists a measure $\tilde{P}^{(N)}$ which is equivalent to the risk-neutral measure $\tilde{P}$.Under the measure $\tilde{P}^{(N)}$, the process

$$S_t^{(N)} = \frac{S_t}{N_t} \tag{4}$$

is a martingale. Derivative pricing under the new measure $\tilde{P}^{(N)}$ is similar as risk-neutral pricing. In particular, the time-$t$ price of a derivative that pays $X$ at maturity $T$ is given by

$$V_t = N_t \tilde{E}^{(N)}\{N_T^{-1} X \mid \mathcal{F}_t\} = \tilde{D}_t^{-1} \tilde{E}\{D_T^{-1} X \mid \mathcal{F}_t\} \tag{4}$$

Moreover, derivative pricing does not depend on the choice of numéraire.

2 Proof of the main result

In this section, we give a sketched proof, which includes the following three steps:

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1That is, the set including all null sets does not change.
2.1 Construction of $\tilde{P}^{(N)}$

Recall that from Girsanov theorem that, we can change the measure from $\tilde{P}$ to another via a Radon-Nikodým process. The Radon-Nikodým process is a positive martingale with unit expectation under the current measure $\tilde{P}$. Now we have a positive asset price process $\{N_t\}$, whose discounted price process $\{D_tN_t\}$ is a positive martingale under $\tilde{P}$. We can let $\{D_tN_t/N_0\}$ be a Radon-Nikodým process and define a new measure $\tilde{P}^{(N)}$.

How stochastic processes behave under $\tilde{P}^{(N)}$? Girsanov theorem says, if the Radon-Nikodým process is given by

$$Z_t = \exp \left( \int_0^t \nu(u) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t \|\nu(u)\|^2 dt \right),$$

where $\nu(u) = (\nu_1(u), \ldots, \nu_d(t))^\tau$, and $\tilde{W}_u = (\tilde{W}^1_u, \ldots, \tilde{W}^d_u)^\tau$ is a vector consisting of $d$ independent Brownian motions. Then under the new measure defined by the Radon-Nikodým process $\{Z_t\}$, the vector process

$$\tilde{W}^{(N)}_t = - \int_0^t \nu(u) dt + \tilde{W}_t,$$  \hspace{1cm} (5)

is a $d$-dimensional standard Brownian motion. In our case, because $N_t$ is always positive, we can legally assume\(^2\) that there exist a vector volatility process $\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$ such that

$$\frac{D_tN_t}{N_0} = \exp \left( \int_0^t \nu(u) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t \|\nu(u)\|^2 du \right).$$ \hspace{1cm} (6)

And the new measure generated by the numéraire $N_t$ is the one determined by the above Radon-Nikodým process. We also know the form of standard Brownian motion under $P^{(N)}$. (see (5))

2.2 $\{S_t^{(N)}\}$ is a martingale under $\tilde{P}^{(N)}$

We assume the asset price process $\{S_t\}$ is driven by a $d$-dimensional standard Brownian motion under the risk-neutral measure $\tilde{P}$. That is, the discounted price is given by

$$D_tS_t = S_0 \exp \left( \int_0^t \sigma(u) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t \|\sigma(u)\|^2 dt \right).$$  \hspace{1cm} (7)

From (6) and (7) we obtain that

$$S_t^{(N)} = \frac{S_t}{N_t} = \frac{S_0}{N_0} \exp \left( \int_0^t (\sigma(u) - \nu(u)) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t \left( \|\sigma(u)\|^2 - \|\nu(u)\|^2 \right) du \right).$$  \hspace{1cm} (8)

To show that $S_t^{(N)}$ is a martingale under $\tilde{P}^{(N)}$, let us denote by

$$X_t = \int_0^t (\sigma(u) - \nu(u)) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t \left( \|\sigma(u)\|^2 - \|\nu(u)\|^2 \right) du,$$

so that

$$dX_t = (\sigma(t) - \nu(t)) \cdot d\tilde{W}_t - \frac{1}{2} \left( \|\sigma(t)\|^2 - \|\nu(t)\|^2 \right) dt,$$

$$d(X_t)^2 = (\sigma(t) - \nu(t)) \cdot (\sigma(t) - \nu(t)) dt = \|\sigma(t)\|^2 dt - 2\sigma(t) \cdot \nu(t) dt + \|\nu(t)\|^2 dt.$$ \hspace{1cm} (10)

With $f(x) = \frac{S_0}{N_0} e^x$ we have $S_t^{(N)} = f(X_t)$. Using Itô’s lemma we have

$$dS_t^{(N)} = f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2$$

$$= S_t^{(N)} \left( (\sigma(t) - \nu(t)) \cdot d\tilde{W}_t - \frac{1}{2} \left( \|\sigma(t)\|^2 - \|\nu(t)\|^2 \right) dt + \frac{1}{2} \|\sigma(t)\|^2 dt - \sigma(t) \cdot \nu(t) dt + \frac{1}{2} \|\nu(t)\|^2 \right)$$

$$= S_t^{(N)} (\sigma(t) - \nu(t)) \cdot (-\nu(t) dt + d\tilde{W}_t) = S_t^{(N)} (\sigma(t) - \nu(t)) \cdot d\tilde{W}^{(N)}_t.$$ \hspace{1cm} (12)

This means $\{S_t^{(N)}\}$ is an exponential martingale with instantaneous volatility $\|\sigma(t) - \nu(t)\|$.

\(^2\)For details, refer to Theorem 9.2.1 on page 377.
2.3 Derivative pricing does not dependent on numéraire

From the above discussion we know that, using $N_t$ as the numéraire, a claim that pays $X$ at time $T$ has a time-$t$ price

$$V_t = N_t \tilde{E}^{(N)} \{ N_T^{-1} X | \mathcal{F}_t \}. \quad (13)$$

Is this price the same as the one given by risk-neutral measure? The answer is yes. Recall that for any random variable $Y$, the Radon-Nikodým process links the two conditional expectations of $Y$:

$$\tilde{E}^{(N)} \{ Y | \mathcal{F}_t \} = \frac{N_0}{D_t N_t} \tilde{E} \left\{ \frac{D_T N_T}{N_0} Y | \mathcal{F}_t \right\}. \quad (14)$$

From (13) and (14) we have that

$$V_t = N_t \cdot \frac{N_0}{D_t N_t} \tilde{E} \left\{ \frac{D_T N_T}{N_0} X | \mathcal{F}_t \right\} = D_t^{-1} \tilde{E} \{ D_T^{-1} X | \mathcal{F}_t \}. \quad (15)$$

3 Application: foreign exchange rate

We consider a financial market which includes a domestic money market with rate $R_t$, a domestic stock $S_t$ and a foreign money market with rate $R^f_t$. The domestic money market account value is given in (1), the foreign money market account value is given by

$$M^f_t = e^{\int_0^t R^f_u du}. \quad (16)$$

There is an exchange rate $Q_t$, which represents units of domestic currency per unit of foreign currency. Let us assume that, under the actual measure $P$, the processes $S_t$ and $Q_t$ satisfy

$$\frac{dS_t}{S_t} = \alpha(t) dt + \sigma_1(t) dW^1_t, \quad (17)$$
$$\frac{dQ_t}{Q_t} = \gamma(t) dt + \sigma_2(t) \left[ \rho(t) dW^1_t + \sqrt{1-\rho^2(t)} dW^2_t \right], \quad (18)$$

where $W^1_t$ and $W^2_t$ are two independent standard Brownian motion under $P$. We want to find the domestic risk-neutral measure $\tilde{P}$ generated by the domestic money market. Under this measure,

- $M^t_{M_t} = \frac{M_t}{M_t} = 1$ is always a martingale;
- $M^t_{S_t} = \frac{S_t}{M_t} = D_t S_t$ is a martingale;
- the value of foreign money market $M^t_{M_t} = D_t M_t Q_t$ is a martingale.

To find this measure, we consider the first market price of risk equation: find a $\Theta(t)$ such that

$$\sigma_1(t) \Theta(t) = \alpha(t) - R_t. \quad (19)$$

Then we construct a process

$$\tilde{W}_t^1 = \int_0^t \Theta_1(u) du + W_t^1, \quad (20)$$

which allows us to rewrite (17) as

$$\frac{d(D_t S_t)}{D_t S_t} = \sigma_1(t) d\tilde{W}_t^1. \quad (21)$$

So under the risk-neutral measure, $\tilde{W}_t^1$ should be a standard Brownian motion.

Let us now consider the foreign money market account. First note that from (16) we have

$$dM^f_t = R^f_t M^f_t dt. \quad (22)$$
Using Itô’s product rule we have
\[ d(M_t^f Q_t) = M_t^f Q_t \left( (R_t^f + \gamma(t))dt + \sigma_2(t)\rho(t)dW_t^1 + \sigma_2(t)\sqrt{1-\rho^2(t)}dW_t^2 \right), \] (23)
and finally,
\[ d(D_t M_t^f Q_t) = D_t M_t^f Q_t \left( (R_t^f - R_t + \gamma(t))dt + \sigma_2(t)\rho(t)dW_t^1 + \sigma_2(t)\sqrt{1-\rho^2(t)}dW_t^2 \right). \] (24)

Now let us construct a process
\[ \hat{W}_t^2 = \int_0^t \Theta_2(u)du + W_t^2, \] (25)
so that we can rewrite (24) as
\[ d(D_t M_t^f Q_t) = \sigma_2(t)D_t M_t^f Q_t \left( \rho(t)d\hat{W}_t^1 + \sqrt{1-\rho^2(t)}d\hat{W}_t^2 \right). \] (26)

This requires that,
\[ \sigma_2(t) \left( \rho(t)\Theta_1(t) + \sqrt{1-\rho^2(t)}\Theta_2(t) \right) = R_t^f - R_t + \gamma(t). \] (27)

One can solve for \( \Theta_1(t) \) first using (19), and then solve for \( \Theta_2(t) \) using (27). Let us consider an exponential martingale
\[ Z_t = \exp \left( -\int_0^t \Theta_1(u)dW_u^1 - \int_0^t \Theta_2(u)dW_u^2 - \frac{1}{2} \int_0^t (\Theta_1(u))^2 du - \frac{1}{2} \int_0^t (\Theta_2(u))^2 du \right). \] (28)

Then the risk-neutral measure \( \hat{P} \) is defined by
\[ \left. \frac{d\hat{P}}{dP} \right|_{\mathcal{F}_t} = Z_t. \] (29)

And under \( \hat{P} \), \( \{\hat{W}_t^1\} \) and \( \{\hat{W}_t^2\} \) are independent standard Brownian motions. As we know, the stock price dynamics now becomes
\[ \frac{dS_t}{S_t} = R_t dt + \sigma_1(t)d\hat{W}_t. \] (30)

Moreover, the exchange rate \( Q_t \) satisfies
\[ \frac{dQ_t}{Q_t} = [R_t - R_t^f]dt + \sigma_2(t)\rho(t)d\hat{W}_t^1 + \sigma_2(t)\sqrt{1-\rho^2(t)}d\hat{W}_t^2 \] (31)
\[ = [R_t - R_t^f]dt + \sigma_2(t)\hat{W}_t^3, \] (32)
where
\[ \hat{W}_t^3 = \int_0^t \rho(u)d\hat{W}_u^1 + \int_0^t \sqrt{1-\rho^2(u)}d\hat{W}_u^2, \] (33)
is a standard Brownian motion under the risk-neutral measure \( \hat{P} \).\(^3\)

**Remark 1.** We notice from (33) that, the exchange rate \( Q_t \)’s drift is not \( R_t \) under the risk-neutral measure. This is very different from stocks. The reason is that, one unit of foreign currency can be invested in foreign money market to earn foreign interest rate, and this is effectively a dividend. The dividend of \( Q_t \) is paid at a continuous rate \( R_t^f \), which reduces the drift of \( Q_t \).

**Remark 2.** The drift of \( \frac{1}{Q_t} \) under the foreign risk-neutral measure (using \( D_t M_t^f Q_t \) as the numéraire) is \( R_t^f - R_t \). But it is not the case under the domestic risk-neutral measure. For details, check 9.3.4.

\(^3\)This is by Le\'vy’s characterization of Brownian motion.