Lecture 21: Interest Rate and Bonds

November 29, 2010

1 Interest rate models

Interest rate models are relevant when we want to incorporate interest rate risk for stock derivative pricing and interest rate derivative pricing. The traditional way in modeling interest rate is to model the short rate. In particular, let \( R_t \) be the instantaneous interest rate realized at time \( t \), then we assume \( \{ R_t \}_{t \geq 0} \) satisfies a stochastic differential equation

\[
dR_t = \beta(t, R_t)dt + \gamma(t, R_t)d\tilde{W}_t,
\]

where \( \tilde{W} \) is a standard Brownian motion under a risk-neutral measure \( \tilde{P} \).

By the definition of \( R_t \), we immediately know that, the money market account value process \( \{ M_t \} \) satisfies

\[
dM_t = R_t dt,
\]

so we have

\[
M_t = e^{\int_0^t R_u du}, \quad \text{and} \quad D_t = \frac{1}{M_t} = e^{\int_0^t R_u du}.
\]

Here \( D_t \) is the usual discount factor.

A zero-coupon bond is a contract promising to pay a certain “face” amount, which we take to be 1, at a fixed maturity \( T \). Prior to that, the bond makes no payments. We denote the time-\( t \) price of the bond by \( B(t, T) \). We can express it using the money market numéraire:

\[
B(t, T) = \frac{M_t}{M_T}\tilde{E}\{B(T, T) \mid \mathcal{F}_t\} = \frac{1}{D_t}\tilde{E}\{D_T \mid \mathcal{F}_t\},
\]

as \( B(T, T) = 1 \). If we have a model for the short rate \( R_t \), then we can proceed to have

\[
B(t, T) = \tilde{E}\{e^{-\int_t^T R_u du} \mid \mathcal{F}_t\}.
\]

We can define the yield of the bond between \( t \) and \( T \) as

\[
Y(t, T) = -\frac{1}{T-t}\log B(t, T), \quad \text{or} \quad B(t, T) = e^{-Y(t, T)(T-t)}.
\]

The yield \( Y(t, T) \) is the constant rate of continuously compounding interest between \( t \) and \( T \) that is consistent with the bond price \( B(t, T) \).

The computation of bond price in (5) is similar as before. Since \( \{ R_t \} \) is a Markov process, we can assume that there exists a function \( f(t, r) \) such that

\[
B(t, T) = f(t, R_t).
\]

Notice that under the risk-neutral measure \( \tilde{P} \), the discounted bond price \( D_t B(t, T) \) is a martingale, so the drift of \( d(D_t B(t, T)) \) must be zero. This leads to a PDE approach for bond pricing, which we illustrate below using the Hull-White interest rate model.

Example 1. In the Hull-White interest rate model, we assume that the short rate \( R_t \) satisfies

\[
dR_t = (a(t) - b(t) R_t)dt + \sigma(t) d\tilde{E}_t,
\]
where \( a(t), b(t), \sigma(t) \) are nonrandom positive functions of time. Let us consider the differential

\[
d(D_t f(t, R_t)) = f_t(t, R_t) dD_t + f_r(t, R_t) dR_t + \frac{1}{2} f_{rr}(t, R_t) (dR_t)^2
\]

\[
d(D_t f(t, R_t)) = D_t \left[ - R_t f(t, R_t) dt + \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial r} dR_t + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} (dR_t)^2 \right]
\]

\[
= D_t \left[ - R_t f(t, R_t) + \frac{\partial f}{\partial t} + (a(t) - b(t) R_t) \frac{\partial f}{\partial r} + \frac{1}{2} \sigma^2(t) \right] dt + \text{“martingale”}.
\]

By imposing zero drift in (9) we obtain the following PDE:

\[
f_t(t, r) + (a(t) - b(t) r)f_r(t, r) + \frac{1}{2} \sigma^2(t)f_{rr}(t, r) = rf(t, r).
\]

We assume the function \( f(t, r) \) has the following form:

\[
f(t, r) = e^{-r C(t, T) - A(t, T)},
\]

for some deterministic functions \( C \) and \( A \). If these two functions are found, then the yield is

\[
Y(t, T) = -\frac{1}{T-t} \log f(t, r) = \frac{1}{T-t} (C(t, T)r + A(t, T)),
\]

which is an affine function of \( r \). Using (11) we can rewrite (10) as

\[
\left[ (-C_t(t, T) + b(t)C(t, T) - 1) r - A_t(t, T) - a(t)C(t, T) + \frac{1}{2} \sigma^2(t)C^2(t, T) \right] f(t, r) = 0.
\]

Since (13) must hold for any \( r \), we must have

\[
-C_t(t, T) + b(t)C(t, T) - 1 = 0, \quad A_t(t, T) + a(t)C(t, T) - \frac{1}{2} \sigma^2(t)C^2(t, T) = 0.
\]

From (14) we can solve for \( C(t, T) \), and then from (15) we can solve for \( A(t, T) \). The results are given by

\[
C(t, T) = \int_t^T e^{-\int_t^s b(v) dv} ds,
\]

\[
A(t, T) = \int_t^T \left( a(s)C(s, T) - \frac{1}{2} \sigma^2(s)C^2(s, T) \right) ds.
\]

As a result, the time-\( t \) price of the bond is given by

\[
B(t, T) = e^{-R_t C(t, T) - A(t, T)}, \quad 0 \leq t \leq T.
\]

**Remark 1.** One can similarly derive the PDE for pricing options on a bond. For details refer to Example 6.5.3 in the textbook.

### 2 Forward measures

We incorporate interest rate risk in stock option pricing. Recall that the time-\( t \) value of a forward contract with maturity \( T \) and strike \( K \) is given by

\[
V_t = \frac{1}{D_t} \tilde{E} \{ D_T (S_T - K) | \mathcal{F}_t \} = S_t - K \frac{1}{D_t} \tilde{E} \{ D_T | \mathcal{F}_t \} = S_t - KB(t, T).
\]

The forward price of the stock at time \( t \) is the break-even strike price \( K \):

\[
\text{For}_S(t, T) = \frac{S_t}{B(t, T)}.
\]
We choose the zero-coupon bond maturing at $T$ to be our numéraire, the advantage of this is that we do not need to know the dependence between the interest rate and the stock price. In particular, to price a stock derivative that pays $V_T$ at maturity, we have using money-market numéraire that

$$V_t = \frac{1}{D_t} \tilde{E}\{D_T V_T | \mathcal{F}_t\},$$

(21)

which requires the joint distribution of $D_T$ and $V_T$. However, if we use bond price as the numéraire, and denote by $\tilde{P}^T$ the measure generated by the bond price, then we have

$$V_t = B(t, T) \tilde{E}^T\{V_T | \mathcal{F}_t\},$$

(22)

which just requires the distribution of $V_T$ under $\tilde{P}^T$.

Under the measure $\tilde{P}^T$, the process $\tilde{S}_t$ is a Brownian motion under $\tilde{P}^T$. We need the distribution of $\tilde{S}_t$. Now let us price a call option on $\tilde{S}_T$ using change of numéraire. The time-0 price is given by the risk-neutral pricing formula

$$V_0 = \tilde{E}\{D_T (S_T - K)^+ \} = \tilde{E}\{D_T (S_T - K) \mathbb{1}_{S_T > K} \}$$

$$= \tilde{E}\{D_T S_T \mathbb{1}_{S_T > K} \} - K \tilde{E}\{D_T \mathbb{1}_{S_T > K} \}$$

$$= \tilde{E}\{D_T S_T \mathbb{1}_{S_T > K} \} - K B(0, T) \tilde{E}^T\{ \mathbb{1}_{S_T > K} \}$$

$$= \tilde{E}\{D_T S_T \mathbb{1}_{S_T > K} \} - K B(0, T) \tilde{P}^T(\tilde{S}_T(T) > K).$$

(25)

Using (24) we obtain that

$$\tilde{P}^T(\tilde{S}_T(T) > K) = \tilde{P}^T\left(\sigma W_T^T - \frac{1}{2} \sigma^2 T > \log \frac{KB(0, T)}{S_0}\right)$$

$$= \tilde{P}^T\left(- \frac{W_T^T}{\sqrt{T}} < \frac{1}{\sigma \sqrt{T}} \left[\log \frac{S_0}{KB(0, T)} - \frac{1}{2} \sigma^2 T\right]\right)$$

$$= N(d_-(0)),$$

(26)

where

$$d_\pm(t) = \frac{1}{\sigma \sqrt{T - t}} \left[\log \frac{\tilde{S}_T(t, T)}{K} \pm \frac{1}{2} \sigma^2 (T - t)\right].$$

(27)

To compute $\tilde{E}\{D_T S_T \mathbb{1}_{S_T > K} \}$, we also use the idea of change of measure. This time we use the stock price as the numéraire, and denote by $\tilde{P}^S$ the measure generated by the stock price. Then we have

$$\tilde{E}\{D_T S_T \mathbb{1}_{S_T > K} \} = S_0 \tilde{E}\left\{\frac{D_T S_T}{S_0} \mathbb{1}_{S_T > K} \right\} = S_0 \tilde{E}^S\{ \mathbb{1}_{S_T > K} \} = S_0 \tilde{P}^S(\tilde{S}_T(T) > K).$$

(28)

We need the distribution of $\tilde{S}_T(T)$ under measure $\tilde{P}^S$. For this reason we check the dynamics of $1/\tilde{S}_T(t, T) = \frac{B(t, T)}{B(t, T)}$, which should be a martingale under $\tilde{P}^S$. We have from (23) that

$$d\left(\frac{1}{\tilde{S}_T(t, T)}\right) = -\frac{\sigma}{\tilde{S}_T(t, T)} dW_t^T + \frac{\sigma^2}{\tilde{S}_T(t, T)} dt = -\frac{\sigma}{\tilde{S}_T(t, T)}(-\sigma dt + dW_t^T).$$

(29)

Therefore, under $\tilde{P}^S$, the process $\tilde{W}_t^S = -\sigma t + \tilde{W}_t^T$ is a Brownian motion. And we can rewrite (24) to have

$$\frac{S_t}{B(t, T)} = \tilde{S}_T(t, T) = \frac{S_0}{B(0, T)} \exp\left(\sigma \tilde{W}_t^S + \frac{1}{2} \sigma^2 t\right).$$

(30)
As a result, we have

\[
\tilde{P}^S \left( \text{For}_{S}(T, T) > K \right) = \tilde{P}^S \left( -\sigma \tilde{W}_T^S - \frac{1}{2} \sigma^2 T < \log \frac{S_0}{KB(0, T)} \right)
\]

\[
= \tilde{P}^S \left( -\frac{\tilde{W}_T^S}{\sqrt{T}} < \frac{1}{\sigma \sqrt{T}} \left[ \log \frac{S_0}{KB(0, T)} + \frac{1}{2} \sigma^2 T \right] \right)
\]

\[
= N(d_+(0)). \tag{31}
\]

In conclusion, the call price at time 0 is given by

\[
V_0 = S_0 N(d_+(0)) - KB(0, T) N(d_-(0)). \tag{32}
\]

In general, one can similarly obtain that

\[
V_t = S_t N(d_+(t)) - KB(t, T) N(d_-(t)), \tag{33}
\]

or equivalently,

\[
\frac{V_t}{B(t, T)} = \text{For}_{S}(t, T) N(d_+(t)) - K N(d_-(t)). \tag{34}
\]