1 Forward rates

We fix a time-horizon $T$, say 50 years, and we consider all bonds that mature at or before $T$. The time-$t$ price of a bond maturing at $T \leq T$ is denoted by $B(t, T)$, which is assumed to be strictly less than one whenever $t < T$. Similar as using forward contract to lock the price of an asset in the future, bonds with various maturities can be used to lock interest rate for borrowing and lending at a later time. In particular, if we want to save money at time $T$, and withdraw the money at time $T + \delta$ for some $\delta > 0$, we can lock the interest rate now (time $t$):

- short a bond maturing at $T$, this generates $B(t, T)$ in our pocket;
- at the same time, long $\frac{B(t, T)}{B(t, T + \delta)}$ units of bonds maturing at $T + \delta$, this costs $\frac{B(t, T)}{B(t, T + \delta)} \times \frac{B(t, T)}{B(t, T + \delta)} = B(t, T)$.

Then at time $T$, we need to close the short position by paying one dollar. At time $T + \delta$, we receive $1 \times \frac{B(t, T)}{B(t, T + \delta)} \times \frac{B(t, T)}{B(t, T + \delta)} = B(t, T)$. So by using the above strategy, we will definitely receive an interest of $B(t, T) - 1$ during period $[T, T + \delta]$. Recall that the yield is the constant rate of continuous compounding interest that is consistent with the bond price. The yield that explains the above interest received at $T + \delta$ is

$$Y(T, T + \delta) = \frac{1}{\delta} \log \frac{B(t, T)}{B(t, T + \delta)},$$

or $B(t, T) = e^{\delta Y(T, T + \delta)}$. (1)

Since the bond maturing at $T + \delta$ is less expensive than the bond maturing at $T$, the above yield is always positive. This is a forward-looking yield. We define the forward rate at time $t$ for investing at time $T$ to be

$$f(t, T) = \lim_{\delta \to 0^+} \frac{1}{\delta} \log \frac{B(t, T)}{B(t, T + \delta)} = -\frac{\partial}{\partial T} \log B(t, T).$$

The forward rate can be interpreted as the instantaneous interest rate at time $T$ that can be locked in at the earlier time $t$.

The forward rate $f(t, T)$ can be regarded as a function of $T$, when $t$ is fixed. The graph of this function for all $t \leq T \leq T$ is called the forward curve. When the forward curve is known, we can recover the bond price $B(t, T)$ for all maturity $t \leq T \leq T$:

$$\int_t^T f(t, v) dv = -[\log B(t, T) - \log B(t, t)] = -\log B(t, t),$$

since $B(t, t) = 1$. Therefore, we have

$$B(t, T) = \exp \left( -\int_t^T f(t, v) dv \right) t \leq T \leq T.$$  (4)

Notice that (4) is different from the formula which we have seen in the last lecture. Last time we computed the bond price using short rate:

$$B(t, T) = \hat{E} \left\{ \exp \left( -\int_t^T R_v dv \right) \bigg| \mathcal{F}_t \right\}$$

(5)
The short rate \( R_v \) can only be realized at a later time \( v \), whereas the forward rate \( f(t, v) \) is already known at time \( t \leq v \). Obviously, we have
\[
R_t = f(t, t).
\] (6)

This is the instantaneous rate we can lock in at time \( t \) for borrowing at time \( t \).

On the other hand, we can build the forward curve at time \( t \) using all available bond prices \( B(t, T) \). Since not all maturities are available, we need to adopt various interpolation techniques (linear, or B-spline). However, our focus is not on this topic.

We will now proceed to model the dynamics of forward rate and bond price.

## 2 HJM model

Different from short rate models (Vasicek, Hull-White, CIR, etc), HJM model directly models the forward curve \( \{f(t, T)\}_{T \geq t} \). In particular, it is assumed that
\[
f(t, T) = f(0, T) + \int_0^t \alpha(u, T)du + \int_0^t \sigma(u, T)dW_u,
\] (7)
or in other words,
\[
df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t,
\] (8)
where \( \{W_t\} \) is a Brownian motion.

The coefficients \( \alpha(t, T) \) and \( \sigma(t, T) \) can be random, but when \( T \) is fixed, they are adapted in \( t \). This means, at time \( t \), we should be known \( \alpha(t, T) \) and \( \sigma(t, T) \).

In view of (4), to study the dynamics of bond price \( B(t, T) \), \( 0 \leq t \leq T \), we need
\[
d\left(-\int_t^T f(t, v)dv\right) = R_t dt - \int_t^T \left[\alpha(t, v)dt + \sigma(t, v)dW_v\right]dv
\] (9)
\[
= \left(R_t - \int_t^T \alpha(t, v)dv\right)dt - \left(\int_t^T \sigma(t, v)dv\right)dW_t.
\] (9)
Let us denote by
\[
\alpha^*(t, T) = \int_t^T \alpha(t, v)dv,
\] (10)
\[
\sigma^*(t, T) = \int_t^T \sigma(t, v)dv.
\] (11)
Then we have
\[
d\left(-\int_t^T f(t, v)dv\right) = (R_t - \alpha^*(t, T)) dt - \sigma^*(t, T)dW_t.
\] (12)

We use Itô’s lemma to (4), we have that
\[
\frac{dB(t, T)}{B(t, T)} = \left[R_t - \alpha^*(t, T) + \frac{1}{2}(\sigma^*(t, T))^2\right] dt - \sigma^*(t, T)dW_t.
\] (13)

## 3 No-arbitrage condition

HJM model models the whole forward curve using one Brownian motion. To ensure there is no arbitrage, or to ensure the existence of a risk-neutral measure, we have to check the consistency of the model for various maturities \( T \). To this effect, we check the differential of the discounted bond price \( D_tB(t, T) \):
\[
d(D_tB(t, T)) = -R_tD_tB(t, T)dt + D_tdB(t, T)
\] (14)
\[
= D_tB(t, T)\left[\left(-\alpha^*(t, T) + \frac{1}{2}(\sigma^*(t, T))^2\right) dt - \sigma^*(t, T)dW_t\right].
\]
To construct a risk-neutral measure if possible, we need to find a process $\Theta_t$ so that the terms in the above brackets can be expressed as

$$-\sigma^*(t,T)\Theta_t dt + dW_t,$$

and we can apply Girsanov theorem to change the measure, so that under the new measure, we have a Brownian motion which is given by

$$\tilde{W}_t = \int_0^t \Theta_t dt + W_t.$$  

As a result, we ask for the existence of a $\Theta_t$ such that

$$-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2 = -\sigma^*(t,T)\Theta_t.$$  

Taking derivative of both sides with respect to $T$, we have the following consistency equation

$$-\alpha(t,T) + \sigma^*(t,T)\sigma(t,T) = -\sigma(t,T)\Theta_t.$$  

If the volatility $\sigma(t,T)$ is always positive whenever it is defined, then we must have

$$\Theta_t = \frac{\alpha(t,T)}{\sigma(t,T)} - \sigma^*(t,T).$$  

Note that each term on the right hand side can change with $T$.

4 HJM under risk-neutral measure

Let assume that (18) is solvable, so there exists a risk-neutral measure $\tilde{P}$. Under the risk-neutral measure $\tilde{P}$, we have from (7), (16) and (18) that

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}_t = \sigma(t,T)d\tilde{W}_t.$$  

Moreover, the discounted bond price satisfies

$$\frac{d(D_tB(t,T))}{D_tB(t,T)} = -\sigma^*(t,T)[\Theta_t dt + dW_t] = -\alpha(t,T)d\tilde{W}_t.$$  

It follows that the bond price satisfies

$$\frac{dB(t,T)}{B(t,T)} = R_t dt - \sigma^*(t,T)d\tilde{W}_t.$$  

Thus, we can solve for $B(t,T)$:

$$B(t,T) = B(0,T) \exp \left( \int_0^t R_u du - \int_0^t \sigma^*(u,T)d\tilde{W}_u - \frac{1}{2} \int_0^t (\sigma^*(u,T))^2 du \right)$$  

$$= \frac{B(0,T)}{D_t} \exp \left( - \int_0^t \sigma^*(u,T)d\tilde{W}_u - \frac{1}{2} \int_0^t (\sigma^*(u,T))^2 du \right).$$

5 Relation to affine-yield models

In the last lecture we derived the bond price using a short rate model - the Hull-White model. In particular, the short rate $R_t$ is assumed to satisfy the following stochastic differential equations:

$$dR_t = (a(t) - b(t)R_t)dt + \sigma(t)d\tilde{W}_t.$$
Then the time-
t price of a bond maturing at $T$ is given by

$$B(t, T) = \tilde{E}\{e^{-\int_t^T R_u du|F_t}\} = e^{-R_t C(t, T) - A(t, T)},$$

(25)

where

$$C(t, T) = \int_t^T e^{-\int_t^T b(v)dv},$$

(26)

$$A(t, T) = \int_t^T \Big( a(s) C(s, T) - \frac{1}{2} \sigma^2(s) C^2(s, T) \Big) ds.$$  

(27)

We would like to verify (20) for the Vasicek model, which is the Hull-White model with constant $a, b$ and $\sigma$. We then have from (26) and (27) that

$$C(t, T) = \frac{1}{b} \left( 1 - e^{-b(T-t)} \right),$$

(28)

$$A_t(t, T) = -aC(t, T) + \frac{1}{2} \sigma^2 C^2(t, T).$$

(29)

Moreover, we compute the forward rate

$$f(t, T) = -\frac{\partial}{\partial T} B(t, T) = R_t \frac{\partial}{\partial T} C(t, T) + \frac{\partial}{\partial T} A(t, T).$$

(30)

Thus,

$$df(t, T) = \frac{\partial}{\partial T} C(t, T) dR_t + R_t \frac{\partial}{\partial T} C_t(t, T) dt + \frac{\partial}{\partial T} A_t(t, T) dt$$

$$= \left[ \frac{\partial}{\partial T} C(t, T) \left( a - bR_t \right) + R_t \frac{\partial}{\partial T} C_t(t, T) + \frac{\partial}{\partial T} A_t(t, T) \right] dt + \sigma \frac{\partial}{\partial T} C(t, T) d\tilde{W}_t$$

$$= \left[ e^{-b(T-t)} (a - bR_t) + bR_t e^{-b(T-t)} + \left( \frac{\sigma^2}{b} - a \right) e^{-b(T-t)} - \frac{\sigma^2}{b} e^{-2b(T-t)} \right] dt + \sigma e^{-b(T-t)} d\tilde{W}_t$$

$$= \frac{\sigma^2}{b} (e^{-b(T-t)} - e^{-2b(T-t)}) dt + \sigma e^{-b(T-t)} d\tilde{W}_t.$$  

(31)

From (31) we know that

$$\sigma(t, T) = \sigma e^{-b(T-t)}.$$  

(32)

And (20) now becomes

$$\frac{\sigma^2}{b} (e^{-b(T-t)} - e^{-2b(T-t)}) = \sigma(t, T) \sigma^*(t, T)$$

$$= \sigma e^{-b(T-t)} \int_t^T e^{-b(v-t)} dv$$

$$= \frac{\sigma^2}{b} (e^{-b(T-t)} - e^{-2b(T-t)}).$$  

(33)

So (20) is verified. Vasicek model has no arbitrage under HJM framework.