Lecture 23: Poisson Process

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1 Drawbacks of continuous models

We have only considered continuous-time models with continuous sample paths so far. But there is obviously some drawbacks of this setting. For example, market crash or big downward price jumps (“flash crash”) caused by news or uncontrolled market impact, defaultable bonds, all cannot be captured properly using continuous models. To address the above problem in derivative pricing, we consider jump models. In this lecture we study the simplest jump model, the Poisson process.

2 Definition of Poisson process

Poisson process has been historically used to model the number of clients waiting in line during some period, or the number of buses that will arrive in the next two hours. This number is a random variable that can be modeled by a Poisson process. Let us denote this time by $\tau$. We will assume that $\tau$ is an exponentially distributed random variable. That is, for any $t > 0$, the probability that the bus will arrive after $t$ is given by

$$P(\tau \geq t) = e^{-\lambda t}. \quad (1)$$

The reason why exponential distribution is appropriate for $\tau$ is because this distribution has memory-less property. In particular, we have

$$P(\tau > t + s | \tau > s) = \frac{P(\tau > t + s, \tau > s)}{P(\tau > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(\tau > t). \quad (2)$$

In words, this means, if the bus has not arrived in the past $s$ minutes, then we do not lose any information if we forget about the past.

Let us denote by $N(t)$ the number of buses that will arrive in $[0, t]$, then clearly, for any integer $k \geq 1$,

$$\{N(t) \geq k\} = \{\tau_1 + \tau_2 + \ldots + \tau_k \leq t\}, \quad (3)$$

where $\tau_i$’s are independent identically distributed exponential random variables with intensity $\lambda$. We can use (1) and (3) to derive the tail probability of $N(t)$:

$$P(N(t) \geq k) = \int \ldots \int_{t_i \geq 0, \sum_{i=1}^{k} t_i \leq t} \prod_{i=1}^{k} \left(-\frac{\partial}{\partial t_i} P(\tau > t_i)\right) dt_i$$

$$= \int \ldots \int_{t_i \geq 0, \sum_{i=1}^{k} t_i \leq t} \lambda^k e^{-\lambda(t_1 + t_2 + \ldots + t_k)} dt_i$$

$$= \int \ldots \int_{t_i \geq 0, \sum_{i=1}^{k-1} t_i \leq t} \lambda^{k-1} e^{-\lambda(t_1 + t_2 + \ldots + t_{k-1})} dt_1 \cdots \left(\lambda e^{-\lambda t} dt_k \right)$$

$$= \int \ldots \int_{t_i \geq 0, \sum_{i=1}^{k-1} t_i + t_k \leq t} \lambda^{k-1} e^{-\lambda(t_1 + t_2 + \ldots + t_{k-1})} [1 - e^{-\lambda(t-t_1 - t_2 - \ldots - t_{k-1})}] dt_1 dt_2 \ldots dt_{k-1}$$

$$= P(N(t) \geq k - 1) - \lambda^{k-1} e^{-\lambda t} \int \ldots \int_{t_i \geq 0, \sum_{i=1}^{k-1} t_i \leq t} dt_1 dt_2 \ldots dt_{k-1}$$

$$= P(N(t) \geq k - 1) - \frac{1}{(k-1)!} (\lambda t)^{k-1} e^{-\lambda t}. \quad (4)$$
From (4) we obtain that
\[ P(N(t) = k) = \frac{1}{(k-1)!} (\lambda t)^{k-1} e^{-\lambda t}. \] (5)

In other words, the distribution of the random variable \( N(t) \) is given as
\[ P(N(t) = k) = \frac{1}{k!} (\lambda t)^k e^{-\lambda t}, \quad k \geq 0, \ t > 0. \] (6)

One can immediately obtain the distribution of \( N(t+s) - N(s) \) given \( F_s \). In particular, using memory-less property of exponential random variables, we have that
\[ N(t+s) - N(s) \mid F_s \overset{i.i.d.}{=} N(t-s). \] (7)

Moreover, let \( 0 = t_0 < t_1 < \ldots < t_n \) be an increasing sequence, then the increments
\[ N(t_1) - N(t_0), \ N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1}) \]
are stationary and independent, and
\[ P(N(t_{j+1}) - N(t_j) = k) = \frac{(\lambda(t_{j+1} - t_j))^k}{k!} e^{-\lambda(t_{j+1} - t_j)}. \] (8)

## 3 Mean and variance of Poisson increments

Since we know that
\[ P(N(t) - N(s) = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \] (9)
the mean of increment \( N(t) - N(s) \) can be computed directly. In particular, we have
\[
E(N(t) - N(s)) = \sum_{k=0}^{\infty} k \cdot \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}
\]
\[ = \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{(\lambda(t-s))^k}{k!}
\]
\[ = \lambda(t-s) e^{-\lambda(t-s)} \cdot e^{\lambda(t-s)} = \lambda(t-s). \] (10)

Similarly, we have
\[
E(N(t) - N(s))^2 = \sum_{k=0}^{\infty} k^2 \cdot \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}
\]
\[ = \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=0}^{\infty} (k+1) \frac{(\lambda(t-s))^k}{k!}
\]
\[ = \lambda^2 (t-s)^2 e^{-\lambda(t-s)} \cdot \sum_{k=0}^{\infty} \frac{(\lambda(t-s))^k}{k!} + \lambda(t-s) e^{-\lambda(t-s)} \cdot \lambda(t-s) e^{-\lambda(t-s)} \cdot \sum_{k=0}^{\infty} \frac{(\lambda(t-s))^k}{k!}
\]
\[ = \lambda^2 (t-s)^2 e^{-\lambda(t-s)} \cdot e^{\lambda(t-s)} + \lambda(t-s) e^{-\lambda(t-s)} \cdot e^{\lambda(t-s)}
\]
\[ = \lambda^2 (t-s)^2 + \lambda(t-s). \] (11)

As a result, we see that the variance is the same as the mean:
\[ \text{Var}(N(t) - N(s)) = E(N(t) - N(s))^2 - (E(N(t) - N(s)))^2 = \lambda(t-s). \] (12)
4 Martingale properties

Similar as Brownian motion, Poisson process has stationary and independent increments. However, it is clearly seen that Poisson process is not a martingale anyway. Poisson process is a non-decreasing jump process. If we add a negative drift to it, then it may be a martingale though. In particular, we consider the following process

\[ M(t) = N(t) - \lambda t. \] (13)

We want to show that \( \{M(t)\} \) is a martingale. Indeed, for any \( t > s \), we have

\[
E(M(t) - M(s)|\mathcal{F}_s) = E(N(t) - N(s)|\mathcal{F}_s) - \lambda(t - s)
= E(N(t - s)) - \lambda(t - s) = \lambda(t - s) - \lambda(t - s) = 0.
\] (14)

Therefore,

\[ E(M(t)|\mathcal{F}_s) = M(s). \] (15)

Recall that if \( \{W_t\} \) is a Brownian motion, then we can construct an exponential martingale:

\[ Z(t) = \exp\left(\gamma W_t - \frac{1}{2}\gamma^2 t\right), \forall \gamma. \] (16)

One proof of this fact is based on Itô’s formula.

Now we have \( M(t) = N(t) - \lambda t \) as a martingale. We can similar construct an exponential martingale:

\[ Y(t) = \exp\left(\gamma N(t) - \lambda(1 - e^\gamma)t\right), \forall \gamma. \] (17)

One can prove the above process is a martingale by noticing that

\[
dY(t) = Y(t) - Y(t^-) = Y(t^-)\left( e^{\gamma(N(t^-) - N(t^-))} \cdot e^{-\lambda(1 - e^\gamma)(t^- - t)} - 1 \right)
= Y(t^-)\left( (e^\gamma - 1)dN(t) + 1|1 - \lambda(1 - e^\gamma)dt| - 1 \right)
= Y(t^-)(e^\gamma - 1)(dN(t) - \lambda dt),
\] (18)

which is a martingale.\footnote{We also used the fact that \( dN(t)dt = 0 \), since jumps are very infrequent, and the set of all jump times has zero Lebesgue measure.}

This enables us to consider jump-diffusion type model like

\[ S_t = \exp\left( \sigma W_t + \left( r - \lambda(1 - e^\gamma) - \frac{1}{2}\sigma^2 \right)t + \gamma N(t) \right). \] (19)

And the discounted process \( \{D_t S_t\} \) is a martingale with a diffusion part and a jump part.