Lecture 24: Introduction to credit risk

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1 Introduction

Credit derivatives are financial securities that facilitate transfer of credit risk from one agent to another. Because credit defaults are very rare event, the model used in the credit derivative pricing is usually very different from the models used in other fields. In this lecture we will briefly give a short introduction to various approaches. They are ratings transition matrix, structural models, reduced form models.

2 Rating transaction matrix

The Altman Z-score is a way to use financial statements to gauge credit risk. Most trading firms let ratings agencies do this type of analysis, and rely on firms ratings as rough-and-ready prediction of historical (not risk-neutral) default probabilities. A convenient way to work with ratings data is through published ratings transition matrix, typically spanning a period of 1 year. Here is a typical example (from Moody’s data)

\[
M(1) = \begin{array}{cccccccc}
\text{Grade} & \text{Aaa} & \text{Aa} & \text{A} & \text{Baa} & \text{Ba} & \text{B} & \text{Default} \\
\text{Aaa} & 91.027\% & 6.998\% & 1.003\% & 0.650\% & 0.238\% & 0.059\% & 0.025\% \\
\text{Aa} & 7.003\% & 85.823\% & 5.997\% & 0.704\% & 0.266\% & 0.147\% & 0.060\% \\
\text{A} & 2.000\% & 10.865\% & 80.251\% & 6.159\% & 0.397\% & 0.238\% & 0.090\% \\
\text{Baa} & 0.299\% & 0.999\% & 3.798\% & 90.624\% & 3.680\% & 0.400\% & 0.200\% \\
\text{Ba} & 0.151\% & 0.902\% & 3.701\% & 7.002\% & 72.855\% & 12.889\% & 2.500\% \\
\text{B} & 0.007\% & 0.047\% & 0.217\% & 0.405\% & 8.898\% & 78.849\% & 11.576\% \\
\text{Default} & 0.000\% & 0.000\% & 0.000\% & 0.000\% & 0.000\% & 0.000\% & 100.000\% \\
\end{array}
\]

Notice that Default is an absorbing state: once you reach it, you will be stuck forever. By the properties of transition matrices, we can find the n-year transition matrix by matrix multiplication:

\[
M(n) = [M(1)]^n. \tag{1}
\]

By looking at the right-most column in the resulting matrix, we can get default estimates at all annual horizons, for all ratings.

Remark 1. In linear algebra we know that there exists a unique matrix \( G \) such that

\[
M(1) = \exp(G) = \sum_{k=0}^{\infty} \frac{1}{k!} G^k. \tag{2}
\]

Using this \( G \) we are able to obtain the transition matrix at all horizons, not just annual ones:

\[
M(t) = \exp(tG), \forall t > 0. \tag{3}
\]

3 Structural models

We consider the simple one-period structural model: a firm has assets \( V \), debt \( B \) and equity \( S \). Then we have

\[
V(t) = B(t) + S(t). \tag{4}
\]
Let us assume that the firm debt consists of T-maturity zero coupon bonds with notional D. At time T, the debt holders will receive

\[ B(T) = \min\{D, V(T)\} = V(T) - (V(T) - D)^+. \] (5)

The equity holders will receive

\[ S(T) = V(T) - B(T) = (V(T) - D)^+. \] (6)

In the simple Merton approach we assume that the assets pay no dividends and satisfy a simple risk-neutral diffusion

\[ \frac{dV(t)}{V(t)} = rdt + \sigma_V d\tilde{W}_t. \] (7)

Then using Black-Scholes, we have

\[ S(0) = V(0)N(d_+) - De^{-rT}N(d_-), \]
\[ B(0) = V(0) - S(0), \] (8) (9)

where

\[ d_\pm = \frac{1}{\sigma_V \sqrt{T}} \left[ \log \frac{V(0)}{D} + \left( r \pm \frac{1}{2} \sigma_V^2 \right) T \right]. \] (10)

In the Merton model, the risk-neutral default probability over the horizon \([0, T]\) is

\[ P(\text{default}) = P(V(T) < D) = N(-d_-). \] (11)

One extension over the above setting is to consider the default event as the first time that the asset value \(V\) drops below \(D\). Then the default probability over the horizon \([0, T]\) is

\[ P(\text{default}) = P(\tau_V \leq T), \] (12)

where \(\tau_V\) is the first hitting time of the asset value process \(V\) to the fixed barrier \(D\). While this extension allows for a full term structure of default probabilities, it is typically not realistic. For instance, it can be demonstrated that the credit spread term structure generated by this model has collapsing spreads for short maturities. This is essentially a consequence of the fact that defaults generated by diffusion processes are “predictable” with slowly deteriorating firm assets, rather than “surprising” (e.g., as in Enron).

4 Reduced-form models: Poisson processes

We consider the first jump of a Poisson process to be the default time. In particular, we have

\[ \tilde{P}(\tau > t) = e^{-\lambda t}, \] (13)

where \(\tilde{P}\) is a risk-neutral probability.

4.1 Pricing risky bond

Now we consider a defaultable unit bond maturing at time \(T\). We denote by \(B(t, T)\) the time-\(t\) price of the bond, and denote the survival probability of the issuer of the bond by

\[ X(t, T) = \tilde{P}(\tau > T|\mathcal{F}_t) = \mathbb{1}_{\tau > t} e^{-\lambda(T-t)}. \] (14)

Moreover, we denote the default-free unit bond price at time \(t\) by \(P(t, T)\). It is known that, \(\{R_u\}\) is the (short) interest rate, then

\[ P(t, T) = \tilde{E}(e^{-\int_t^T R_u du}|\mathcal{F}_t). \] (15)
As the defaultable bond pays one dollar only if the issuer has not defaulted until \( T \), the risk-neutral pricing formula suggests that

\[
B(t, T) = M_t \tilde{E} \left( \mathbb{1}_{T > T} \frac{1}{M_T} \big| \mathcal{F}_t \right) = \frac{1}{D_t} \tilde{E} (\mathbb{1}_{T > T} D_T | \mathcal{F}_t).
\]  

(16)

Assuming that the interest rate process \( \{ R_t \} \) is independent of the indicator \( \mathbb{1}_{T > T} \), then from (14), (15) and (16), we have

\[
B(t, T) = \mathbb{1}_{T > t} \tilde{E} (e^{-\int_t^T (R_u + \lambda) du} | \mathcal{F}_t) = \mathbb{1}_{T > t} P(t, T) X(t, T).
\]  

(17)

In general, if we know the distribution of \( \tau \) under the forward measure \( \tilde{P}_T \), which is the measure generated by the default-free zero-coupon bonds, then we have

\[
B(t, T) = \mathbb{1}_{T > t} \tilde{E} (\mathbb{1}_{\tau > t} \tilde{P}_T (\tau > T | \tau > t)) = \mathbb{1}_{T > t} \tilde{P}_T (\tau > t) X(t, T).
\]  

(18)

The above identity holds without assuming the independence between \( \mathbb{1}_{T > T} \) and the interest rate process.

4.2 Pricing of recovery amounts

We just looked at a risky discount bond, which paid nothing whatsoever in case a default takes place. In reality, upon default the assets of the firm are divided among the various claim holders. As a consequence claim holders will recover a certain fraction of what is owed to them (senior claim holders will recover more than junior ones)

To incorporate this into pricing, we need to figure out the value of receiving \( R \) at time \( \tau \). That is, we need to find the expectation.

\[
C(0, R) = \tilde{E} \left( R \exp \left( - \int_0^\tau R_u du \right) \mathbb{1}_{\tau \leq T} \right) = \int_0^T \tilde{E} \left( R \exp \left( - \int_0^s R_u du \right) \mathbb{1}_{\tau \in [s, s+ds]} \right) ds.
\]  

(19)

where \( R \) is the recovery rate (usually 40\%). As we know that for any \( t \geq s \),

\[
\tilde{P}(\tau \geq t | \tau > s) = e^{-\lambda(t-s)},
\]  

(20)

and

\[
\tilde{P}(\tau \in [t, t+dt] | \tau > s) = \lambda e^{-\lambda(t-s)} dt.
\]  

(21)

We again assume that \( tau \) is independent of the interest rate process, then we have

\[
\tilde{E} \left( R \exp \left( - \int_0^s R_u du \right) \mathbb{1}_{\tau \in [s, s+ds]} \right) = \tilde{E} \left( R \exp \left( - \int_0^s R_u du \right) \mathbb{1}_{\tau \in [s, s+ds]} \bigg| \tau > s \right) \tilde{P}(\tau > s) = \tilde{E} \left[ R \exp \left( - \int_0^s R_u du \right) \right] \lambda e^{-\lambda s} ds.
\]  

(22)

Finally, we have

\[
C(0, R) = \tilde{E} \left( \int_0^T \lambda R \exp \left( - \int_0^s (R_u + \lambda) du \right) ds \right).
\]  

(23)

4.3 Pricing of corporate bonds

Consider now a corporate bond \( B_c \) that promises to pay a coupon \( c \) on a schedule \( \{T_i\}_{i=1}^N \), and returns to the bond investor the notional ($1 for simplicity) at final maturity \( T_N \).

The present value of the coupon payment promised at time \( T_i \) will then be

\[
c \delta_i \tilde{E} (\mathbb{1}_{T > T_i} e^{-\int_0^{T_i} R_u du}) = c \delta_i B(0, T_i).
\]  

(24)
where $B$ is the price of a defaultable zero-coupon bond discussed in the previous section, and $\delta_i$ is the day-count fraction for period $[T_{i-1}, T_i)$ ($\approx T_i - T_{i-1}$).

The present value of the coupon the return of notional ($1$) at time $T_N$ is just $B(0, T_N)$.

The value of receiving recovery $R$ at the time of default (if $\tau < T_N$) is $C(0, R)$, which is given in (23).

Adding up all the components gives us the present value of a corporate bond as

$$B_c(0) = c \sum_{i=1}^{N} \delta_i B(0, T_i) + B(0, T_N) + C(0, R).$$

(25)

### 4.4 Pricing of credit default swaps (CDS)

CDS is the most important credit derivative. It is used by the holders of risky bonds to get back the loss from a default of the issuer. Under the terms of a CDS, one party (the “protection buyer”) pays a running coupon to another (the “protection seller”), in return for protection against default on some firm.

Specifically, if default on the specified firm takes place before the final CDS maturity, the protection seller pays the protection buyer $1 - R$ at the time of default. $R$ is the observed recovery rate on senior debt of the firm in question. At time of default, coupon payments cease.

The value of the protection buyers coupon payment stream (“the coupon leg” or “the fixed leg”) is

$$PV_{coup}(0) = c \sum_{i=1}^{N} \delta_i B(0, T_i), \quad \delta_i \approx T - i - T_{i-1}.$$  

(26)

The value the protection seller’s payment (“the asset leg” or “the floating leg”) is

$$PV_{prot}(0) = C(0, 1 - R).$$

(27)

So the total value of the CDS (as seen from the protection buyer) is

$$PV_{CDS}(0) = PV_{prot}(0) - PV_{coup}(0) = C(0, 1 - R) - c \sum_{i=1}^{N} \delta_i B(0, T_i).$$

(28)

The breakeven coupon $c$ is called the par CDS spread, i.e,

$$c_{par} = \frac{C(0, 1 - R)}{\sum_{i=1}^{N} \delta_i B(0, T_i)}.$$  

(29)

When both the intensity $\lambda$ and interest rate $R_t$ are roughly constants, a good approximation is

$$c_{par} \approx \lambda (1 - R).$$

(30)