Lecture 1 – Part II

Interest Rate Fundamentals

Topics in Quantitative Finance: Inflation Derivatives

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Fundamentals of Interest Rates

In part II of this lecture we will consider fundamental concepts of interest rates, including risk-free rate and money market account, zero coupon bonds, spot and forward interest rates, day-count and compound conventions, and general interest rate curves. We will also review standard market conventions for bond and swap pricing, as well as pricing of interest rate caps, swaptions and bond options.

The Money Market Account and the Short Rate

The first definition we will consider is the (continuously compounded) bank account, or money market account. A money-market account represents a (locally) risk-less investment, with profits accruing continuously at the risk-free rate prevailing in the market at any instant.

**Money Market Account:** We define $B(t)$ to be the value of (continuously compounded) bank account at time $t \geq 0$. We assume that $B(0) = 1$ and that the bank account evolves according to the following differential equation:

$$ dB(t) = r_s B(t) dt, \quad B(0) = 1 $$  \hspace{1cm} (EQ 1)

where $r_s$ is a positive (possibly stochastic) real-valued process. As a consequence

$$ B(t) = \exp\left( \int_0^t r_s ds \right) $$  \hspace{1cm} (EQ 2)

The above definition tell us that investing a unit of currency at time 0 yields at time $t$ the value in Equation 2, and $r_t$ is the instantaneous rate at which the bank account accrues. This instantaneous rate is usually referred to as instantaneous spot rate or, more simply, the short rate. In fact, in an arbitrary small time interval $[t, t+\Delta t]$ we have:

$$ \frac{B(t+\Delta t) - B(t)}{B(t)} = r_t \Delta t $$  \hspace{1cm} (EQ 3)

Hence, the bank account grows at a rate $r(t)$. It follows that the value at time $t$ of a unit of currency payable at time $T$ is given by $B(t) / B(T)$. This leads to the following definition:
**Stochastic Discount Factor**: The (stochastic) discount factor $D_t(T)$ between two times $t$ and $T$ is the amount at time $t$ that is equivalent to one unit of currency payable at time $T$, and is given by:

$$D_t(T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s ds\right)$$  \hspace{1cm} (EQ 4)

When applying the Black-Scholes to equity or foreign-exchange (FX) markets, $r$ is assumed to be deterministic function of time, so that the bank account and the discount factors at any future time are deterministic functions of time. However, when dealing with interest-rate products, the probabilistic nature of $r$ is important since it affects the nature of the bank-account numeraire $B$, and hence the pricing of interest-rate products. It is therefore necessary in those contexts to drop the deterministic setup and to start modeling the evolution of $r$ in time through a *stochastic* process.

**Zero-Coupon Bonds and Spot Interest Rates**

**Zero-Coupon Bond**: A $T$-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time $T$ with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t,T)$. Clearly $P(T,T) = 1$ for all $T$.

If the current time is $t$, a zero-coupon bond for maturity $T$ is a contract that establishes the present value of one unit of currency to be paid at time $T$, the maturity of the contract. If rates $r_t$ are deterministic, then $D$ is deterministic as well and necessarily $D(t,T) = P(t,T)$ for all $(t,T)$.

However, if rates are stochastic, $D(t,T)$ is a random quantity at time $t$ depending on the future evolution of $r$ between $t$ and $T$. Since the zero-coupon bond price, being the time $t$-value of a contract with payoff at time $T$, $P(t,T)$, has to be known (deterministic) at time $t$. We will see that $P(t,T)$ can be viewed as expectation of the random variable $D(t,T)$ under a particular probability measure (risk-neutral measure).

**Time to Maturity**: The time to maturity $T - t$ is the amount of time (in years) from the present time $t$ to the maturity $T > t$.

The definition makes clear and definite sense when $t$ and $T$ are real numbers associated with two instances of time. However, if $t$ and $T$ denote two dates expressed as day/month/year, say $D_1 = (d_1, m_1, y_1)$ and $D_2 = (d_2, m_2, y_2)$, we need to define the amount of time between the two dates of number of days (or years) between them. This choice, however, is not unique, and the market evaluates the time between $t$ and $T$ in different ways. Indeed, the number of days between $D_1$ and $D_2$ is calculated according to the relevant market convention, which tells us how to count days (day count convention), whether to include holidays (holiday conventions) and so on.
**Day-count Convention / Year Fraction:** We denote by $\tau(t, T)$ the chosen time measure between $t$ and $T$, which is usually referred as the year fraction between $t$ and $T$. When $t$ and $T$ are less than one day, $\tau(t, T)$ is to be interpreted as $T - t$ (in years).

Otherwise we require a choice of *day-count convention*. Here are some examples of day-count conventions:

**Actual / 365 Convention (ACT/365):** With this convention the year is 365 days long, and the year fraction between two dates is the actual number of days between them divided by 365. If we denote by $D_2 - D_1$ the actual number of days between the two dates above, we have the year fraction in this case as

$$\frac{D_2 - D_1}{365}$$

For example, the year fraction between January 4, 2000 and July 4, 2000 is $182 / 365 = 0.49863$

**Actual / 360 Convention (ACT/360):** A year in this case is assumed to be exactly 360 days long. The corresponding fraction is

$$\frac{D_2 - D_1}{360}$$

Therefore the year fraction between January 4, 2000 and July 4, 2000 is $182 / 360 = 0.50556$

**30 / 360 Convention:** With this convention, months are assumed to be exactly 30 days long and years are assumed 360 days long. The year fraction in this case between the dates is defined by:

$$\frac{\max(30 - d_1, 0) + \min(d_2, 30) + 360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)}{360}$$

For example, the year fraction between January 4, 2000 and July 4, 2000 is now:

$$\frac{(30 - 4) + 4 + 360 \times 0 + 30 \times 5}{360} = 0.5$$

**Actual / Actual Convention (ACT/ACT):** In this case the year fraction between two dates is the actual number of days between them divided by number of days in a year beginning on a *reference date*, normally chosen to be equal to the earlier of the two dates. Setting the reference date to $D_3 = (d_1, m_1, y_1 + 1)$. The year fraction in this case between the dates is defined by:
\[ \frac{D_2 - D_1}{D_3 - D_1} \]

Therefore the year fraction between January 4, 2000 and July 4, 2000 is \( 182 / 366 = 0.497268 \)

As already hinted at above, adjustments may be included in the conventions in order to leave out holidays. These are generally referred to as holiday conventions. If a date is a holiday it can be replaced with the first working date following it (following), or possibly when the following working date falls in a new month replaced with the previous working date (modified following).

There are also month-end conventions which affect treatment of months with 28, 29 or 31 days in different variants of 30/360 convention. These additional conventions can materially impact the evaluation of year-fraction, and are important consideration in fixed-income markets.

In moving from zero-coupon bond prices to interest rates, we need to know another fundamental feature of the rates, the compound convention to be applied in the rate definition. We will now define different compounding conventions in achieving this.

**Continuously-compounded Spot Interest Rate**: The continuously-compounded spot interest rate (or continuously compounded zero-coupon rate) prevailing at time \( t \) for the maturity \( T \) is denoted by \( R(t,T) \) and is the constant rate at which an investment of \( P(t,T) \) units of currency at time \( t \) accrues continuously to yield a unit of amount of currency at time \( T \). In formulas:

\[
R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)} \quad (\text{EQ 5})
\]

The zero-coupon bond price can therefore be expressed in terms of the continuously-compounded spot rate (zero-coupon rate) as

\[
P(t,T) = e^{-R(t,T)\tau(t,T)} \quad (\text{EQ 6})
\]

**Simply-compounded Spot Interest Rate**: The simply compounded spot interest rate (or simply-compounded zero-coupon rate) prevailing at time \( t \) for maturity \( T \) is denoted \( L(t,T) \) and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from \( P(t,T) \) units of currency at time \( t \), when accruing is proportional to the investment time. In formulas:

\[
L(t,T) = \frac{1 - P(t,T)}{\tau(t,T)P(t,T)} \quad (\text{EQ 7})
\]
The market LIBOR rates are *simply-compounded* rates, which motivates the notation $L$ and are typically linked to zero-coupon bond prices by the Actual/360 day-count convention for computing $\tau(t,T)$.

It follows that zero-coupon bond prices can be expressed in terms of $L$ as

$$P(t,T) = \frac{1}{1 + \tau(t,T)L(t,T)}$$  \hspace{1cm} (EQ 8)

**Annually-compounded Spot Interest Rate**: The *annually-compounded spot interest rate* (or *annually compounded zero-coupon rate*) prevailing at time $t$ for maturity $T$ is denoted by $Y(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of unit of currency at maturity, starting from $P(t,T)$ units of currency at time $t$, reinvesting the proceeds *once a year*. In formulas

$$Y(t,T) = \frac{1}{[P(t,T)]^{1/\tau(t,T)}} - 1$$  \hspace{1cm} (EQ 9)

This implies that bond prices can be expressed in terms of annually-compounded rates as

$$P(t,T) = \frac{1}{(1 + Y(t,T))^{\tau(t,T)}}$$  \hspace{1cm} (EQ 10)

A straightforward extension of annual compounding leads to the following definition, which is based on reinvesting $k$ times per year.

**k-Times-Per-Year Compounding Spot Interest Rate**: The *$k$-times-per-year -compounded spot interest rate* (or *$k$-times-per-year -compounded zero-coupon rate*) prevailing at time $t$ for maturity $T$ is denoted by $Y^k(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of unit of currency at maturity, starting from $P(t,T)$ units of currency at time $t$, when reinvesting the proceeds $k$-times a year. In formulas

$$Y^k(t,T) = \frac{k}{[P(t,T)]^{1/(k \tau(t,T))}} - k$$  \hspace{1cm} (EQ 11)

so that we can write

$$P(t,T) = \frac{1}{\left(1 + \frac{Y^k(t,T)}{k}\right)^{k \tau(t,T)}}$$  \hspace{1cm} (EQ 12)
Note that continuously-compounded rates can be obtained as the limit of $k$-times-per-period-compounded rates for the number $k$ of compounding times (also called compound frequency) going to infinity. Indeed, we can readily show that

$$\lim_{k \to \infty} \frac{k}{\tau(t,T)} \ln \left( \frac{P(t,T)}{1 + \frac{k}{T}} \right) = R(t,T)$$

(EQ 13)

Noting that for each fixed $Y$:

$$\lim_{k \to \infty} \left( 1 + \frac{Y}{k} \right)^{k \cdot \tau(t,T)} = e^{Y \cdot \tau(t,T)}$$

(EQ 14)

Also, note that all definitions of spot interest rates are equivalent in infinitesimal time intervals. Indeed, it can be easily proved that the short rate is obtainable as a limit of all different rates defined above:

$$r(t) = \lim_{T \to t^+} R(t,T) = \lim_{T \to t^+} L(t,T) = \lim_{T \to t^+} Y(t,T) = \lim_{T \to t^+} Y^k(t,T) \text{ for each } k$$

(EQ 15)

**Fundamental (Market) Interest Rate Curves**

A Fundamental curve that can be obtained from the market data of interest rates is the zero-coupon curve at a given date $t$. This curve is the graph of the function mapping maturities into rates at different times $t$. More precisely, we have the following definition:

**Zero-coupon Curve:** The zero-coupon curve (sometimes referred to as the yield curve) at time $t$ is the graph of the function

$$T \to \begin{cases} L(t,T) & t < T \leq t + 1 \text{ (years)} \\ Y(t,T) & T > t + 1 \text{ (years)} \end{cases}$$

(EQ 16)

Such a zero-coupon curve (see Figure 1) is also called a term structure of interest rates at time $t$. It is the plot of simply-compounded interest rates for all maturities $T$ up to one year and of annually compounded rates for maturities $T$ larger than one year.

The term yield curve is often used to denote several different curves deduced from interest rate market quotes, and is in fact ambiguous. Unless differently specified, when we use this term we mean the zero-coupon curve.

Finally, at times we may consider the same plot for rates with different compounding conventions, such as for example $T \to R(t,T), T > t$. The term interest rate curve will be used to describe all such curves, no matter what compounding convention is being used.
Zero-bond Curve: the zero-bond curve at time $t$ is the graph of the function $T \rightarrow P(t,T)$, $T > t$ which because of positivity of interest rates, is a $T$-decreasing function, starting with $P(t,t) = 1$. Such a curve is also referred to as the term structure of discount factors.

The graph of the zero-bond curve (Figure 2) is less informative than the zero-coupon curve. The former is monotonic, while the latter can show several possible shapes.
Forward Rates

Forward rates are characterized by three time instants, namely the current time \( t \) at which the rate is considered, its expiry \( T \), and its maturity \( S \) with \( t \leq T \leq S \). Forward rates are interest rates that can be locked in today for an investment in a future time period, and are set consistently with the current term structure of discount factors.

We can define a forward rate through a (prototypical) forward-rate agreement (FRA). A FRA is a contract gives the holder an interest rate payment based on the spot rate \( L(T,S) \) resetting in \( T \) and with maturity \( S \). Formally, at time \( S \) one receives \( N \tau(T,S)K \) units of currency and pays the amount \( N \tau(T,S)L(T,S) \), where \( N \) is the contract nominal value. The value of the contract in \( S \) is therefore:

\[
N \tau(T,S)(K - L(T,S)) \quad \text{(EQ 17)}
\]

Here we assume that that both rates have the same day-count convention (in practice the two legs can have different day-count conventions). Clearly, if \( L \) is larger than \( K \) then the value of the contract is negative, and in the other case it is positive. Recalling the expression for \( L \) we can write this as

\[
N \left[ \tau(T,S)K - \frac{1}{P(T,S)} + 1 \right] \quad \text{(EQ 18)}
\]

To calculate the present value of this expression, we note that the second term \( 1/ P(T,S) \) payable at time \( S \) is equivalent to holding one unit of currency at time \( T \). In turn one unit of currency at time \( T \) is worth \( P(t,T) \) units of currency at time \( t \). Similarly, the remaining terms \( \tau(T,S)K + 1 \) at time \( S \) is worth \( P(t,T)(\tau(T,S)K + 1) \) at time \( t \). The total value of the contract at time \( t \) is therefore

\[
\text{FRA}(t,T,S,\tau(T,S),N,K) = N[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)] \quad \text{(EQ 19)}
\]

There is one value of \( K \) that renders the contract fair at time \( t \), i.e. such that the contract value is 0 at time \( t \). The resulting rate defines the simply-compounded forward rate, as defined below.

**Simply-compounded Forward Interest Rate:** The simply-compounded forward interest rate prevailing at time \( t \) for expiry \( T > t \) and maturity \( S > T \) is denoted by \( F(t,T,S) \) and is defined by

\[
F(t,T,S) = \frac{1}{\tau(t,T)} \left( \frac{P(t,T)}{P(t,S)} - 1 \right) \quad \text{(EQ 20)}
\]
It is the value of the fixed rate in a prototypical FRA with expiry \( T \) and maturity \( S \) that renders the FRA a fair contract at time \( t \).

Notice that we can write the value of the FRA above in terms of the simply-compounded forward rate just defined as

\[
\text{FRA}(t, T, \tau(t, T), N, K) = NP(t, S)\tau(t, T)(K - F(t, T, S)) \quad \text{(EQ 21)}
\]

Therefore, to value a FRA, we just have to replace the LIBOR rate \( L(T, S) \) in the payoff with the corresponding forward rate \( F(t, T, S) \), and then take the present value of the resulting deterministic quantity.

We will see later that \( F(t, T, S) \) is the expectation of \( L(T, S) \) at time \( t \) under a suitable probability measure.

When the maturity of the forward rate collapses towards its expiry, we have the notion of instantaneous forward rate. Indeed let us consider the limit:

\[
\lim_{s \to T} \frac{1}{P(t, S)} \frac{P(t, S) - P(t, T)}{S - T} = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T} = -\frac{\partial \ln P(t, T)}{\partial T} \quad \text{(EQ 22)}
\]

Where we have use the convention that \( \tau(T, S) = S - T \) when \( S \) is extremely close to \( T \). This leads to the following definition:

**Instantaneous Forward Interest Rate**: The instantaneous forward interest rate prevailing at time \( t \) for maturity \( T > t \) is denoted by \( f(t, T) \) and is defined by

\[
f(t, T) = \lim_{s \to T} F(t, T, S) = -\frac{\partial \ln P(t, T)}{\partial T} \quad \text{(EQ 23)}
\]

so that we also have

\[
P(t, T) = \exp \left( -\int_t^T f(t, u)du \right) \quad \text{(EQ 24)}
\]

Clearly, for this notion to make sense, we need to assume smoothness of the zero-coupon price function \( T \to P(t, T) \) for all \( T \). Intuitively, the instantaneous forward rate \( f(t, T) \) is the forward rate at time \( t \) whose maturity is very close to its expiry \( T \), say \( f(t, T) \approx F(t, T, T + \Delta T) \) with \( \Delta T \) very small.

Instantaneous forward rates are fundamental quantities in the theory of interest rates. Indeed, as it turns out the most general way to express fairness (absence of arbitrage) interest rate model is to relate certain quantities in the expression for the evolution of \( f \), as
expressed in the celebrated Heath, Jarrow and Morton framework, in which is based on the instantaneous forward rates \( f(t,T) \) as the fundamental quantities to be modeled.

**Interest-Rate Swaps and Forward Swap Rates**

A generalization of the FRA is the Interest Rate Swap (IRS). Here is the definition of a prototypical (forward start) interest rate swap.

**Payer (Forward Start) Interest Rate Swap**: A (prototypical) payer (forward-start) interest rate swap (PFS) is a contact that exchanges the payments between two differently indexed legs, starting from a future time instant. At every instant \( T_i \) in a prespecified set of dates \( T_{a+1}, \ldots, T_{\beta} \), the fixed leg pays out the amount \( N \tau_i K \) corresponding to a fixed interest rate \( K \), a nominal value \( N \) and year fraction \( \tau_i \) between \( T_{i-1} \) and \( T_i \), whereas the floating leg pays the amount \( N \tau_i L(T_{i-1},T_i) \) corresponding to the interest rate \( L(T_{i-1},T_i) \) resetting at the previous instant \( T_{i-1} \) for maturity given by the current payment instant \( T_i \). Clearly, the floating-leg resets at dates \( T_a, T_{a+1}, \ldots, T_{\beta-1} \) and pays at dates \( T_{a+1}, \ldots, T_{\beta} \). We set \( \Gamma = \{ T_a, \ldots, T_{\beta} \} \) and \( \tau = \{ \tau_{a+1}, \ldots, \tau_{\beta} \} \).

In this description we are considering that the fixed-rate payments and floating-rate payments occur at the same dates and with the same year fractions. This is clearly to simplify exposition and is generally not true for realistic IRS.

The discounted payoff at a time \( t < T_a \) of a PFS can be expressed as

\[
N \sum_{i=a+1}^{\beta} D(t,T_i) \tau_i \left( L(T_{i-1},T_i) - K \right) \tag{EQ 25}
\]

For a receiver (forward-start) interest Rate swap (RFS) the discounted payoff at \( t \) is

\[
N \sum_{i=a+1}^{\beta} D(t,T_i) \tau_i \left( K - L(T_{i-1},T_i) \right) \tag{EQ 26}
\]

If we view this last contract as the portfolio of FRAs, we can value each FRA through the formula above and then add up the resulting values to obtain:

\[
RFS(t, \Gamma, \tau, N, K) = \sum_{i=a+1}^{\beta} FRA(t,T_{i-1},T_i, \tau_i, N, K) \]

\[
= N \sum_{i=a+1}^{\beta} \tau_i P(t,T_i) \left( K - F(t,T_{i-1},T_i) \right) \tag{EQ 27}
\]

\[
= N \sum_{i=a+1}^{\beta} \tau_i KP(t,T_i) - NP(t,T_a) + NP(t,T_{\beta})
\]
The two legs of an IRS can be seen as two fundamental (prototypical) contracts. The fixed leg can be thought of as a coupon-bearing bond, and the floating leg can be thought of as a floating-rate note. Therefore, IRS can be viewed as a contract exchanging the coupon-bearing bond with a floating-rate note, defined formally as follows.

**Coupon-bearing Bond:** A (prototypical) *coupon-bearing bond* is a contract that ensures the payment at future times $T_{α+1},...,T_β$ of deterministic amounts of currency (cash-flows) $c = \{c_{α+1},...,c_β\}$. Typically, the *cash flows* are defined as $c_i = Nτ_iK$ for $i < β$ and $c_β = Nτ_βK + N$ where $K$ is a fixed interest rate (coupon rate) and $N$ is the bond nominal value (principal). The last cash flow includes the *reimbursement* of the notional value of the bond. If $K = 0$, the bond reduces to a zero-coupon bond with maturity $T_β$.

Since each cash flow has to be discounted back to the current time $t$ from the payment time $T$, the current value of the bond is

$$CB(t,Γ,c) = \sum_{i=α+1}^{β} c_i P(t,T_i) \tag{EQ 28}$$

**Floating-rate Note:** A (prototypical) *floating-rate note* is a contract ensuring the payment at future times $T_{α+1},...,T_β$ of the LIBOR rates that reset at the previous instants $T_α,...,T_{β-1}$. Moreover, the note pays a last cash flow consisting of *reimbursement* of the nominal value of the note at fixed time $T_β$.

The value of the floating-rate note is obtained by reversing the sign of RFS, setting $K = 0$ and adding the notional payment at maturity, $-RFS(t,Γ,N,0) + NP(t,T_β) = NP(t,T_α)$:

$$FN(t,Γ,c) = NP(t,T_α) \tag{EQ 29}$$

This means that the (prototypical) floating-rate note is always equivalent to $N$ units of currency at its first reset date $T_α$. In particular if $t = T_α$, the value of the note is $N$, so that the floating rate note at its first reset is always equal to its nominal value. This holds as well for $t = T_i$, $i = α + 1,...,β - 1$ in that the value of the note on all the reset dates is $N$. This is sometimes referred to as saying that floating-rate note always trade at par.

We have seen that requiring FRA to be fair leads to the definition of the forward rates. Analogously, we may require that above IRS to be fair at time $t$, and look for a particular $K$ such that the above contract value is zero. This defines the forward swap rate.
Forward Swap Rate: the forward swap rate $S_{\alpha,\beta}(t)$ at time $t$ for the set of times $\Gamma$ and year fractions $\tau$ is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time $t$, i.e., it is the fixed rate $K$ for which $RFS(t, \Gamma, \tau, N, K) = 0$. We easily obtain:

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

(EQ 30)

Letting $F_j(t) = F(t, T_{j-1}, T_j)$ it is simple to see that

$$\frac{P(t, T_k)}{P(t, T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{P(t, T_j)}{P(t, T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1 + \tau_j F_j(t)}$$

(EQ 31)

Therefore, we can also express forward swap rate in terms of the forward rates as

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1 + \tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1 + \tau_j F_j(t)}}$$

(EQ 32)

Figure 3 shows an example of the (spot-start) swap rate curve representing (spot-start) swap rates for different maturities.

Figure 3: US LIBOR Swap (Par) and Zero-Coupon Rate Curves (1/4/2007)
Interest Rate Caps / Floors

We now introduce two main derivative products of the interest rate market, namely caps and swaptions. These products are covered more extensively later on in the chapter devoted to market models.

Interest Rate Caps / Floors

An interest rate cap is a contract that can be viewed as payer IRS where each exchange payment is executed only if it has positive value. The cap discounted payoff is therefore given by

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)N\tau_i\left(L(T_{i-1},T_i) - K\right)^+$$  \hspace{1cm} (EQ 33)

Analogously, an interest rate floor is equivalent to a receiver IRS where the exchange payment is executed only if it has positive value. The floor discounted payoff is therefore given by

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)N\tau_i\left(K - L(T_{i-1},T_i)\right)^+$$  \hspace{1cm} (EQ 34)

Caps can be viewed as mechanism for a company to ensure that its payments under its LIBOR based obligations are capped at a maximum cap rate $K$, as is evident from the equality $L - (L - K)^+ = \min(L, K)$. Similarly, a floor can be used to offer a minimum floor rate on the floating payments of LIBOR based obligations.

A cap can be decomposed additively, as a sum of term $D(t,T_i)N\tau_i\left(L(T_{i-1},T_i) - K\right)^+$. Each such term defines a contract that is termed caplet. The floorlet contracts are also defined in an analogous way.

It is market practice to price a cap with the following sum of Black’s formulas at time 0:

$$\text{Cap}^{\text{Black}}(0,\Gamma,\tau,N,K,\sigma,\beta) = N \sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i \text{Bl}(K,F(0,T_{i-1},T_i),\nu_i,1)$$  \hspace{1cm} (EQ 35)

where, denoting by $\Phi$ the standard Gaussian cumulative distribution function, we have

$$\text{Bl}(K,F,\nu,w) = Fw\Phi\left(w d_1(K,F,\nu)\right) - Kw\Phi\left(w d_2(K,F,\nu)\right)$$  \hspace{1cm} (EQ 36)

$$d_1(K,F,\nu) = \frac{\ln(F/K) + \nu^2/2}{\nu}$$
\[ d_2(K, F, \nu) = \frac{\ln(F/K) - \nu^2/2}{\nu} = d_1 - \nu \]

\[ \nu = \sigma_{\alpha, \beta} \sqrt{T_{i-1}} \]

with a common volatility parameter \( \sigma_{\alpha, \beta} \) that is retrieved from the market quotes (actually there are different types of caps that are quoted in the market than the prototypical one discussed here). Analogously, the corresponding floor is priced according to the formula

\[
\text{Flr}^{\text{Black}}(0, \Gamma, \tau, N, K, \sigma_{\alpha, \beta}) = N \sum_{i=0}^{\beta} P(0, T_i) \tau_i \text{Bl}(K, F(0, T_{i-1}, T_i), \nu_i, -1) \quad \text{(EQ 37)}
\]

An example of the cap volatility curve is shown in the Figure 4 below:

![Figure 4: US LIBOR Cap Volatility Term Structure](image)

The Black’s formulas for caps and floors have both historical and formal justifications which will be explained when we discuss market models.

**Definition** – Consider a cap (floor) with payments dates \( T_{\alpha+1}, \ldots, T_{\beta} \), associated year fractions \( \tau_{\alpha+1}, \ldots, \tau_{\beta} \) and strike \( K \). The cap (floor) is said to be at-the-money (ATM) if and only if

\[
K = K_{\text{ATM}} = S_{\alpha, \beta}(0) = \frac{P(0, T_{\alpha}) - P(0, T_{\beta})}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)} \quad \text{(EQ 38)}
\]
The cap is instead said to be in-the-money (ITM) if $K < K_{ATM}$ and out-of-the-money (OTM) if $K > K_{ATM}$, with the converse holding for a floor.

Using the equality $(L - K)^+ - (K - L)^+ = L - K$ we can note that the difference between a cap and the corresponding floor is equivalent to a forward-start swap. It is, therefore, easy to prove that a cap (floor) is ATM if and only if its price is equal to the corresponding floor (cap).

Notice that in case the cap has only one payment date ($\alpha + 1 = \beta$) then cap collapses to a single caplet. In such a case, the at-the-money caplet strike is $K_{ATM} = F(0, T_\alpha, T_{\alpha+1})$.

**Swaptions**

The second class of basic derivatives we consider are swaptions (or swap options). They are options on an IRS. There are two basic types of swaptions, a payer version and a receiver version.

A *European payer swaption* is an option giving the right (and not the obligation) to enter a payer IRS at a given future time, the swaption maturity. Usually the swaption maturity coincides with the first reset date of the underlying IRS. The underlying IRS length $(T_\beta - T_\alpha)$ is called the tenor of the swaption, and the set of reset and payment dates is called the tenor structure.

The payer swaption will be exercised only if the value of the underlying IRS at the exercise date is positive. Hence, the payer swaption payoff, discounted to the first reset date $T_\alpha$ is given by

$$
N\left( \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) (F(T_\alpha, T_{i-1}, T_i) - K) \right)^+ 
$$

(EQ 39)

Similarly, the payer swaption payoff, discounted to the current time $t$ is given by

$$
ND(t, T_\alpha) \left( \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) (F(T_\alpha, T_{i-1}, T_i) - K) \right)^+ 
$$

(EQ 40)

Contrary to the cap case, this payoff can not be decomposed in more elementary products, and this is the fundamental difference between these two main interest rate derivatives. Indeed, since the positive operator is a piece-wise linear convex function we have that

$$
\left( \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) (F(T_\alpha, T_{i-1}, T_i) - K) \right)^+ \leq \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) (F(T_\alpha, T_{i-1}, T_i) - K)^+ 
$$

(EQ 41)
Therefore in order to value and manage the swaption contracts, we will need to consider joint action of the rates involved in the contract payoff. From the mathematical viewpoint, this implies that, contrary to the cap case, terminal correlations (which are related but more appropriate than instantaneous correlations) between different rates could be fundamental in handling swaptions.

It is the market practice to value swaptions with Black-like formula. Precisely, the price of the above *payer swaption* at time 0 is

$$\mathbf{PS}_{\text{Black}}(0, \Gamma, \tau, N, K, \sigma_{\alpha,\beta}) = N \cdot \text{Bl}(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta} \sqrt{T_a}, 1) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \quad \text{(EQ 42)}$$

where $\sigma_{\alpha,\beta}$ is the volatility parameter in the market that is different from the corresponding volatility for caps / floors. A similar formula is used for *receiver swaption*, which gives the holder the right to enter at time $T_a$ a receiver IRS, with payment dates $\Gamma$:

$$\mathbf{RS}_{\text{Black}}(0, \Gamma, \tau, N, K, \sigma_{\alpha,\beta}) = N \cdot \text{Bl}(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta} \sqrt{T_a}, -1) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \quad \text{(EQ 43)}$$

An example of the market swaption volatility surface is shown in the Figure 5 below where we plot $\sigma_{\alpha,\beta}$ against their corresponding maturities $T_a$ and the swap lengths (tenors) $T_\beta - T_a$:

![Figure 5: US LIBOR ATM Swaption Volatility Surface](image-url)
Similarly to the cap/floor case, the Black formulas above have historical and formal justification which we will analyze in more detail when we discuss market models.

**Definition** – The payer (receiver) swaption defined above is said to be *at-the-money* (ATM) if and only if

\[
K = K_{ATM} = S_{\alpha,\beta}(0) = \frac{P(0, T_\alpha) - P(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)}
\]  
(EQ 44)

The payer swaption is said to be instead *in-the-money* (ITM) if \( K < K_{ATM} \) and *out-of-the-money* (OTM) if \( K > K_{ATM} \), with the converse holding for receiver swaptions.

Finally, note that an alternative expression for the above discounted payer-swaption payoff, expressed in terms of the relevant forward swap rate, is at time 0:

\[
ND(0, T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)
\]  
(EQ 45)

This alternative expression confirms the intuitive meaning of the ITM and OTM expressions. Indeed, if you evaluate this payoff by substituting the forward swap rate \( S_{\alpha,\beta}(T_\alpha) \) with its current forward value \( S_{\alpha,\beta}(0) \) you obtain a positive multiple of \( (S_{\alpha,\beta}(0) - K)^+ \) which is strictly positive if and only if \( S_{\alpha,\beta}(0) > K \).
Homework Assignment 1:

1. In our first exercise we would like to compute spot (zero-coupon) interest rates, forward rates and swap rates with various maturities and tenors from the market supplied discount factors (zero-coupon bond prices). The following table represents prevailing discount factors for various maturities, specified here for simplicity in years (hence no need to specify a day-count convention):

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount Factors</td>
<td>0.985</td>
<td>0.956</td>
<td>0.917</td>
<td>0.874</td>
<td>0.828</td>
<td>0.784</td>
<td>0.740</td>
<td>0.697</td>
<td>0.656</td>
<td>0.616</td>
</tr>
</tbody>
</table>

**Exercise 1:** Using the table above,

- Compute continuously-compounded spot (zero-coupon) interest rates for all the maturities $T_i = 1, 2, \ldots, 10$ given above.

- Similarly, compute simply-compounded, annually-compounded ($k = 1$), semiannually-compounded ($k = 2$), quarterly compounded ($k = 4$), and monthly compounded ($k = 12$) spot (zero-coupon) interest rates.

- Compute the simply-compounded forward rates $F(0, T_{i-1}, T_i)$, $i = 1, 2, \ldots, 10$ with expiry $T_{i-1}$ and maturity $T_i$, assuming $T_0 = 0$.

- Similarly, compute simply-compounded forward rates for 2-year expiry and different maturities, i.e. $F(0, T_2, T_i)$, $i = 3, 4, \ldots, 10$.

- Compute (simply-compounded) forward swap rates $S_{\alpha\beta}$ for $T_\alpha = 0$ and $T_\beta = 1, 2, \ldots, 10$ (Note that in our conventions this means that the first fixed and floating payments occur at $T_{\alpha+1} = T_1$). Furthermore, since $T_\alpha = 0$ we can refer to these rates as “spot” swap rates.

- Repeat the last exercise for $T_\alpha = 2, T_\beta = 3, \ldots, 10$.

**NOTE:** It is the market convention to describe forward rates and forward swap rates in terms of their *expiry* and *tenor*, with “tenor” defined as maturity minus expiry. For example, forward rates $F(0, T_{i-1}, T_i)$ may be described as “$T_{i-1} \times 1$ forward rates” (e.g. $F(0, 1, 2)$ is referred to as “1 x 1” forward rate and $F(0, 3, 7)$ as 3 x 4 forward rate). Similarly, forward swap rates $S_{\alpha\beta}$ are referred to as “$T_\alpha \times (T_\beta - T_\alpha)$ forward swap rates”, (e.g. the forward swap rate $S_{1,5}$ is referred to as “1 x 4 forward swap rate” and $S_{5,9}$ as “5 x 4 forward swap rate”).
2. In our second exercise we would like to compute various forward and swap rates from a market-supplied \textit{annually-compounded} zero-coupon rate curve (yield curve). Here we assume ACT/365 day-count convention (ignoring holiday and month-end conventions) and specify maturities as dates, assuming that the current date is Jan 1, 2007 or “1/1/07”.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1/1/08</th>
<th>1/1/09</th>
<th>1/1/10</th>
<th>1/1/11</th>
<th>1/1/12</th>
<th>1/1/13</th>
<th>1/1/14</th>
<th>1/1/15</th>
<th>1/1/16</th>
<th>1/1/17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero-Coupon Rates</td>
<td>4.45</td>
<td>5.59</td>
<td>6.38</td>
<td>6.85</td>
<td>7.16</td>
<td>7.38</td>
<td>7.55</td>
<td>7.69</td>
<td>7.79</td>
<td>7.88</td>
</tr>
</tbody>
</table>

**Exercise 2:** Using the table above,

- Compute \textit{year-fractions} between the current date and dates $D_i$, $i = 1,2,...,10$ in above table, assuming ACT/365 convention.
- Compute \textit{zero-coupon bond prices} (discount factors) for $T_i = 1,2,...,10$ given above.
- Compute the \textit{simply-compounded} forward rates $F(0,T_{i-1},T_i)$, $i = 1,2,...,10$.
- Compute (simply-compounded) spot swap rates $S_{a,\beta}$ for $T_a = 0$, $T_\beta = 1,2,...,10$.

3. In our third exercise we would like to compute prices of ATM caps/floors and payer/receiver swaptions (using Black’s formulas) for various tenors (specified for simplicity in years) using the yield curve shown below and assuming constant volatility parameter $\sigma = 0.15$ in all cases:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero-coupon Rates</td>
<td>3.18</td>
<td>3.55</td>
<td>3.77</td>
<td>3.95</td>
<td>4.11</td>
<td>4.25</td>
<td>4.38</td>
<td>4.49</td>
<td>4.59</td>
<td>4.68</td>
</tr>
</tbody>
</table>

**Exercise 3:** Using the table above,

- Applying Black’s cap/floor formula, calculate prices of ATM \textit{caplets} and \textit{floorlets} (i.e., with strikes equal to the respective forward rates) for all expirations $T_i = 1,2,...,10$ given the assumptions above.
- Using these, compute the price of ATM \textit{caps} and \textit{floors} (i.e. sum of caplets/floorlets) with the first reset at $T_a = 1$ and maturities $T_\beta = 2,3,...,10$.
- Repeat the last exercise for 10% ITM and OTM caps and floors, i.e. setting the caplet/floorlet strikes equal to 0.9 (or 1.1) times the corresponding forward rates.
- Applying Black’s swaption formula, calculate the price of ATM payer and receiver swaptions with expiration (i.e., the first reset of the underlying forward-start swap at) $T_a = 1$ and maturities $T_{\beta} = 2, 3, \ldots, 10$.

- Repeat the last exercise for 10% ITM and OTM payer as well as receiver swaptions.

**Extra Credit:** Using the same inputs as in Exercise 3, compare the price and sensitivities (delta, gamma, theta, vega) of 2 x 5 caps and payer swaptions (i.e, $T_a = 2$ and $T_{\beta} = 7$) for strikes in the range [0.01, 0.03, … 0.10]. Note that you can compute the all sensitivities by numerically shifting the relevant parameters.