1 The simplex method for uncapacitated network flow problems

1. A typical iteration starts with a basic feasible solution $f$ associated with a tree $T$. (What is a tree? A tree is a graph with no cycles)

2. To compute the dual vector $p$, solve the system of equation:

$$
\begin{align*}
p_i - p_j &= c_{ij} \forall (i,j) \in T \\
p_n &= 0
\end{align*}
$$

By proceeding from the root towards the leaves.

3. Compute the reduced costs $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$ of all arcs $(i,j) \notin T$. If they are all nonnegative, the current basic feasible solution is optimal and the algorithm terminates; else, choose some $(i,j)$ with $\bar{c}_{ij} < 0$ to be brought into the basis.

4. The entering arc $(i,j)$ and the arcs in $T$ form a unique cycle. If all arcs in the cycle are oriented the same way as $(i,j)$ then the optimal cost is $-\infty$ and the algorithm terminates.

5. Let $B$ be the set of arcs in the cycle that are oriented in the opposite direction from $(i,j)$. Let $\theta^* = \min_{(k,l) \in B} f_{kl}$, and push $\theta^*$ units of flow around the cycle. A new flow vector is determined:

$$
\hat{f}_{kl} = \begin{cases} 
  f_{kl} + \theta & \text{if } (k,l) \in F \\
  f_{kl} - \theta & \text{if } (k,l) \in B \\
  f_{kl} & \text{Otherwise}
\end{cases}
$$

Remove from the basis one of the old basic variables whose new value is equal to zero.

2 The simplex method for capacitated network flow problems

1. A typical iteration starts with a basic feasible solution $f$ associated with a tree $T$ and a partition of remaining arcs into two sets $D, U$, such that $f_{ij} = d_{ij}$ for $(i,j) \in D$, and $f_{ij} = u_{ij}$ for $(i,j) \in U$.

2. To compute the dual vector $p$, solve the system of equation:

$$
\begin{align*}
p_i - p_j &= c_{ij} \forall (i,j) \in T \\
p_n &= 0
\end{align*}
$$

By proceeding from the root towards the leaves.

3. Compute the reduced costs $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$ of all arcs $(i,j) \notin T$. If $\bar{c}_{ij} \geq 0$ for all $(i,j) \in D$, and $\bar{c}_{ij} \leq 0$ for all $(i,j) \in U$, the current basic feasible solution is optimal and the algorithm terminates.

4. Let $(i,j)$ be an arc such that $\bar{c}_{ij} < 0$ and $(i,j) \in D$, or such that $\bar{c}_{ij} > 0$ and $(i,j) \in U$. This arc $(i,j)$ together with the tree $T$ forms a unique cycle. Choose the orientation of the cycle as follows. If $(i,j) \in D$, then $(i,j)$ should be a forward arc. If $(i,j) \in U$, then $(i,j)$ should be a backward arc.
5. Let $F$ and $B$ be the forward and backward arcs respectively, in the cycle. Determine $\theta^*$:

$$\theta^* = \min \left\{ \min_{(k,l) \in B} \{ f_{kl} - d_{kl} \}, \min_{(k,l) \in F} \{ u_{kl} - f_{kl} \} \right\}$$  \hfill (4)$$

A new flow vector is determined:

$$\hat{f}_{kl} = \begin{cases} 
  f_{kl} + \theta & \text{if } (k,l) \in F \\
  f_{kl} - \theta & \text{if } (k,l) \in B \\
  f_{kl} & \text{Otherwise}
\end{cases}$$  \hfill (5)$$

Finally update the set $T, D, U$.

3 Example

Consider the following network flow diagram.

Figure 1: The network

Using this diagram, determine the primal and dual basic solutions when

a. $n_B = \{8, 7, 2, 5\}$, $n_0 = \{1, 3, 6\}$, $n_1 = \{4\}$.

b. $n_B = \{8, 1, 2, 4\}$, $n_0 = \{6, 7\}$, $n_1 = \{3, 5\}$.

c. $n_B = \{8, 1, 2, 4\}$, $n_0 = \{6, 7\}$, $n_1 = \{3, 5\}$.

Classify each as feasible, infeasible or no solution. Classify feasible solution as optimal or non optimal. In each case, indicate the reason for your classification.
Figure 2: part a.

Marginal costs: \( c_1 = 7, c_2 = -4, c_3 = -10, c_4 = 8 \). The solution is feasible, but not optimal. Arc 3 is a candidate to enter the basis.

Figure 3: part b.

Marginal costs: \( c_1 = 3, c_2 = 3, c_3 = 8, c_4 = -7 \). The solution is feasible, but not optimal. Arcs 3, 5 and 7 are candidates to enter the basis.

Figure 4: part c.

The solution is not feasible because some flows are above the upper bound and some flows are negative.