IEOR 4703: Homework 7

Here we explore estimating some Greeks for some derivatives of GBM. The basic framework is a payoff $C_T$ at expiration date $T$, and its price $C_0 = e^{-rT}E(C_T)$, where we are using the risk-neutral measure (probability) for the underlying GBM. We wish to estimate, for example, $\Delta = \frac{dC_0}{dS(0)}$, the sensitivity to the initial stock price. In what follows, we will be considering a sample-path approach, which works only when it can be verified (a priori) that the operations of taking derivative (wrt a parameter $\alpha$) and taking expected value can be interchanged, that is, that

$$\frac{dE(C_T)}{d\alpha} = E\left[\frac{dC_T}{d\alpha}\right]. \quad (1)$$

For if this is so, then we can use Monte Carlo simulation to estimate the right-hand side in (1): Simulate $n$ iid copies of $\frac{dC_T}{d\alpha}$ and take the empirical average.

To dispense with the notion that such an interchange as in (1) is always possible (no, it is not!) one merely need consider a digital option with payoff $C_T = I\{S(T) > K\}$. In this case, for example, $\frac{dC_T}{dS(0)} = 0$ since the indicator is a piecewise constant function of $S(0)$, thus $E\left[\frac{dC_T}{dS(0)}\right] = 0$. But $E(C_T) = P(S(T) > K)$ is a nice smooth function of $S(0) > 0$, with a non-zero derivative. (Yes, $C_T$ is not differentiable (nor continuous even) at the value of $S(0)$ for which $S(T) = K$, but $P(S(T) = K) = 0$, so this point can be ignored.) Since taking a derivative is actually taking a limit ($f'(x) = \lim_{h \to 0} (f(x + h) - f(x))/h$), the needed condition for the interchange (1) is in fact uniform integrability.

In a typical application of the above sample-path method, we could, using the same Monte Carlo simulation runs, estimate both the price and its Greek at the same time; another advantage of the method.

Exercises

Consider GBM under its risk-neutral measure, for the purposes of pricing our derivatives. Suppose the risk-free interest rate $r = 0.05$, and $\sigma = 0.04$. Thus the $\mu$ in the Brownian motion is to be set to $\mu^* = r - \sigma^2/2 = 0.05 - 0.0008 = 0.0492$ yielding $X(t) = (0.04)B(t) + (0.0492)t$, and $S(t) = S(0)e^{X(t)}$.

1. The discounted payoff of the Asian call option, that you priced in Homework 3, is given by

$$Y = e^{-0.20}E\left(1 + \sum_{i=1}^{4} S(i) - 40\right)^+. \quad (i)$$

The price of this option is thus $C_0 = E(Y)$. Letting $\Sigma = \sum_{i=1}^{4} S(i)$, we can re-write the payoff as

$$Y = e^{-0.20}E(\Sigma - 40)^+.$$

The sample-path derivative of $Y$ with respect to the initial price of the stock, $S(0)$, is easily shown\(^\dagger\) to be (wp1)

$$\frac{dY}{dS(0)} = e^{-0.20}I\{\Sigma > 40\} \frac{\Sigma}{S(0)}.$$

\(^\dagger\)if $h(y) = (y - K)^+$, then for $y < K$, $h'(y) = 0$, and for $y > K$, $h'(y) = 1$; thus (except for the point $y = K$ where the derivative does not exists) $h'(y) = I\{y > K\}$. More generally (chain rule) if $f(x) = (g(x) - K)^+$, then $f'(x) = g'(x)I\{g(x) > K\}$. There is non-differentiability at the point $g(x) = K$, but in our case, $(S(T) - K)^+$, $P(S(T) = K) = 0$ since $S(T)$ has a continuous distribution.
It is known\(^2\) that this is an unbiased estimator for the \(\Delta\) of this option, that is,

\[
\Delta \overset{\text{def}}{=} \frac{dC_0}{dS(0)} = E(D).
\]

By simulating \(n\) (large) iid copies of \(D\), we can thus estimate the \(\Delta\) via

\[
\Delta \approx \frac{1}{n} \sum_{i=1}^{n} D_i.
\]

For \(S(0) = 35\), carry out such a simulation with \(n = 5000\) to obtain the \(\Delta\) estimate.

2. Lookback call option:

Consider the discounted payoff

\[
Y = e^{-0.20} (S(4) - \min\{S(1), S(2), S(3), S(4)\}).
\]

In this case, we can re-write this by pulling out the \(S(0)\) to obtain

\[
Y = S(0) \frac{Y}{S(0)},
\]

where \(Y/S(0)\) eliminates the \(S(0)\) from each piece \(S(i)\) and thus has no more \(S(0)\) in it. Consequently

\[
D = \frac{dY}{dS(0)} = \frac{Y}{S(0)},
\]

\(D\) can also be shown to be an unbiased estimate for the \(\Delta\).

For \(S(0) = 35\), carry out such a simulation with \(n = 5000\) to obtain the \(\Delta\) estimate.

3. Vega for the Asian call:

Here we want the derivative of the price with respect to the volatility \(\sigma\), \(vega = \frac{dC_0}{d\sigma}\). Here we illustrate a nice recursive method for deriving the sample-path derivative \(D = \frac{dY}{d\sigma}\).

Let us consider a general Asian call with discounted payoff \(Y = e^{-rT} (S - K)^+\), where

\[
S = \frac{1}{k} \sum_{i=1}^{k} S(t_i); \quad 0 < t_1 < \cdots < t_k = T.
\]

First of all, direct calculation yields

\[
D = \frac{dY}{d\sigma} = e^{-rT} \left[ \frac{1}{k} \sum_{i=1}^{k} \frac{dS(t_i)}{d\sigma} \right] I\{S > K\}.
\]

(2)

So it remains to derive the \(\frac{dS(t_i)}{d\sigma}\).

But given iid unit normals, \(Z_1, \ldots, Z_k\), we can recursively construct

\[
S(t_i) = S(t_{i-1}) e^{X(t_i) - X(t_{i-1})}
= S(t_{i-1}) e^{\sigma \sqrt{t_i - t_{i-1}} Z_i + (r - \sigma^2/2)(t_i - t_{i-1})}.
\]

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\(^2\)In this case uniform integrability is easily verified directly: Let \(f(S(0)) = (S - K)^+\), where \(S = \frac{1}{k} \sum_{i=1}^{k} S(t_i)\). Then it is easily verified that \(f(S(0) + h) - f(S(0))/h \leq X = S/S(0)\) (independent of \(h\)) and since \(E(X) < \infty\), the result follows; \(\frac{dC_0}{dS(0)} = \lim_{h \to 0} e^{-rT} E((f(S(0) + h) - f(S(0)))/h) = E(\Delta)\).
Differentiating both sides wrt $\sigma$ then yields the recursion

$$\frac{dS(t_i)}{d\sigma} = \frac{dS(t_{i-1})}{d\sigma} \frac{S(t_i)}{S(t_{i-1})} + S(t_i)(-\sigma[t_i - t_{i-1}] + \sqrt{t_i - t_{i-1}} Z_i),$$

where we already know that $\frac{dS(0)}{d\sigma} = 0$ since $S(0)$ is a constant. Thus solving the recursion yields

$$\frac{dS(t_i)}{d\sigma} = S(t_i)(-\sigma t_i + \sum_{j=1}^{i} \sqrt{t_j - t_{j-1}}Z_j).$$

We can if we so wish (via algebra) re-write this so as to get rid of the dependence on specific $Z_j$:

$$\frac{dS(t_i)}{d\sigma} = S(t_i) \frac{1}{\sigma}(- (r + \sigma^2/2)t_i + \ln \left( \frac{S(t_i)}{S(0)} \right)).$$

Plugging in to (2) then gives us our $D$. (And it can be shown to be unbiased.)

For $S(0) = 35$, use the same parameters as in Exercise 1 to carry out such a simulation with $n = 5000$ to obtain the *vega* estimate.